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COMMON FIXED POINT THEOREMS FOR INTERPOLATIVE MAPPINGS IN BICOMPLEX-VALUED b-METRIC SPACES WITH AN APPLICATION TO NON-LINEAR MATRIX EQUATIONS

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ABSTRACT. This manuscript plans to prove a common fixed point theorem for interpolative contraction mappings in Bicomplex-valued b-metric spaces. Our results generalize and extend several works in literature. We provide an example for verification of our results. To demonstrate the effectiveness of our main result, we give an application to non-linear matrix equations.

1. Introduction

Bakhtin [3] and Czerwik [7] generalized metric space to b-metric spaces and developed Banach's contraction principle [4] to these spaces. The study of bicomplex numbers was initiated in 1982 by [33] who gave some properties of bicomplex numbers. In 1934, Dragoni [10] established the first rudiments of function theory on bicomplex numbers. In 1991, Price [27] gave an introduction to multicomplex spaces and functions. In 2011, Azam *et al.* [2] gave the concepts of new spaces called complex valued metric spaces and established the existence of fixed point theorems under the contraction condition in rational expression. Marzouki *et al.* [24] gave a generalized common fixed point theorem in complex-valued b-metric spaces. Rao *et al.* [30] proved a common fixed point theorem in complex-valued b-metric spaces. In 2020, Datta *et al.* [9] by combining the concepts mentioned above proved some common fixed point theorems for contractive mappings in bicomplex valued b-metric spaces. In 2021, Beg *et al.* [5] proved the fixed point in bicomplex valued metric spaces. Mani *et al.* [23] proved the results for the solution of a Fredholm integral equation via a common fixed point theorem on bicomplex valued b-metric space.

Recently, the study of nonlinear matrix equations was given by Garai and Dey [11] who gave a common solution to a pair of non-linear matrix equations via fixed point results. Nashine *et al.* [26] found a common positive solution of two nonlinear matrix equations using fixed point Results. Joseph *et al.* [14] gave some results by solving a system of linear equations via a bicomplex valued metric. The complex-valued metric has several applications in the branches of Mathematics, including algebraic geometry, number theory, applied Mathematics, hydrodynamics, mechanical engineering, thermodynamics and electrical engineering.

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Furthermore, Karapinar [18] converted the classical Kannan [17] contraction to interpolative Kannan mapping in metric spaces. Since then, several finding has been presented for various type of interpolative mappings in different spaces. Karapinar *et al.* [19] generalized the results on interpolative Hardy-Rogers type contractions. Yesikaya [37] gave the results on interpolative Hardy-Rogers Suzuki-type contractive mappings. Gautam *et al.* [12, 13] proved fixed point results for ω-interpolative Chatterjea type contraction in quasi-partial b-metric spaces. Mishra *et al.* [25] introduced an interpolative Reich-Rus-Ciric and Hardy-Rogers contraction on quasi-partial b-metric spaces and related fixed point results. Alansari and Ali [1] gave some results on interpolative presic type contractions. Wangwe and Kumar [35] proved fixed point results for interpolative ψ-Hardy-Rogers type contraction mappings in quasi-partial b-metric spaces with applications.

This manuscript aims to prove some common fixed point theorems for interpolative contraction mappings in bicomplex valued b-metric spaces. In particular, we generalize and extend the results proved by Beg *et al.* [5], Mani *et al.* [23], Datta *et al.* [9], Nashine *et al.* [26] and Joseph *et al.* [14].

2. Preliminaries

In this section, we give some preliminaries of definitions and theorems for developing new results.

We denote the set of complex numbers and bicomplex numbers as $\mathbb{C}_0, \mathbb{C}_1, \mathbb{C}_2$.

Segre [33] defined the complex number as follows:

$$z = z_1 + z_2 i_1,$$

where $z_1, z_2 \in \mathbb{C}_0$, $i_1^2 = -1$. Define

$$\mathbb{C}_1 = \{z : z = z_1 + z_2 i_1, z_1, z_2 \in \mathbb{C}_0\}.$$

Let $z \in \mathbb{C}_1$. The norm $\|.\| : \mathbb{C}_1 \to \mathbb{C}_0^+$ is then defined by

$$\|z\| = \sqrt{z_1^2 + z_2^2}.$$

The concept of bicomplex number was given by

$$\theta = \kappa_1 + \kappa_2 i_1 + \kappa_3 i_2 + \kappa_4 i_1 i_2, \tag{1}$$

where $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \in \mathbb{R}$ and the independent units i_1, i_2 are such that

$$\begin{aligned} i_1^2 &= i_2^2 = -1, \\ i_1 i_2 &= i_2 i_1. \end{aligned}$$

We denote

$$i_1 i_2 = j,$$

which is known as a hyperbolic unit and such that

$$\begin{aligned} j^2 &= (i_1 i_2)^2 = i_1^2 i_2^2 = 1, \\ i_1 j &= i_1(i_1 i_2) = i_1^2 i_2 = j i_1 = -i_2, \\ i_2 j = j i_2 &= -i_1. \end{aligned}$$

The set of bicomplex numbers $\mathbb{C}_2$ is defined as:

$$\mathbb{C}_2 = \{\theta : \theta = \kappa_1 + \kappa_2 i_1 + \kappa_3 i_2 + \kappa_4 i_1 i_2, \kappa_1, \kappa_2, \kappa_3, \kappa_4 \in \mathbb{C}_0\}.$$

In another way:

$$\mathbb{C}_2 = \{\theta : \theta = z_1 + i_2 z_2, z_1, z_2 \in \mathbb{C}_0\}, \tag{2}$$

where

$$\begin{aligned} z_1 &= \kappa_1 + \kappa_2 i_1 \in \mathbb{C}_1, \\ z_2 &= \kappa_3 + \kappa_4 i_1 \in \mathbb{C}_1. \end{aligned}$$

The operations on bicomplex numbers. If $\theta = z_1 + i_1 z_2$ and $\sigma = \varpi_1 + i_2 \varpi_2$, then the sum and subtraction is given by

$$\begin{aligned} \theta + \sigma &= z_1 + i_1 z_2 + \varpi_1 + i_2 \varpi_2 = (z_1 + \varpi_1) + i_2(z_2 + \varpi_2), \\ \theta - \sigma &= z_1 - i_1 z_2 + \varpi_1 + i_2 \varpi_2 = (z_1 - \varpi_1) + i_2(z_2 - \varpi_2), \end{aligned}$$

and the product is given by

$$\theta.\sigma = (z_1 + i_1 z_2).(\varpi_1 + i_2 \varpi_2) = (z_1 \varpi_1 - z_2 \varpi_2) + i_2(z_1 \varpi_2 - z_2 \varpi_1).$$

In $\mathbb{C}_2$ there are four idempotents (unchanged) elements, that are $0, 1, e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$ out of which e_1 and e_2 are non-trivial such that $e_1 + e_2 = 1$ and $e_1.e_2 = 0$.

So, every bicomplex number $z_1 + i_1 z_2$ can uniquely be expressed as the combination of e_1 and e_2, namely

$$\theta = z_1 + i_1 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2. \tag{3}$$

This representation of θ is known as the idempotent representation of bicomplex numbers and the complex coefficient $\theta_1 = (z_1 - i_1 z_2)$ and $\theta_2 = (z_1 + i_1 z_2)$ are known as idempotent components of the bicomplex number θ. That is

$$\theta = z_1 + i_1 z_2 = \theta_1 e_1 + \theta_2 e_2. \tag{4}$$

An element $\theta = z_1 + i_1 z_2 \in \mathbb{C}_2$ is non-singular if and and only if $|z_1^2 + z_{2|}^2| \neq 0$ and singular if and only if $|z_1^2 + z_{2|}^2| = 0$. The inverse of θ is defined as

$$\theta^{-1} = \frac{z_1 - i_1 z_2}{z_1^2 + z_2^2}.$$

The norm $\|.\|$ of $\mathbb{C}_2$ is a positive real valued function $\|.\| : \mathbb{C}_2 \to \mathbb{C}_0^+$ which is defined by

$$\begin{aligned} \|\theta\| &= \|z_1 + i_1 z_2\| = \sqrt{z_1^2 + z_2^2}, \\ &= \sqrt{\frac{|z_1 - i_1 z_2|^2 + |z_1 + i_1 z_2|^2}{2}}, \\ &= \sqrt{\kappa_1^2 + \kappa_2^2 + \kappa_3^2}, \end{aligned}$$

where

$$\begin{aligned} \theta &= \kappa_1 + \kappa_2 i + \kappa_3 i_2 + \kappa_4 i_1 i_2, \\ &= z_1 + i_2 z_2. \end{aligned}$$

The vector space $\mathbb{C}_2$ for the defined norm is a normed linear space, and $\mathbb{C}_2$ is complete. Therefore $\mathbb{C}_2$ is a Banach space. If $\theta, \sigma \in \mathbb{C}_2$, then

$$\|\theta.\sigma\| \quad \leq \quad \sqrt{2}\|\theta\|\|\sigma\|,$$

hold instead of

$$\|\theta.\sigma\| \quad \leq \quad \|\theta\|\|\sigma\|,$$

and therefore $\mathbb{C}_2$ is not a Banach algebra.

