DOI: 10.28919/ejma.2023.3.9

Eur. J. Math. Appl. (2023)3:9
URL: http://ejma.euap.org

ON A CLASS OF FRACTIONAL $p(.,.)$-KIRCHHOFF-SCHRÖDINGER SYSTEM TYPE

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Abstract. In the present article, we study the existence of a weak solution to an elliptic system of Kirchhoff-Shrödinger type, driven by the fractional $p(.,.)$-Laplacian operator. We use the direct variational method and Ekeland variational principle to claim our results.

1. Introduction

In this paper, we discuss the existence of a weak solutions to the following nonhomogeneous fractional $p(.,.)$-Laplacian system of Kirchhoff-Schrödinger type

$$(1)\qquad \begin{cases} A_1\Big(\mathcal{F}_1(u)\Big)\Big((-\Delta)^{s(.)}_{p(.)}u + a_1(x)|u|^{q(x)-2}u\Big) = F_u(u,v) + b_1(x) & in\,\Omega,\\ A_2\Big(\mathcal{F}_2(v)\Big)\Big((-\Delta)^{s(.)}_{p(.)}v + a_2(x)|v|^{q(x)-2}v\Big) = F_v(u,v) + b_2(x) & in\,\Omega,\\ u = v = 0 & on\;\mathbb{R}^N\backslash\Omega, \end{cases}$$

where

$$(2)\qquad \mathcal{F}_i(w) := \int_{\Omega\times\Omega} \frac{|w(x)-w(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s(x,y)p(x,y)}}dxdy + \int_\Omega \frac{a_i(x)}{q(x)}|w|^{q(x)}dx,$$

$\Omega \subset \mathbb{R}^N$ is a bounded smooth domain with $N \geq s(x,y)p(x,y)$ for any $(x,y) \in \overline{\Omega}\times\overline{\Omega}$, F_u (respectively, F_v) denotes the partial derivative of F with respect to u (respectively, v) and the nonlocal operator $(-\Delta)^{s(.)}_{p(.)}$ is the fractional $p(.,.)$-Laplacian operator given by

$$(3)\qquad (-\Delta)^{s(.)}_{p(.)}u(x) := P.V\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(\overline{u}(x)-\overline{u}(y))}{|x-y|^{N+s(x,y)p(x,y)}}dy, \quad x\in\Omega,$$

where $\overline{u} \in C_0^\infty$, P.V stands for Cauchy's principal value. To state our result, we assume that (B): $b_{i=1,2} \in L^{q(x)}$, $\dfrac{1}{q(x)} + \dfrac{1}{\overline{p}(x)} = 1$, $1 < q(x) < p_s^*(x) = N\overline{p}/(N-\overline{s}(x)\overline{p}(x))$, $\overline{p}(x) = p(x,x)$, $\overline{s}(x) = s(x,x)$, $p(.)$ and $s(.)$ are symmetric, that is, $p(x,y) = p(y,x)$ and $s(x,y) = s(y,x)$ for any $(x,y) \in D := \overline{\Omega}\times\overline{\Omega}$.

• $p(.) : D \to (1,\infty)$ is Lipschitz continuous functions and $q(.) : \overline{\Omega} \to R$ is continuous functions such that

(Ps) : $0 < s^- = \inf\limits_{(x,y)\in D} s(x,y) < s^+ = \sup\limits_{(x,y)\in D} s(x,y) < 1 < p^- = \inf\limits_{(x,y)\in D} p(x,y) < p^+ =$

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Key words and phrases. fractional $p(.,.)$-Laplacian operator; variational method; Ekeland's principle.

Received 22/03/2023.