For any two bicomplex numbers $\theta, \sigma \in \mathbb{C}_2$. It follows that

(1) $\theta \precsim_{i_2} \sigma \Longleftrightarrow \|\theta\| \leq \|\sigma\|$,
(2) $\|\theta + \sigma\| \leq \|\sigma\| + \|\sigma\|$,
(3) $\|\lambda\theta\| \leq \lambda\|\sigma\|$, where $\lambda \in \mathbb{R}$,
(4) $\|\theta.\sigma\| \leq \sqrt{2}\|\theta\|\|\sigma\|$, and the equality holds only when at least one of θ and σ is degenerated.
(5) $\|\theta^{-1}\| = \|\theta\|^{-1}$ if θ is a degenerated with $0 \prec \theta \in \mathbb{C}_2$.
(6) $\|\frac{\theta}{\sigma}\| = \frac{\|\theta\|}{\|\sigma\|}$, if $\theta \in \mathbb{C}_2$ is degenerated.

Now, we give the partial order relation $\precsim_{i_2}$ on $\mathbb{C}_2$ as below: Let $\mathbb{C}_2$ be the set of bicomplex numbers and $\theta = z_1 + i_2 z_2, \sigma = \varpi_1 + i_2\varpi_2 \in \mathbb{C}_2$. Then $\theta \preceq_{i_2} \sigma$ if and only if $z_1 \preceq_{i_2} \varpi_1$ and $z_2 \preceq_{i_2} \varpi_2$ i.e., $\theta \precsim_{i_2} \sigma$ if one of the following axioms satisfied:

(1) $z_1 = \varpi_1, z_2 = \varpi_2$,
(2) $z_2 \preceq \varpi_1, z_2 = \varpi_2$,
(3) $z_1 = \varpi_1, z_2 \preceq \varpi_2$,
(4) $z_1 \preceq \varpi_1, z_2 \preceq \varpi_2$.

In particular we can write $\theta \precneqq_{i_2} \sigma$ if $\theta \precsim_{i_2} \sigma$ and $\theta \neq \sigma$, i.e., one of $(2) - (4)$ is satisfied. We write $\theta \prec_{i_2} \sigma$, if only (4) is satisfied.

The metric function in bicomplex-valued b-metric spaces is as follows:

Definition 2.1. [9] Let Υ be a non-empty set and $s \geq 1$ be a given real number. A function $d_B : \Upsilon \times \Upsilon \rightarrow \mathbb{C}_2$ is called a bicomplex b-metric on Υ, such that

(BCM1) $0 \preceq_{i_2} d_B(\vartheta, \varpi)$ and $d_B(\vartheta, \varpi) = 0$ if and only if $\vartheta = \varpi$ for all $\vartheta, \varpi \in \Upsilon$,
(BCM2) $d_B(\vartheta, \varpi) = d_B(\varpi, \vartheta)$ for all $\theta, \vartheta \in \Upsilon$,
(BCM3) $d_B(\vartheta, \varpi) \preceq_{i_2} s[d_B(\vartheta, \varrho) + d(\varrho, \varpi)]$ for all $\vartheta, \varpi, \varrho \in \Upsilon$.

Then (Υ, d_B) is called a bicomplex-valued b-metric on Υ.

The following are examples which satisfy the axioms of bicomplex-valued b-metric spaces.

Example 2.1. [9] Let $\Upsilon = [0, 1] \in \mathbb{C}_2$ be a set of bicomplex b-metric. Define $d_B : \mathbb{C}_2 \times \mathbb{C}_2 \rightarrow \mathbb{C}_2$ by

$$d_B(\vartheta, \varpi) = \|\vartheta - \varpi\|^2 + i_2\|\vartheta - \varpi\|^2.$$

Then $(\mathbb{C}_2, d_B)$ is a bicomplex-valued b-metric space.

Example 2.2. [23] Let $\Upsilon = \mathbb{C}_2$. Define a metric $d_B : \mathbb{C}_2 \times \mathbb{C}_2 \rightarrow \mathbb{C}_2$ by

$$d_B(\vartheta, \varpi) = |\vartheta - \varpi|e^{i_2 k},$$

where $k \in [0, \frac{\pi}{2}]$. Then $(\mathbb{C}_2, d_B)$ is a complex-valued b-metric space with $s = 2$.

Example 2.3. [9] Let $\Upsilon = [0,1]$. Consider a metric $d_B : \Upsilon \times \Upsilon \to \mathbb{C}_2$ by

$$d_B(\vartheta, \varpi) = (1 + i_1 + i_2 + i_1 i_2)|\vartheta - \varpi|^2.$$

Then $(\mathbb{C}_2, d_B)$ is a complex-valued b-metric space with $s = 2$.

Definition 2.2. *[9] Let (Υ, d_B) be a bicomplex valued b-metric space. A point $\vartheta \in \Upsilon$ is said to be an interior point of a set $\mathcal{D} \subseteq \Upsilon$ whenever we can find $0 < \varpi \in \mathbb{C}$ satisfying $B(\vartheta, \varpi) = \{\varpi \in \Upsilon : d_B(\vartheta, \varpi) \preceq_{i_2} \varpi\} \in \mathcal{D}$, where $B(\vartheta, \varpi)$ is an open ball. Then, $B(\vartheta, \varpi) = \{\varpi \in \Upsilon : d_B(\vartheta, \varpi) \preceq_{i_2} \varpi\}$ is a closed ball.*

We give the fundamental properties on bicomplex-valued b-metric spaces.

Definition 2.3. [9] Let (Υ, d_B) be a bicomplex-valued b-metric space. Let $\{\vartheta_n\}$ be a sequence in Υ and $\vartheta \in \Upsilon$.

(i) If for every sequence $\{\vartheta_n\}$ is said to be a convergent sequence and converge to a point ϑ if, for any $0 \prec_{i_2} r \in \mathbb{C}_2$, there is a natural number $n_0 \in \mathbb{N}$ such that

$$d_B(\vartheta_n, \vartheta) \prec_{i_2} r,$$

for all $n \geq n_0$.

We write this by

$$\lim_{n\to\infty} \vartheta_n = \vartheta \textit{ or } \vartheta_n \to \vartheta \textit{ as } n \to \infty.$$

(ii) A sequence ϑ_Ω is said to be a Cauchy sequence in (Υ, d_B) if for any $0 \prec_{i_2} r \in \mathbb{C}_2$ there is a natural number $n_0 \in \mathbb{R}$ such that

$$d_B(\vartheta_n, \vartheta_{n+m}) \prec_{i_2} r,$$

for all $n, m \in \mathbb{N}$ an $n > n_0$.

(iii) If every Cauchy sequence in Υ is convergent in Υ then (Υ, d_B) is said to be a complete bicomplex-valued b-metric space.

Lemma 2.1. [9] Let (Υ, d_B) be a bicomplex-valued b-metric space. A sequence and (ϑ_n) is convergences to $\vartheta_n \in \Upsilon$ if and only if

$$\lim_{n\to\infty} \|d_B(\vartheta_n, \vartheta)\| \;=\; 0.$$

Lemma 2.2. [9] Let (Υ, d_B) be a bicomplex-valued b-metric space. $\{\vartheta_n\}$ is a Cauchy sequence in Υ if and only if

$$\lim_{n,m\to\infty} \|d_B(\vartheta_n, \vartheta_{n+m})\| \;=\; 0.$$

Definition 2.4. *[15, 34] Let $\mathcal{S}$ and $\mathcal{T}$ be two self-mapping of a non-empty set Υ.*

(i) *A point $\vartheta \in \Upsilon$ is called a fixed point of $\mathcal{S}$ if $\mathcal{S}\vartheta = \vartheta$.*

(ii) *A point $\vartheta \in \Upsilon$ is called a coincidence point of $\mathcal{S}$ and $\mathcal{T}$ if $\mathcal{S}\vartheta = \mathcal{T}\vartheta$, and the point $\vartheta \in \Upsilon$ such that $\vartheta = \mathcal{S}\vartheta = \mathcal{T}\vartheta$ is called point of coincidence of $\mathcal{S}$ and $\mathcal{T}$.*

(iii) *A point $\vartheta \in \Upsilon$ is called a common fixed point of $\mathcal{S}$ and $\mathcal{T}$ if $\vartheta = \mathcal{S}\vartheta = \mathcal{T}\vartheta$.*

(iv) *Let $\mathcal{S}, \mathcal{T} : \Upsilon \to \Upsilon$ be two self mappings then $\mathcal{S}$ and $\mathcal{T}$ are said to be weakly compatible if $\mathcal{S}\mathcal{T}\vartheta = \mathcal{T}\mathcal{S}\vartheta$ whenever $\mathcal{S}\vartheta = \mathcal{T}\vartheta$.*

We establish some preliminary results:

Joseph *et al.* [14] proved a common fixed point theorem on a bicomplex valued metric space, as follows:

Theorem 2.1. *[14] Let (Υ, d_B) be a complete bicomplex valued metric space and $\mathcal{S}, \mathcal{T} : \Upsilon \to \Upsilon$ be self mappings such that*

$$d_B(\mathcal{S}\vartheta, \mathcal{S}\varpi) \quad \preceq_{i_2} \quad \alpha d_B(\vartheta, \varpi) + \frac{\beta d_B(\vartheta, \mathcal{S}\vartheta) d_B(\varpi, \mathcal{T}\varpi) + \gamma d_B(\varpi, \mathcal{S}\vartheta) d_B(\vartheta, \mathcal{T}\varpi)}{1 + d_B(\vartheta, \varpi)},$$

for all $\vartheta, \varpi \in \Upsilon$, where α, β, γ are non-negative reals with $\alpha + \sqrt{2\beta} + \sqrt{2\gamma} < 1$. Then $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point.

Azam *et al.* [2] generalized Dass and Gupta [8] contraction mapping from metric space to complex-valued metric space as follows:

Theorem 2.2. *[2] Let (Υ, d) be a complete complex-value metric space and $\mathcal{S}, \mathcal{T} : \Upsilon \to \Upsilon$ be two mappings. If $\mathcal{S}$ and $\mathcal{T}$ satisfy*

$$d(\mathcal{S}\vartheta, \mathcal{T}\varpi) \quad \preceq \quad \lambda d(\vartheta, \varpi) + \mu \frac{d(\vartheta, \mathcal{S}\vartheta) d(\varpi, \mathcal{T}\varpi)}{1 + d(x, y)},$$

for all $\vartheta, \varpi \in \upsilon$, where λ, μ are non-negative real with $\lambda + \mu < 1$. Then $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point in Υ.

Mani *et al.* [23] proved the following theorem in bicomplex valued b-metric spaces.

Theorem 2.3. *[23] Let (Υ, d_B) be a complete bicomplex valued b-metric space with coefficient $s \geq 1$ and $\mathcal{S}, \mathcal{T} : \Upsilon \to \Upsilon$. If there exists $\lambda_1, \lambda_2, \lambda_3, \lambda_4 : \Upsilon \to [0, \frac{1}{s}]$ such that for all $\vartheta, \varpi \in \Upsilon$;*

(i) *$\lambda_1(\mathcal{T}\sigma) \leq \lambda_1(\vartheta), \lambda_2(\mathcal{T}\vartheta) \leq \lambda_2(\vartheta), \lambda_3(\mathcal{T}\vartheta) \leq \lambda_3(\vartheta)$ and $\lambda_4(\mathcal{T}\vartheta) \leq \lambda_4(\vartheta)$,*

(ii) *$\lambda_1(s\vartheta) \leq \lambda_1(\vartheta), \lambda_2(s\vartheta) \leq \lambda_2(\vartheta), \lambda_3(s\vartheta) \leq \lambda_3(\vartheta)$ and $\lambda_4(s\vartheta) \leq \lambda_4(\vartheta)$,*

(iii) *$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 < 1$,*

(iii) *Also,*

$$\begin{aligned} d_B(\mathcal{S}\vartheta, \mathcal{T}\varpi) \quad \preceq_{i_2} \quad & \lambda_1(\vartheta) d_B(\vartheta, \varpi) + \lambda_2(\vartheta) \frac{d_B(\vartheta, \mathcal{S}\varpi)(1 + d_B(\vartheta, \mathcal{T}\vartheta))}{1 + d_B(\vartheta, \varpi)} \\ & + \lambda_3(\vartheta) \frac{d_B(\varpi, \mathcal{S}\varpi)(1 + d_B(\vartheta, \mathcal{T}\vartheta))}{1 + d_B(\vartheta, \varpi)} + \\ & \lambda_4(\vartheta) \max\{d_B(\vartheta, \mathcal{S}\vartheta), d_B(\varpi, \mathcal{T}\varpi)\}. \end{aligned}$$

Then $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point in Υ.

Applying the notions from Kannan [16] and Krein *et al.* [21], Karapinar [18] introduced the following results for interpolative Kannan contraction as follows:

Definition 2.5. [18] Let (Υ, d) be a metric space, the mapping $\mathcal{T} : \Upsilon \to \Upsilon$ is said to be interpolative Kannan contraction mappings if

$$d(\mathcal{T}\vartheta, \mathcal{T}\varpi) \quad \leq \quad c[d(\vartheta, \mathcal{T}\vartheta)]^{\delta}.[d(\varpi, \mathcal{T}\varpi)]^{1-\delta}, \tag{5}$$

for all $\vartheta, \varpi \in \Upsilon$ with $\vartheta \neq \mathcal{T}\vartheta$, where $c \in [0, 1)$ and $\delta \in (0, 1)$.

Theorem 2.4. *[18] Let (Υ, d) be a complete metric space and $\mathcal{T}$ be an interpolative Kannan type contraction. Then $\mathcal{T}$ has a unique fixed point in Υ.*

Reich [31], Rus [32] and Ćirić [6] improved Banach [4] and Kannan [16] fixed point theorem as

Theorem 2.5. *[20] Let (Υ, d) be a complete metric space. $\mathcal{T} : \Upsilon \to \Upsilon$ be a mapping such that*

$$d(\mathcal{T}\vartheta, \mathcal{T}\varpi) \leq \lambda d(\vartheta, \varpi) + \zeta d(\vartheta, \mathcal{T}\vartheta) + \eta d(\varpi, \mathcal{T}\varpi), \tag{6}$$

for all $\vartheta, \varpi \in \Upsilon$ where $\lambda + \zeta + \eta < 1$. Then $\mathcal{T}$ possesses a unique fixed point in Υ.

Karapinar *et al.* [20] proved the results for Interpolative Reich-Rus-Ćirić type contractions on partial metric spaces as follows.

Theorem 2.6. *[20] Let (Υ, p) be a complete metric space. $\mathcal{T} : \Upsilon \to \Upsilon$ be a mapping such that*

$$p(\mathcal{T}\vartheta, \mathcal{T}\varpi) \leq c[p(\vartheta, \varpi)]^{\delta}.[p(\vartheta, \mathcal{T}\vartheta)]^{\alpha}.[p(\varpi, \mathcal{T}\varpi)]^{1-\alpha-\delta}, \tag{7}$$

for all $\vartheta, \varpi \in \Upsilon \backslash Fix(\mathcal{T})$ where $Fix(\mathcal{T}) = \{\vartheta \in \Upsilon, \mathcal{T}\vartheta = \vartheta\}$. Then $\mathcal{T}$ has a fixed point in Υ.