$\sup_{(x,y)\in D} p(x,y)$.
(Q): $1 < q^- = \inf_{x\in\overline{\Omega}} q(x) < p^+ = \sup_{x\in\overline{\Omega}} q(x) < +\infty$.
• The potential function $a_{i=1,2}$ satisfy:
(P): $a_i \in C(\mathbb{R}^N)$, $\inf_{x\in\Omega} a_i(x) = a_i^- > 0$ and $\lim_{|x|\to+\infty} a_i(x) = +\infty$.
•: The Kirchhoff functions $A_{i=1,2}$ satisfy:
(L) there exist $k_1 > 0$ and $\theta > \frac{1}{p^-}$ such that

$$A_i(t) > k_1 t^{\theta-1} \; for\ all\ t > 0.$$

•: The non-linear term $F : \mathbb{R}^2 \to R$ is a C^1-function such that:
(F_1)

$$F(0,0) = 0, \quad \frac{\partial F}{\partial u} = F_u(u,v) \quad and \quad \frac{\partial F}{\partial v} = F_v(u,v) \quad for\ all\ (u;v) \in \mathbb{R}^2.$$

(F_2) There exists $K > 0$ such that $F(u,v) = F(u+K, v+K)$ for all $(u;v) \in \mathbb{R}^2$.

The stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{4}$$

presented by Kirchhoff [13] in 1883. Later (4) was developed to form

$$u_{tt} - A\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u = H(x,u) \quad x \in \Omega. \tag{5}$$

After that, many authors studied the following nonlocal elliptic boundary value problem

$$-A\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u = H(x,u) \quad x \in \Omega, \tag{6}$$

and other authors like, Yong Wu et al in [15] were interested in studying the following elliptic Kirchhoff system, driven by fractional variable-order exponente:

$$\begin{cases} A_1\Big(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s(x,y)p(x,y)}}dxdy\Big)\Big((-\Delta)^{s(.)}_{p(.)}u\Big) = F_u(u,v) + b_1(x) & in\,\Omega, \\ A_2\Big(\int_{\Omega\times\Omega} \frac{|v(x)-v(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+s(x,y)p(x,y)}}dxdy\Big)\Big((-\Delta)^{s(.)}_{p(.)}v\Big) = F_u(u,v) + b_2(x) & in\,\Omega, \\ u = v = 0 & on\,\mathbb{R}^N\backslash\Omega, \end{cases}$$

Rabil Ayazoglu et al in [14] were interested in studying the following fractional $p(.,.)$-Laplacian equation of Kirchhoff-Schrödinger type

$$A\Big(\mathcal{F}(u)\Big)\Big((-\Delta)^{s(.)}_{p(.)}u + V(x)|u|^{p(x)-2}\Big) = f(x,y) \quad x \in \mathbb{R}^N. \tag{7}$$

Azroul et al in [3] studied the following nonlocal fractional (p,q)-Schrodinger-Kirchhoff system type:

$$\begin{cases} A_1\Big(I_{K,p}(u)\Big)\Big(\mathcal{L}^K_p u + V(x)|u|^{p-2}u\Big) = \lambda F_u(x,u,v) + \nu G_u(x,u,v) & in\,\mathbb{R}^N, \\ A_2\Big(I_{K,q}(v)\Big)\Big(\mathcal{L}^K_q v + V(x)|v|^{q-2}v\Big) = \lambda F_v(x,u,v) + \nu G_v(x,u,v) & in\,\mathbb{R}^N \\ (u,v) \in W^p \times W^q \end{cases}$$

where
$$I_{K,p}(w)=\int_{\mathbb{R}^N\times\mathbb{R}^N}|w(x)-w(y)|^pK_p(x-y)dxdy+\int_{\mathbb{R}^N}V(x)|w|^pdx.$$
and $\mathcal{L}_r^K$ is a nonlocal integro-differential operator of elliptic type defined as:
$$\mathcal{L}_r^Ku(x)=\int_{\mathbb{R}^N\setminus B_\epsilon(x)}|u(x)-u(y)|^{r-2}(u(x)-u(y))K_r(x-y)dy.$$
and K_r is a measurable function satisfies some properties. Problems which involve the $p(.)$-Kirchhoff type have been intensively studied in the recent years, because of their numerous and relevant applications in many fields of mathematics, for example, electrorheological fluids (see [2]), elastic mechanics ([1]), image restoration ([7]). For this type of operator combined with a system of Kirchhoff functions we recall [1,5,11]. Inspired by the above articles, we aim in this paper to prove, under minoration conditions on $A_{i=1,2}$ and periodic conditions on F , the existence of solutions for the system (1) by applying variational method and Ekeland's principle.