3. Main Results

The following are our main results:

Theorem 3.1. *Let (Υ, d_B, s) be a complete bicomplex valued b-metric space with $s \geq 2$ and let $\mathcal{S}, \mathcal{T} : \Upsilon \to \Upsilon$ be two-self interpolative Ćirić rational type mappings, such that*

$$\begin{aligned} d_B(\mathcal{S}\vartheta, \mathcal{T}\varpi) \preceq_{i_2} \; & \tau\Big[d_B(\vartheta, \varpi)\Big]^{\beta_1}.\Big[d_B(\vartheta, \mathcal{S}\vartheta)\Big]^{\beta_2}. \\ & \Big[d_B(\varpi, \mathcal{T}\varpi)\Big]^{\beta_3}.\Big[\frac{d_B(\vartheta, \mathcal{S}\vartheta)(1 + d_B(\vartheta, \mathcal{T}\vartheta))}{1 + d_B(\vartheta, \varpi)}\Big]^{\beta_4} \\ & .\Big[\frac{d_B(\varpi, \mathcal{S}\varpi)(1 + d_B(\vartheta, \mathcal{T}\vartheta))}{1 + d_B(\vartheta, \varpi)}\Big]^{1-\beta_1-\beta_2-\beta_3-\beta_4}, \end{aligned} \tag{8}$$

for all $\vartheta, \varpi \in \Upsilon$, $\sum_{\Omega=1}^{4} \beta_i < 1$ and $\tau \in [0, \frac{1}{s}]$. Then, $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point in Υ.

Proof. Let ϑ_0 be an arbitrary point in Υ. We can construct a sequence $\{\vartheta_{2n}\}$ in Υsatisfying

$$\vartheta_{2n+1} = \mathcal{S}\vartheta_{2n}, \tag{9}$$

and

$$\vartheta_{2n+2} = \mathcal{T}\vartheta_{2n+1}, \tag{10}$$

for all $n \in \mathbb{N}_0$. By definition 2.4, a point $\vartheta_{2n} \in \Upsilon$ is called a common fixed point of $\mathcal{S}$ and $\mathcal{T}$ if $\vartheta_{2n} = \mathcal{S}\vartheta_{2n} = \mathcal{T}\vartheta_{2n+1}$.

Again, we can choose

$$\vartheta_{2n+2} = \mathcal{S}\vartheta_{2n+1}, \tag{11}$$

and

$$\vartheta_{2n+3} = \mathcal{T}\vartheta_{2n+2}, \tag{12}$$

for all $n \in \mathbb{N}_0$. Using definition 2.4, a point $\vartheta_{2n+1} \in \Upsilon$ is called a common fixed point of $\mathcal{S}$ and $\mathcal{T}$ if $\vartheta_{2n+1} = \mathcal{S}\vartheta_{2n+1} = \mathcal{T}\vartheta_{2n+2}$.

Suppose that $\vartheta_{2n+1} \neq \mathcal{S}\vartheta_{2n}$ and $\vartheta_{2n+2} \neq \mathcal{T}\vartheta_{2n+1}$. Using (9) and (10), we have

$$d_B(\vartheta_{2n+1}, \vartheta_{2n+2}) = d_B(\mathcal{S}\vartheta_{2n}, \mathcal{T}\vartheta_{2n+1}) \neq 0, \tag{13}$$

for all $n \in \mathbb{N}_0$.

Let $\vartheta = \vartheta_{2n}$ and $\varpi = \vartheta_{2n+1}$ in (8), we obtain

$$\begin{aligned}
d_B(\mathcal{S}\vartheta_{2n}, \mathcal{T}\vartheta_{2n+1}) &\preceq_{i_2} \tau\Big[d_B(\vartheta_{2n}, \vartheta_{2n+1})\Big]^{\beta_1}.\Big[d_B(\vartheta_{2n}, \mathcal{S}\vartheta_{2n})\Big]^{\beta_2}. \\
&\quad \Big[d_B(\vartheta_{2n+1}, \mathcal{T}\vartheta_{2n+1})\Big]^{\beta_3}. \\
&\quad \Big[\frac{d_B(\vartheta_{2n}, \mathcal{S}\vartheta_{2n})(1 + d_B(\vartheta_{2n}, \mathcal{T}\vartheta_{2n}))}{1 + d_B(\vartheta_{2n}, \vartheta_{2n+1})}\Big]^{\beta_4} \\
&\quad .\Big[\frac{d_B(\vartheta_{2n+1}, \mathcal{S}\vartheta_{2n+1})(1 + d_B(\vartheta_{2n}, \mathcal{T}\vartheta_{2n}))}{1 + d_B(\vartheta_{2n}, \vartheta_{2n+1})}\Big]^{1-\beta_1-\beta_2-\beta_3-\beta_4}, \\
&\preceq_{i_2} \tau\Big[d_B(\vartheta_{2n}, \vartheta_{2n+1})\Big]^{\beta_1}.\Big[d_B(\vartheta_{2n}, \vartheta_{2n+1})\Big]^{\beta_2}. \\
&\quad \Big[d_B(\vartheta_{2n+1}, \vartheta_{2n+2})\Big]^{\beta_3}. \\
&\quad \Big[\frac{d_B(\vartheta_{2n}, \vartheta_{2n+1})(1 + d_B(\vartheta_{2n}, \vartheta_{2n+1}))}{1 + d_B(\vartheta_{2n}, \vartheta_{2n+1})}\Big]^{\beta_4} \\
&\quad .\Big[\frac{d_B(\vartheta_{2n+1}, \vartheta_{2n+2})(1 + d_B(\vartheta_{2n}, \vartheta_{2n+1}))}{1 + d_B(\vartheta_{2n}, \vartheta_{2n+1})}\Big]^{1-\beta_1-\beta_2-\beta_3-\beta_4}, \\
&\preceq_{i_2} \tau\Big[d_B(\vartheta_{2n}, \vartheta_{2n+1})\Big]^{\beta_1}.\Big[d_B(\vartheta_{2n}, \vartheta_{2n+1})\Big]^{\beta_2}. \\
&\quad \Big[d_B(\vartheta_{2n+1}, \vartheta_{2n+2})\Big]^{\beta_3}.\Big[d_B(\vartheta_{2n}, \vartheta_{2n+1})\Big]^{\beta_4} \\
&\quad .\Big[d_B(\vartheta_{2n+1}, \vartheta_{2n+2})\Big]^{1-\beta_1-\beta_2-\beta_3-\beta_4}, \\
&\preceq_{i_2} \tau\Big[d_B(\vartheta_{2n}, \vartheta_{2n+1})\Big]^{\beta_1+\beta 2+\beta_4} \\
&\quad .\Big[d_B(\vartheta_{2n+1}, \vartheta_{2n+2})\Big]^{1-\beta_1-\beta_2-\beta_4}, \\
\Big[d_B(\vartheta_{2n+1}, \vartheta_{2n+2})\Big]^{1-(1-\beta_1-\beta_2-\beta_4)} &\preceq_{i_2} \tau\Big[d_B(\vartheta_{2n}, \vartheta_{2n+1})\Big]^{\beta_1+\beta_2+\beta_4}, \\
d_B(\vartheta_{2n+1}, \vartheta_{2n+2}) &\preceq_{i_2} \tau^{\frac{1}{\beta_1+\beta_2+\lambda\beta_4}} d_B(\vartheta_{2n}, \vartheta_{2n+1}). \qquad (14)
\end{aligned}$$

Let $\omega = \tau^{\frac{1}{\beta_1+\beta_2+\lambda\beta_4}} < 1$ in (14), we have

$$d_B(\vartheta_{2n+1}, \vartheta_{2n+2}) \preceq_{i_2} \omega d_B(\vartheta_{2n}, \vartheta_{2n+1}). \qquad (15)$$

Repeating the above procedures n-times by induction the inequality (14) deduce to

$$\|d_B(\vartheta_{2n+1}, \vartheta_{2n+2})\| \preceq_{i_2} \omega^n \|d_B(\vartheta_{2n}, \vartheta_{2n+1})\|. \qquad (16)$$

From Lemma 2.1 and Definition 2.3, we get

$$\lim_{n\to\infty} \|d_B(\vartheta_{2n+1}, \vartheta_{2n+2})\| \to 0 \ as \ n \to \infty,$$

which is a contradiction.

Let $n, m \in \mathbb{N}$, $m > n$, such that

$$\begin{aligned}
\|d_B(\vartheta_{2n+1},\vartheta_{2n+m})\| &\preceq_{i_2} s[d_B(\vartheta_{2n+1},\vartheta_{2n+2})+d(\vartheta_{2n+2},\vartheta_{2n+m})],\\
&\preceq_{i_2} sd_B(\vartheta_{2n+1},\vartheta_{2n+2})+s^2d_B(\vartheta_{2n+2},\vartheta_{2n+3})+d(\vartheta_{2n+3},\vartheta_{2n+4})+\dots,\\
&\preceq_{i_2} s\omega^n\|d_B(\vartheta_{2n},\vartheta_{2n+1})\|+s^2\omega^{n+1}\|d_B(\vartheta_{2n},\vartheta_{2n+1})\|\\
&\quad +s^3\omega^{n+2}\|d_B(\vartheta_{2n},\vartheta_{2n+1})\|.+\dots,\\
&\preceq_{i_2} [s\omega^n+s^2\omega^{n+1}+s^3\omega^{n+2}+\dots,]\|d_B(\vartheta_{2n},\vartheta_{2n+1})\|\\
&\preceq_{i_2} s^n[1+s\omega^n+s^2\omega^{n+1}+\dots,]\|d_B(\vartheta_{2n},\vartheta_{2n+1})\|\\
&\preceq_{i_2} \frac{s\omega^n}{1-s\omega^n}\|d_B(\vartheta_{2n},\vartheta_{2n+1})\|. \qquad (17)
\end{aligned}$$

By taking $n,m\to\infty$ in (17), using Lemma 2.2 we get

$$\|d_B(\vartheta_{2n},\vartheta_{2n+m})\|\to 0,$$

Hence, $\{\vartheta_{2n}\}$ is a Cauchy sequence.

Since (Υ,d_B) is a complete bicomplex valued b-metric space, there exists a fixed point $\vartheta^\star\in\Upsilon$ such that

$$\lim_{n\to\infty} d_B(\vartheta_{2n},\vartheta^\star)=0.$$
$$\lim_{n\to\infty}\vartheta_{2n}=\vartheta^\star.$$

Suppose that $r\in\Upsilon$ such that

$$\|d_B(\vartheta^\star,\mathcal{S}\vartheta^\star)\|=\|r\|>0.$$

By $(BCM3)$, Definition 2.3 and 2.4, we get

$$\begin{aligned}
r=d_B(\vartheta^\star,\mathcal{S}\vartheta^\star) &\preceq_{i_2} s[d_B(\vartheta^\star,\vartheta_{2n+2})+d_B(x_{2n+2},\mathcal{S}\vartheta^\star)],\\
&\preceq_{i_2} sd_B(\vartheta^\star,\vartheta_{2n+2})+sd_B(x_{2n+2},\mathcal{S}\vartheta^\star),\\
&\preceq_{i_2} sd_B(\vartheta^\star,\vartheta_{2n+2})+sd_B(\mathcal{T}\vartheta_{2n+1},\mathcal{S}\vartheta^\star),\\
&\preceq_{i_2} sd_B(\vartheta^\star,\vartheta_{2n+2})+sd_B(\mathcal{S}\vartheta^\star,\mathcal{T}\vartheta_{2n+1}), \qquad (18)
\end{aligned}$$

Let $\vartheta=\vartheta^\star$ and $\varpi=\vartheta_{2n+1}$ in (8) and (18), we obtain

$$\begin{aligned}
d_B(\vartheta^\star,\mathcal{S}\vartheta^\star) &\preceq_{i_2} sd_B(\vartheta^\star,\vartheta_{2n+2})+\tau s\Big[d_B(\vartheta^\star,\vartheta_{2n+1})\Big]^{\beta_1}.\Big[d_B(\vartheta^\star,\mathcal{S}\vartheta^\star)\Big]^{\beta_2}.\\
&\quad \Big[d_B(\vartheta_{2n+1},\mathcal{T}\vartheta_{2n+1})\Big]^{\beta_3}.\Big[\frac{d_B(\vartheta^\star,\mathcal{S}\vartheta^\star)(1+d_B(\vartheta^\star,\mathcal{T}\vartheta^\star))}{1+d_B(\vartheta^\star,\vartheta_{2n+1})}\Big]^{\beta_4}\\
&\quad .\Big[\frac{d_B(\vartheta_{2n+1},\mathcal{S}\vartheta_{2n+1})(1+d_B(\vartheta^\star,\mathcal{T}\vartheta^\star))}{1+d_B(\vartheta^\star,\vartheta_{2n+1})}\Big]^{1-\beta_1-\beta_2-\beta_3-\beta_4},\\
\|d_B(\vartheta^\star,\mathcal{S}\vartheta^\star)\| &\preceq_{i_2} \|r\|=0.
\end{aligned}$$

which is a contradiction. Therefore $\|d_B(\vartheta^\star,\mathcal{S}\vartheta^\star)\|\precneq_{i_2}\|r\|=0$. Thus $\vartheta^\star=\mathcal{S}\vartheta^\star$. Similarly, we can show that $\vartheta^\star=\mathcal{T}\vartheta^\star$. Hence $\vartheta^\star$ is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$.

For the uniqueness of common fixed point of $\mathcal{S}$ and $\mathcal{T}$. Let $\varpi^\star$ be another common fixed point in Υ such that $\varpi^\star\neq\vartheta^\star$ by definition 2.4, $\varpi^\star=\mathcal{S}\varpi^\star=\mathcal{T}\varpi^\star$. Let $\vartheta=\vartheta^\star$ and $\varpi=\varpi^\star$ in

(8), then we have

$$\begin{aligned}
d_B(\vartheta^\star,\varpi^\star) &= d_B(\mathcal{S}\vartheta^\star,\mathcal{T}\varpi^\star),\\
d_B(\mathcal{S}\vartheta^\star,\mathcal{T}\varpi^\star) &\preceq_{i_2} \tau\Big[d_B(\vartheta^\star,\varpi^\star)\Big]^{\beta_1}.\Big[d_B(\vartheta^\star,\mathcal{S}\vartheta^\star)\Big]^{\beta_2}.\\
&\Big[d_B(\varpi^\star,\mathcal{T}\varpi^\star)\Big]^{\beta_3}.\Big[\frac{d_B(\vartheta^\star,\mathcal{S}\vartheta^\star)(1+d_B(\vartheta^\star,\mathcal{T}\vartheta^\star))}{1+d_B(\vartheta^\star,\varpi^\star)}\Big]^{\beta_4}\\
&.\Big[\frac{d_B(\varpi^\star,\mathcal{S}\varpi^\star)(1+d_B(\vartheta^\star,\mathcal{T}\vartheta^\star))}{1+d_B(\vartheta^\star,\varpi^\star)}\Big]^{1-\beta_1-\beta_2-\beta_3-\beta_4}, \qquad (19)\\
d_B(\vartheta^\star,\varpi^\star) &\preceq_{i_2} \tau\Big[d_B(\vartheta^\star,\varpi^\star)\Big]^{\beta_1}.\Big[d_B(\vartheta^\star,\vartheta^\star)\Big]^{\beta_2}.\\
&\Big[d_B(\varpi^\star,\varpi^\star)\Big]^{\beta_3}.\Big[\frac{d_B(\vartheta^\star,\vartheta^\star)(1+d_B(\vartheta^\star,\vartheta^\star))}{1+d_B(\vartheta^\star,\varpi^\star)}\Big]^{\beta_4}\\
&.\Big[\frac{d_B(\varpi^\star,\varpi^\star)(1+d_B(\vartheta^\star,\vartheta^\star))}{1+d_B(\vartheta^\star,\varpi^\star)}\Big]^{1-\beta_1-\beta_2-\beta_3-\beta_4},\\
d_B(\mathcal{S}\vartheta^\star,\mathcal{T}\varpi^\star) &\preceq_{i_2} 0. \qquad (20)
\end{aligned}$$

Consequently, we have

$$\|d_B(\vartheta^\star,\varpi^\star)\| = 0,$$

which implies that $\vartheta^\star=\varpi^\star$. Hence $\vartheta^\star$ is a unique common fixed point of $\mathcal{S}$ and $\mathcal{T}$. □

Motivated by Mani *et al.* [23], we proved the following theorem in bicomplex valued b-metric space, for the creativity of Theorem 3.1.