This work is organized as follows. In the second Section, we recall some well-known properties and results on fractional Sobolev spaces with variable exponent and we present the existence of a result and its proof.

2. Some preliminary results

In this section, we set some definitions and properties of the Sobolev spaces with variable exponent (see [9,10]).

Let Ω be a Lipschitz bounded open set in $\mathbb{R}^N$. the function space $C_+(\overline{\Omega})$ is defined as follows:
$$C_+(\overline{\Omega}):=\big\{\tau\in C(\overline{\Omega},\mathbb{R}):1<\tau^-\le\tau^+<\infty\quad\text{for all } x\in\overline{\Omega}\big\}.$$
For $\tau\in C^+(\Omega)$, we define the variable exponent Lebesgue space
$$L^{\tau(x)}(\Omega)=\{w:\Omega\to\mathbb{R}\ \text{ is a mesurable function }\int_\Omega|w|^{\tau(x)}dx<+\infty\},$$
This space is equipped with the Luxemburg norm
$$\|w\|_{L^{\tau(x)}(\Omega)}=\|w\|_{\tau(x)}=\inf\Big\{\nu>0:\int_\Omega\Big|\frac{w}{\nu}\Big|^{\tau(x)}\le1\Big\}. \tag{8}$$
Also, the Hölder inequality holds
$$\Big|\int_\Omega uvdx\Big|\le\Big(\frac{1}{\tau^-}+\frac{1}{\tau'^+}\Big)\|u\|_{\tau(x)}\|v\|_{\tau'(x)}\le2\|u\|_{\tau(x)}\|v\|_{\tau'(x)}$$
for all $u\in L^{\tau(x)}(\Omega)$ and $v\in L^{\tau'(x)}(\Omega)$ where $\dfrac{1}{\tau^-}+\dfrac{1}{\tau'^+}=1$. The modular function $\rho:L^{\tau(x)}(\Omega)\to\mathbb{R}$ is defined as
$$\rho_{\tau(x)}(w)=\int_\Omega|w|^{\tau(x)}dx.$$
An important relationship between the norm $\|w\|_{\tau(x)}$ and the corresponding modular function $\rho_{\tau(x)}(.)$ given in this lemma.

Lemma 2.1. *Let $w\in L^{\tau(x)}(\Omega)$, $\{w_k\}\subset L^{\tau(x)}(\Omega)$, $k\in\mathbb{N}$, then*
(i) $\|w\|_{\tau(x)}<1\ (=1;>1)$ if and only if $\rho_{\tau(x)}(w)<1\ (=1;>1)$
(ii) If $\|w\|_{\tau(x)}>1$, then $\|w\|^{\tau^-}_{\tau(x)}\le\rho_{\tau(x)}(w)\le\|w\|^{\tau^+}_{\tau(x)}$,
(iii) If $\|w\|_{\tau(x)}<1$, then $\|w\|^{\tau^+}_{\tau(x)}\le\rho_{\tau(x)}(w)\le\|w\|^{\tau^-}_{\tau(x)}$.

and these assertion are equivalent
(iv) $\lim_{k\to+\infty} ||w_k - w||_{\tau(x)} \Leftrightarrow \lim_{k\to+\infty} \rho_{\tau(x)}(w_k - w) = 0.$
(v) w_k *converges to* w *in* Ω *in measure and* $\lim_{k\to+\infty} \rho_{\tau(x)}(w_k) = \rho_{\tau(x)}(w)$.

3. Fractional Sobolev spaces with variable exponents.

First, we introduce and recall some properties of the fractional Sobolev spaces with variable exponents. see [4, 12].