Theorem 3.2. Let (Υ,d_B) be a complete bicomplex valued b-metric space with coefficient $s=2$ and $\mathcal{S},\mathcal{T}:\Upsilon\to\Upsilon$, if there exists $\delta_1,\delta_2,\delta_3,\delta_4,\delta_5;\Upsilon\to[0,\frac{1}{s}]$ such that for all $\vartheta,\varpi\in\Upsilon$;

(i) $\delta_1(\mathcal{T}\vartheta)\preceq_{i_2}\delta_1(\vartheta),\delta_2(\mathcal{T}\vartheta)\preceq_{i_2}\delta_2(\vartheta),\delta_3(\mathcal{T}\vartheta)\preceq_{i_2}\delta_3(\vartheta)$ and $\delta_4(\mathcal{T}\vartheta)\preceq_{i_2}\delta_4(\vartheta)$,

(ii) $\delta_1(s\vartheta)\preceq_{i_2}\delta_1(\vartheta),\delta_2(s\vartheta)\preceq_{i_2}\delta_2(\vartheta),\delta_3(s\vartheta)\leq\delta_3(\vartheta)$, $\delta_4(s\vartheta)\preceq_{i_2}\delta_4(\vartheta)$ and $\delta_5(s\vartheta)\preceq_{i_2}\delta_5(\vartheta)$,

(iii) $\sum_{n=1}^{5}\delta_n<1$.

(iv) Also,

$$\begin{aligned}
d_B(\mathcal{S}\vartheta,\mathcal{T}\varpi) &\preceq_{i_2} \delta_1(\vartheta)\Big[d_B(\vartheta,\varpi)\Big]^{\beta_1}.\delta_2(\vartheta)\Big[d_B(\vartheta,\mathcal{S}\vartheta)\Big]^{\beta_2}.\\
&\delta_3(\vartheta)\Big[d_B(\varpi,\mathcal{T}\varpi)\Big]^{\beta_3}.\delta_4(\vartheta)\Big[\frac{d_B(\vartheta,\mathcal{S}\vartheta)(1+d_B(\vartheta,\mathcal{T}\vartheta))}{1+d_B(\vartheta,\varpi)}\Big]^{\beta_4}\\
&.\delta_5(\vartheta)\Big[\frac{d_B(\varpi,\mathcal{S}\varpi)(1+d_B(\vartheta,\mathcal{T}\vartheta))}{1+d_B(\vartheta,\varpi)}\Big]^{1-\beta_1-\beta_2-\beta_3-\beta_4}, \qquad (21)
\end{aligned}$$

for all $\vartheta,\varpi\in\Upsilon$, $\sum_{n=1}^{4}\beta_i<1$ and $\tau\in[0,\frac{1}{s}]$. Then, $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point in Υ.

Proof. The proof of this theorem follow the similar proof of Theorem (3.1). This completes the proof. □

We provide the following example for verification of our results in Theorem 3.1.

Example 3.1. *Let $X=[0,1)$ and $d_B:\Upsilon\times\Upsilon\to\mathbb{C}_2$ given by*

$$d_B(\vartheta,\varpi) = |\vartheta-\varpi|^2 e^{i_2 k},$$

where $k \in [0, 2\pi]$. Then, (Υ, d_B) is a complete complex-valued b-metric space with $s = 2$.

Define $\mathcal{S}, \mathcal{T} : \Upsilon \to \Upsilon$ by

$$\mathcal{S}\vartheta = \frac{\vartheta}{4}.$$

and

$$\mathcal{T}\vartheta = \frac{\vartheta^3}{8},$$

for all $\vartheta, \varpi \in \Upsilon$. Assume that $e^{i_2 k} = \cos k + i_2 \sin k$. To verify the hypothesis used in Theorem 3.1, let us calculate the following bicomplex-valued b-metric of the following points.

$$\begin{aligned}
d_B(\mathcal{S}\vartheta, \mathcal{T}\varpi) &= d_B\Big(\frac{\vartheta}{4}, \frac{\varpi^3}{8}\Big) = \Big|\Big|\frac{\vartheta}{4} - \frac{\varpi^3}{8}\Big|\Big|^2 e^{i_2 2\pi} = \Big|\Big|\frac{2\vartheta - \varpi^3}{8}\Big|\Big|^2 e^{i_2 2\pi} \\
&= \Big|\Big|\frac{2\vartheta - \varpi^3}{8}\Big|\Big|^2 \|\cos 2\pi + i_2 \sin 2\pi\| = \Big|\Big|\frac{2\vartheta - \varpi^3}{8}\Big|\Big|^2 = 0.0081, \\
d_B(\vartheta, \varpi) &= \|\vartheta - \varpi\|^2 e^{i_2 2\pi} = \|\vartheta - \varpi\|^2 \|\cos 2\pi + i_2 \sin 2\pi\| = 0.4, \\
d_B(\vartheta, \mathcal{S}\vartheta) &= d_B\Big(\vartheta, \frac{\vartheta}{4}\Big) = \Big|\Big|\vartheta - \frac{\vartheta}{4}\Big|\Big|^2 e^{i_2 2\pi} = \Big|\Big|\frac{4\vartheta - \vartheta}{4}\Big|\Big|^2 e^{i_2 2\pi} \\
&= \Big|\Big|\frac{4\vartheta - \vartheta}{4}\Big|\Big|^2 \|\cos 2\pi + i_2 \sin 2\pi\| = \Big|\Big|\frac{3\vartheta}{4}\Big|\Big|^2 = 0.09, \\
d_B(\varpi, \mathcal{T}\varpi) &= d_B\Big(\varpi, \frac{\varpi^3}{8}\Big) = \Big|\Big|\varpi - \frac{\varpi^3}{8}\Big|\Big|^2 e^{i_2 2\pi} = \Big|\Big|\frac{8\varpi - \varpi^3}{8}\Big|\Big|^2 e^{i_2 2\pi} \\
&= \Big|\Big|\frac{8\varpi - \varpi^3}{8}\Big|\Big|^2 \|\cos 2\pi + i_2 \sin 2\pi\| = \Big|\Big|\frac{8\varpi - \varpi^3}{8}\Big|\Big|^2 = 0.039601,
\end{aligned}$$

$$\begin{aligned}
d_B(\vartheta, \mathcal{T}\vartheta) &= d_B\Big(\vartheta, \frac{\vartheta^3}{8}\Big) = \Big|\Big|\vartheta - \frac{\vartheta^3}{8}\Big|\Big|^2 e^{i_2 2\pi} = \Big|\Big|\frac{8\vartheta - \vartheta^3}{8}\Big|\Big|^2 e^{i_2 2\pi} \\
&= \Big|\Big|\frac{8\vartheta - \vartheta^3}{8}\Big|\Big|^2 \|\cos 2\pi + i_2 \sin 2\pi\| = \Big|\Big|\frac{8\vartheta - \vartheta^3}{8}\Big|\Big|^2 = 0.153664, \\
d_B(\varpi, \mathcal{S}\varpi) &= d_B\Big(\vartheta, \frac{\varpi}{4}\Big) = \Big|\Big|\varpi - \frac{\varpi}{4}\Big|\Big|^2 e^{i_2 2\pi} = \Big|\Big|\frac{4\varpi - \varpi}{4}\Big|\Big|^2 e^{i_2 2\pi} \\
&= \Big|\Big|\frac{3\varpi}{4}\Big|\Big|^2 \|\cos 2\pi + i_2 \sin 2\pi\| = \Big|\Big|\frac{3\varpi}{4}\Big|\Big|^2 = 0.0225,
\end{aligned}$$