Let $p(.) : \overline{\Omega} \times \overline{\Omega} \to (1,\infty)$, $q(.) : \overline{\Omega} \to (1,\infty)$ be two continuous functions. The fractional Sobolev space with variable exponents defined as follows

$$W^{s(.),p(.)}(\Omega) := \Big\{ u \in L^{p(.)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}} dxdy < +\infty \Big\}$$

which is equipped with the following norm

$$||u||_{W^{s(.),p(.)}(\Omega)} = ||u||_{p(.)} + [u]_{s(.),p(.)}$$

where $[.]_{s(.),p(.)}$ is defined by

$$[.]_{s(.),p(.)} = \inf_{\lambda>0} \Big\{ u \in L^{p(.)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+s(x,y)p(x,y)}} dxdy \leq 1 \Big\}.$$

Remind that $(W^{s(.),p(.)}(\Omega), ||u||_{W^{s(.),p(.)}(\Omega)})$ is a separable reflexive Banach space (see [3]). Now when the weighted (potential) function $a_{i=1,2}$ satisfy (Ps), then we defined the weighted variable exponent Lebesgue space $L^{\tau(.)}_{a_i}(\Omega)$ by

$$L^{\tau(.)}_{a_i}(\Omega) = \Big\{ w : \Omega \to R \ , \ w \text{ is a mesurable function } \int_{\Omega} a_i(x)|w|^{\tau_i(x)} dx < +\infty \Big\}$$

with the norm

$$||.||_{\tau(.),a_i} = \inf_{\lambda>0} \Big\{ \int_{\Omega} a_i(x) \Big| \frac{w}{\lambda} \Big|^{\tau_i(x)} dw \leq 1 \Big\}.$$

$L^{\tau(.)}_{a_i}(\Omega)$ is a Banach space. Moreover, the weighted modular function $\rho_{\tau(),a_i(.)}$ is defined as follows

$$\rho_{\tau(.),a_i(.)}(w) = \int_{\Omega} a_i(x)|w|^{\tau_i(x)} dx.$$

To deal with our problem we define the linear subspace $W_{a_{i=1,2}}(\Omega)$ as follows

$$W_{a_{i=1,2}}(\Omega) = \Big\{ u \in L^{q(.)}_{a_i}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}} dxdy < +\infty \Big\}.$$

It is easy to see that $W_{a_{i=1,2}}(\Omega)$ is a separable reflexive Banach space with the norm

$$||u||_{W_{a_i}} = ||u||_{q(.),a_i} + [u]_{s(.),p(.)}.$$

Defined the modular function $\kappa^{s(.)}_{p(.),q(.)}$ by

$$\kappa^{s(.)}_{p(.),q(.)}(u) = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+s(x,y)p(x,y)}} dxdy + \int_{\Omega} a_i(x)|w|^{q(x)} dx$$

which associated with the linear subspace $X(\Omega)$ defined as follows:

$$X(\Omega) = X := \Big\{ u \in {}^{q(.)}(\Omega) : \kappa^{s(.)}_{p(.),q(.)}(u) < +\infty \Big\},$$

eqquiped with the norm

$$||u||_X = ||u|| := \inf\Big\{\lambda > 0 : \kappa^{s(.)}_{p(.),q(.)}\Big(\frac{u}{\lambda}\Big) \leq 1\Big\}.$$

Remark 3.1. i) $||.||$ is an equivalent norm to the norm $||.||_{W_{a_i}}$ of W_{a_i}.
ii) $(X, ||.||)$ is a separable reflexive Banach space.

The relationship between the norm $||.||$ and the corresponding modular function $\kappa^{s(.)}_{p(.),q(.)}(u)$ is given in the following Lemma

Lemma 3.2. *[14] Let $u \in X$, $\{u_k\} \subset X$ and $p^+ < q^-$, then we have*
(i) $||u|| < 1$ $(= 1; > 1)$ if and only if $\kappa^{s(.)}_{p(.),q(.)}(u) < 1$ $(= 1; > 1)$.
(ii) For $u \in X\backslash\{0\}$, $||u|| = \eta \Leftrightarrow \kappa^{s(.)}_{p(.),q(.)}\Big(\frac{u}{\eta}\Big) = 1$.
(iii) If $||u|| \geq 1$, then $||u||^{p-} \leq \kappa^{s(.)}_{p(.),q(.)}(u) \leq ||u||^{q^+}$,
(iv) If $||u|| \leq 1$, then $||u||^{q^-} \leq \kappa^{s(.)}_{p(.),q(.)}(u) \leq ||u||^{p^+}$.
(v) $\lim\limits_{k\to+\infty} ||u_k - u|| = 0 \Leftrightarrow \lim\limits_{k\to+\infty} \kappa^{s(.)}_{p(.),q(.)}(u_k - u) = 0$.

Proposition 3.3. *[14] Let $u \in L^{s(.)}(\Omega)$, $v \in L^{l(.)}(\Omega)$, $w \in L^{z(.)}(\Omega)$.*
If $\frac{1}{s(x)} + \frac{1}{l(x)} + \frac{1}{z(x)} = 1$, $x \in \overline{\Omega}$, then we have

$$\Big|\int_\Omega u(x)v(x)w(x)dx\Big| \leq \Big(\frac{1}{s^-} + \frac{1}{l^-} + \frac{1}{z^-}\Big)||u||_{s(.)}||v||_{l(.)}||w||_{z(.)}. \tag{9}$$

Now, we present some embedding results in fractional Sobolev spaces with variable exponents.

Theorem 3.4. *[8] Let $s \in (0, 1)$, Ω a Lipschitz bounded domain in $\mathbb{R}^N$. Let $p(x, y)$, $q(x)$ be a continuous variable exponents with $s(x, y)p(x, y) < N$, for $(x, y) \in \overline{\Omega} \times \overline{\Omega}$ and $p(x, y) < q(x)$, $x \in \overline{\Omega}$. If $r : \overline{\Omega} \to (1, +\infty)$ is a continuous function such that*

$$1 < r^- \leq r(x) < p^*_s(x) = \frac{N\bar{p}(x)}{N - s(x, y)\bar{p}(x)} \quad \textit{for all } x, y \in \overline{\Omega} \times \overline{\Omega}.$$

Then the embedding $W^{s(.),p(.)}(\Omega) \hookrightarrow L^{r(.)}(\Omega)$ is compact. i.e, there exist a positive constant k_2 such that

$$||u||_{r(x)} \leq k_2||u||_{W^{s(.),p(.)}(\Omega)}. \tag{10}$$

Lemma 3.5. *[14] Let $s \in (0, 1)$. Let $p(x, y)$, $q(x)$ be a continuous variable exponents with $s(x, y)p(x, y) < N$, for $(x, y) \in \overline{\Omega} \times \overline{\Omega}$ and $p(x, y) \leq q(x) \ll p^*_s(x)$ for $x \in \overline{\Omega}$. If (PS), (Q) and (P) hold true. Then the embedding $X \hookrightarrow L^{q(.)}(\Omega)$ is compact. i.e, there exist a positive constant k3 such that*

$$||u||_{q(x)} \leq k_3||u||_X. \tag{11}$$

Now, we recall the following well-known Ekeland variational principle

Theorem 3.6. *[8] Let $E : Z \to R$ be a bounded and C^1 function in the Banach space Z. Then for any $\epsilon > 0$, there exists $\sigma \in Z$ such that*

$$E(\sigma) \leq \inf_Z E + \epsilon \quad \textit{and} \quad ||E'(\sigma)||_{Z^*} \leq \epsilon.$$

At this point we have all tools to start our study for that we define the working space $W := X \times X$ equipped with the norm $||(u,v)|| = ||u|| + ||v||$. Clearly $(W, ||(.,.)||)$ is a separable, reflexive Banach space. Now we set our main results

Theorem 3.7. *Let $s \in (0,1)$, $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. $N > p(x,y)s(x,y)$ for any $(x,y) \in \overline{\Omega} \times \overline{\Omega}$, where $p(.)$, $s(.)$ verify (Ps). Assume that (B), (P), (L), (F_1) and (F_2) are satisfied. Then, problem (1) admits a weak solution $(u_0, v_0) \in W$. If the energy function E is differentiable at (u_0, v_0)*