Using the above equalities and $\vartheta = 0.4, \varpi = 0.2, \beta_1 = 0.1, \beta_2 = 0.3, \beta_3 = 0.2, \beta_4 = 0.1$ and $\tau = \frac{1}{2}$ in (8), we get

$$\begin{aligned}
0.0081 &\preceq_{i_2} 0.5\Big[0.4\Big]^{0.1}.\Big[0.09\Big]^{0.3}. \\
&\quad \Big[0.039601\Big]^{0.2}.\Big[\frac{0.09(1+0.153664)}{1+0.4}\Big]^{0.1} \\
&\quad .\Big[\frac{0.0225(1+0.153664)}{1+0.4}\Big]^{1-0.1-0.3-0.2-0.1}, \\
0.0081 &\preceq_{i_2} 0.5\Big[0.4\Big]^{0.1}.\Big[0.09\Big]^{0.3}. \\
&\quad \Big[0.039601\Big]^{0.2}.\Big[0.0741641142\Big]^{0.1}.\Big[0.018541028\Big]^{0.3}, \\
0.0081 &\preceq_{i_2} 0.5\Big[0.912443536\Big].\Big[0.485593374\Big]. \\
&\quad \Big[0.524253369\Big].\Big[0.7709378\Big].\Big[0.302301389\Big], \\
\|0.0081\| &\preceq_{i_2} \|0.02706706746\|.
\end{aligned}$$

Hence, the hypothesis of Theorem 3.1 is verified. Thus, a pair of mappings $\mathcal{S}$ and $\mathcal{T}$ has unique common fixed points in Υ.

3.1. **An application to Non-linear Matrix Equations in Bicomplex Valued b-metric space.** In this section, we prove the unique common solution of the non-linear matrix equation using quasi-partial b-metric space for the demonstration of the hypothesis given in Theorem 3.1 in bicomplex valued b-metric space. The study of non-linear matrix equation originated by Ran and Reurings [28, 29] using Banach contraction principle concepts in partially ordered sets. The Hermitian solution of the equation $\vartheta = Q + \mathcal{A}\vartheta^{-1}\mathcal{A}^*$ is the matrix equation arising from the Gaussian process. The equation admits both positive definite solution and negative definite solution if and only if $\mathcal{A}$ is non-singular. If $\mathcal{A}$ is singular, no negative definite solution exists. This type of equation has several applications that arise in the analysis of control theory, optimal solution, ladder networks, dynamic programming and system theory [36, 38–40].

The symbol $\|.\|$ denotes the spectral norm of the matrix $\mathcal{A}$, that is

$$\|\mathcal{A}\| \;=\; \sqrt{\lambda^+(\mathcal{A}^\star\mathcal{A})}$$

such that $\lambda^+(\mathcal{A}^\star A)$ is the largest eigenvalue of $\mathcal{A}^\star\mathcal{A}$ where $\mathcal{A}^\star$ is the conjugate transpose of $\mathcal{A}$.

We give the lemmas for future use.

Lemma 3.1. *[29] If $\mathcal{A}, \mathcal{B} \succeq 0$ are $n \times n$ matrices. Then*

$$0 \le tr(\mathcal{A}, \mathcal{B}) \le \|\mathcal{B}\||tr(\mathcal{A})|. \tag{22}$$

Lemma 3.2. *[22, 29] Let $\mathcal{A} \in H(n)$ $\mathcal{A}, \mathcal{B} \preceq I_n$, then*

$$\|\mathcal{A}\| < 1. \tag{23}$$

Consider the following pairs of non-linear matrix equations motivated from [11, 26].

$$\begin{aligned} \vartheta &= Q + \sum_{\Omega=1}^{m} \mathcal{A}_\Omega^* \mathcal{S}(\vartheta)\mathcal{A}_n, \\ \varpi &= Q + \sum_{n=1}^{m} \mathcal{B}_n^* \mathcal{T}(\varpi)\mathcal{B}_n, \end{aligned} \tag{24}$$

where $\mathrm{Q} \in \mathfrak{p}(n)$, $\mathcal{A}_n$ is $n \times \Omega$ matrices, $\mathcal{A}_n^*, \mathcal{B}_n^*$ stands for conjugate transpose of $\mathcal{A}_n, \mathcal{B}_n \in \mathcal{H}(n)$ and $\mathcal{S}, \mathcal{T}; \mathfrak{p}(n) \to \mathfrak{p}(n)$ are maps from the set of all $n \times n$ Hermitian matrices into itself such that $\mathcal{S}, \mathcal{T}(0) = 0$.

The equations (24) can be written interns of sequence

$$\begin{aligned} \vartheta_{2n+1} = \vartheta &= Q_1 + \sum_{n=1}^{m} \mathcal{A}_n^* \mathcal{S}(\vartheta)\mathcal{A}_n, \\ \vartheta_{2n+2} = \varpi &= Q_2 + \sum_{n=1}^{m} \mathcal{B}_n^* \mathcal{T}(\varpi)\mathcal{B}_n. \end{aligned} \tag{25}$$

Define a bicomplex valued b-metric $d_B : \Upsilon \times \Upsilon \to \mathbb{C}_2$ by

$$d_B(\vartheta, \varpi) \;=\; |\vartheta - \varpi|^2 e^{i_2 k},$$

where $k \in [0, 2\pi]$. Then, (Υ, d_B) is a complete complex-valued b-metric space with $s = 2$.

Theorem 3.3. *Consider the class of non-linear matrix equation (24) and suppose the following condition holds.*

(i) *there exists $Q_1, Q_2 \in \mathfrak{p}(n)$, such that*

$$\sum_{n=1}^{m} \mathcal{A}_n^* \mathcal{S}(Q_1)\mathcal{A}_n \;_{i_2}\succ 0, \; \sum_{n=1}^{m} \mathcal{B}_n^* \mathcal{T}(Q_2)\mathcal{B}_n \;_{i_2}\succ 0.$$

(ii) *for all $\mathcal{A}, \mathcal{B} \in \mathfrak{p}(\Omega)$, we have*

$$\sum_{n=1}^{m} \mathcal{A}_n^* \mathcal{A}_n \prec_{i_2} I_n, \; \sum_{n=1}^{m} \mathcal{B}_n^* \mathcal{B}_n \prec_{i_2} I_n.$$

(iii) *there exists $\vartheta, \varpi \in \mathfrak{p}(n)$, such that*

$$Q_1 + \sum_{n=1}^{m} \mathcal{A}_n^* \mathcal{S}(\vartheta)\mathcal{A}_n \quad \preccurlyeq_{i_2} Q_2 + \textstyle\sum_{n=1}^{m} \mathcal{B}_n^* \mathcal{T}(\varpi)\mathcal{B}_n.$$

For $\mathcal{A} = \mathcal{B}$, we have

$$Q_1 + \sum_{n=1}^{m} \mathcal{A}_n^* \mathcal{S}(\vartheta)\mathcal{A}_n \quad \preccurlyeq_{i_2} Q_2 + \textstyle\sum_{n=1}^{m} \mathcal{A}_\Omega^* \mathcal{T}(\varpi)\mathcal{A}_\Omega.$$

(iv) *For $\vartheta, \varpi \in \mathfrak{p}(n)$ and $\vartheta \preceq_{i_2} \varpi$ with $\tau \in [0, \frac{1}{s}], s = 2$,*

$$\begin{aligned} d_B(\mathcal{S}\vartheta, \mathcal{T}\varpi) &= \|\mathcal{S}\vartheta - \mathcal{T}\varpi\|_{tr}, \\ \|\mathcal{S}\vartheta - \mathcal{T}\varpi\|_{tr} &\preceq_{i_2} \tau\Big[\|\vartheta - \varpi\|_{tr}\Big]^{\beta_1}.\Big[\|\vartheta - \mathcal{S}\vartheta\|_{tr}\Big]^{\beta_2}. \\ &\quad \Big[\|\varpi - \mathcal{T}\varpi\|_{tr}\Big]^{\beta_3}.\Big[\frac{\|\vartheta - \mathcal{S}\varpi\|_{tr}(1 + \|\vartheta - \mathcal{T}\vartheta\|_{tr})}{1 + \|\vartheta - \varpi\|_{tr}}\Big]^{\beta_4} \\ &\quad .\Big[\frac{\|\varpi - \mathcal{S}\varpi\|_{tr}(1 + \|\vartheta - \mathcal{T}\vartheta\|_{tr})}{1 + \|\vartheta - \varpi\|_{tr}}\Big]^{1-\beta_1-\beta_2-\beta_3-\beta_4}. \end{aligned}$$

Then, the non linear matrix equation (24) has a unique common solution in $\mathfrak{p}(n) \subseteq \mathcal{H}(n)$.

Proof. Suppose that $\vartheta, \varpi \in \mathfrak{p}(\Omega)$ in such a way that $\vartheta \preceq_{i_2} \varpi$. Define $\mathcal{S}, \mathcal{T} : \Upsilon \to \Upsilon$ by

$$\begin{aligned} \mathcal{S}\vartheta &= Q_1 + \sum_{\Omega=1}^{m} \mathcal{A}_n^* \mathcal{S}(\vartheta)\mathcal{A}_n, \\ \mathcal{T}\varpi &= Q_2 + \sum_{n=1}^{m} \mathcal{B}_n^* \mathcal{T}(\varpi)\mathcal{B}_n. \end{aligned} \tag{26}$$

By equation (22) and (26) with $(i) - (iv)$, we have

$$\begin{aligned} d_B(\mathcal{S}\vartheta, \mathcal{T}\varpi) &= \|\mathcal{S}\vartheta - \mathcal{T}\varpi\|_{tr} = \|Q_1 + \sum_{n=1}^{m} \mathcal{A}_n^* \mathcal{S}(\vartheta)\mathcal{A}_n - Q_2 - \sum_{n=1}^{m} \mathcal{A}_n^* \mathcal{T}(\varpi)\mathcal{A}_n\|^2 e^{i_2 k}, \\ &= \|Q_1 - Q_2 + \sum_{n=1}^{m} \mathcal{A}_n^* \mathcal{A}_n \|\mathcal{S}(\vartheta) - \mathcal{T}(\varpi)\|\|^2 e^{i_2 k}, \end{aligned}$$

$$
\begin{aligned}
d_B(\vartheta-\varpi) &= \|\vartheta-\varpi\|^2 e^{i_2 k},\\
d_B(\vartheta,\mathcal{S}\vartheta) &= \|\vartheta-\mathcal{S}\vartheta\|_{tr} = \|\vartheta - Q_1 - \sum_{n=1}^{m}\mathcal{A}_n^*\mathcal{S}(\vartheta)\mathcal{A}_n\|^2 e^{i_2 k},\\
&= \|\vartheta-\mathcal{S}\vartheta\|_{tr} = \|\vartheta - Q_1 - \sum_{n=1}^{m}\mathcal{A}_n^*\mathcal{A}_n\|\mathcal{S}(\vartheta)\|\|\|^2 e^{i_2 k},\\
d_B(\varpi,\mathcal{T}\varpi) &= \|\varpi-\mathcal{T}\varpi\|_{tr} = \|\vartheta - Q_2 - \sum_{n=1}^{m}\mathcal{A}_n^*\mathcal{T}(\varpi)\mathcal{A}_n\|^2 e^{i_2 k},\\
&= \|\varpi-\mathcal{S}\varpi\|_{tr} = \|\varpi - Q_2 - \sum_{n=1}^{m}\mathcal{A}_n^*\mathcal{A}_n\|\mathcal{T}(\varpi)\|\|\|^2 e^{i_2 k},\\
d_B(\vartheta,\mathcal{S}\varpi) &= \|\vartheta-\mathcal{S}\varpi\|_{tr} = \|\vartheta - Q_1 - \sum_{n=1}^{m}\mathcal{A}_n^*\mathcal{S}(\varpi)\mathcal{A}_n\|^2 e^{i_2 k},\\
&= \|\vartheta-\mathcal{S}\varpi\|_{tr} = \|\vartheta - Q_1 - \sum_{n=1}^{m}\mathcal{A}_n^*\mathcal{A}_n\|\mathcal{S}(\varpi)\|\|\|^2 e^{i_2 k},\\
d_B(\vartheta,\mathcal{T}\vartheta) &= \|\vartheta-\mathcal{S}\vartheta\|_{tr} = \|\vartheta - Q_2 - \sum_{n=1}^{m}\mathcal{A}_n^*\mathcal{T}(\vartheta)\mathcal{A}_n\|^2 e^{i_2 k},\\
&= \|\vartheta-\mathcal{T}\vartheta\|_{tr} = \|\vartheta - Q_2 - \sum_{n=1}^{m}\mathcal{A}_n^*\mathcal{A}_n\|\mathcal{T}(\vartheta)\|\|\|^2 e^{i_2 k},\\
d_B(\varpi,\mathcal{S}\varpi) &= \|\varpi-\mathcal{S}\varpi\|_{tr} = \|\varpi - Q_1 - \sum_{n=1}^{m}\mathcal{A}_n^*\mathcal{S}(\varpi)\mathcal{A}_n\|^2 e^{i_2 k},\\
&= \|\varpi-\mathcal{S}\varpi\|_{tr} = \|\varpi - Q_1 - \sum_{n=1}^{m}\mathcal{A}_n^*\mathcal{A}_n\|\mathcal{S}(\varpi)\|\|\|^2 e^{i_2 k}.
\end{aligned}
$$

By using the above equalities in (8), we obtain

$$
\begin{aligned}
\|\mathcal{S}\vartheta-\mathcal{T}\varpi\|_{tr} \preceq_{i_2}\ & \tau\Big[\|\vartheta-\varpi\|_{tr}\Big]^{\beta_1}.\Big[\|\vartheta-\mathcal{S}\vartheta\|_{tr}\Big]^{\beta_2}.\\
& \Big[\|\varpi-\mathcal{T}\varpi\|_{tr}\Big]^{\beta_3}.\Big[\frac{\|\vartheta-\mathcal{S}\varpi\|_{tr}(1+\|\vartheta-\mathcal{T}\vartheta\|_{tr})}{1+\|\vartheta-\varpi\|_{tr}}\Big]^{\beta_4}\\
& .\Big[\frac{\|\varpi-\mathcal{S}\varpi\|_{tr}(1+\|\vartheta-\mathcal{T}\vartheta\|_{tr})}{1+\|\vartheta-\varpi\|_{tr}}\Big]^{1-\beta_1-\beta_2-\beta_3-\beta_4},
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
d_B(\mathcal{S}\vartheta,\mathcal{T}\varpi) \preceq_{i_2}\ & \tau\Big[d_B(\vartheta,\varpi)\Big]^{\beta_1}.\Big[d_B(\vartheta,\mathcal{S}\vartheta)\Big]^{\beta_2}.\\
& \Big[d_B(\varpi,\mathcal{T}\varpi)\Big]^{\beta_3}.\Big[\frac{d_B(\vartheta,\mathcal{S}\varpi)(1+d_B(\vartheta,\mathcal{T}\vartheta))}{1+d_B(\vartheta,\varpi)}\Big]^{\beta_4}\\
& .\Big[\frac{d_B(\varpi,\mathcal{S}\varpi)(1+d_B(\vartheta,\mathcal{T}\vartheta))}{1+d_B(\vartheta,\varpi)}\Big]^{1-\beta_1-\beta_2-\beta_3-\beta_4}.
\end{aligned}
$$

Using the conditions in Theorem 3.1, we have applied Theorem 3.3 as an application using two non-linear matrix equations. Thus our proof is completed. □

4. Conclusions

The main contribution of this study to fixed point theory is the fixed point result given in Theorem 3.1. This theorem provides thecommon fixed point theorems for interpolative contraction mappings in bicomplex valued b-metric spaces. This paper, inspired by the results obtained by Beg *et al.* [5], Mani *et al.* [23], Datta *et al.* [9], Nashine *et al.* [26] and Joseph *et al.* [14]. We also provided an illustrative example to support the results and an application to the non-linear matrix equations.

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