We say that a pair of functions $(u,v) \in W$ is the weak solution of (1), if for any $(\varphi, \psi) \in W$ one has

$$A_1\Big(\mathcal{F}_1(u)\Big)\Big(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y)(\varphi(x)-\varphi(y)))}{|x-y|^{N+s(x,y)p(x,y)}}dxdy + \int_\Omega a_1(x)|u|^{q(x)-2}u\varphi dx\Big) = \int_\Omega \Big(F_u(u,v)+b_1(x)\Big)\varphi dx$$

$$A_2\Big(\mathcal{F}_2(v)\Big)\Big(\int_{\Omega\times\Omega} \frac{|v(x)-v(y)|^{p(x,y)-2}(v(x)-v(y)(\psi(x)-\psi(y)))}{|x-y|^{N+s(x,y)p(x,y)}}dxdy + \int_\Omega a_2(x)|v|^{q(x)-2}u\psi dx\Big) = \int_\Omega \Big(F_v(u,v)+b_2(x)\Big)\psi dx.$$

We are now able to claim the result of our existence. First by assumption (F_1) we can see that for all $u, v \in \mathbb{R}^2$:

$$\begin{aligned} F(u,v) &= \int_0^u F_r(r,v)dr + F(0,v) \\ &= \int_0^u F_r(r,v)dr + \int_0^v F_t(0,t)dt + F(0,0). \end{aligned}$$

Moreover, by assumption (F_2) we have $F(u,v) = F(u+K, v+K)$ for all $(u,v) \in \mathbb{R}^2$, then we infer that $|F(u,v)| \le k_4$ for all $(u,v) \in \mathbb{R}^2$. Thus

$$\int_\Omega |F(u,v)|dx \le k_4|\Omega| \tag{12}$$

where $|\Omega|$ is the Lebesgue measure of Ω, and k_4 positive constant. Next we defining the energy functional $E : W \to R$ associated to the problem (1) as follows:

$$\begin{aligned} E(u,v) &= \overline{A}_1\Big[\mathcal{F}_1(u)\Big] + \overline{A}_2\Big[\mathcal{F}_2(v)\Big] - \int_\Omega b_1(x)udx - \int_\Omega b_2(x)vdx \\ &- \int_\Omega F(u,v)dx \end{aligned} \tag{13}$$

for all $u, v \in W$, where $\overline{A}_i(t) = \int_0^t A_i(r)dr$. Obviously, the continuity of A_i yields that E is well defined and of class C^1 on $W\backslash\{0,0\}$. Moreover, for all $\varphi, \psi \in W$ and $u, v \in W$, its Gâteaux derivative is given by

$$\begin{aligned}\langle E'(u,v),(\varphi,\psi)\rangle &= A_1\Big(\mathcal{F}_1(u)\Big)\\ &\times\left(\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y)(\varphi(x)-\varphi(y)))}{|x-y|^{N+s(x,y)p(x,y)}}dxdy\right.\\ &\left.+\int_\Omega a_1(x)|u|^{q(x)-2}u\varphi dx\right)\\ &+A_2\Big(\mathcal{F}_2(v)\Big)\left(\int_{\Omega\times\Omega}\frac{|v(x)-v(y)|^{p(x,y)-2}(v(x)-v(y)(\psi(x)-\psi(y)))}{|x-y|^{N+s(x,y)p(x,y)}}dxdy\right.\\ &\left.+\int_\Omega a_2(x)|v|^{q(x)-2}u\psi dx\right)\\ &-\int_\Omega\Big(F_u(u,v)+b_1(x)\Big)\varphi dx-\int_\Omega\Big(F_v(u,v)+b_2(x)\Big)\psi dx\end{aligned}$$

where $\langle .,.\rangle$ denotes the usual duality between W and its dual space W^*. Note that, the critical points of E are weak solutions of (1).

Lemma 3.8. *The energy function E is coercive and bounded in W .*

Proof. Let $(u,v)\in W$, according to (12) and (13) we have

$$E(u,v)\geq \overline{A}_1\Big[\mathcal{F}_1(u)\Big]+\overline{A}_2\Big[\mathcal{F}_2(v)\Big]-\int_\Omega b_1(x)udx-\int_\Omega b_2(x)vdx-\int_\Omega F(u,v)dx$$

$$\geq \overline{A}_1\Big[\mathcal{F}_1(u)\Big]+\overline{A}_2\Big[\mathcal{F}_2(v)\Big]-\int_\Omega b_1(x)udx-\int_\Omega b_2(x)vdx-k_4|\Omega|.$$

Condition (B) and the Hölder inequality infer that

$$E(u,v)\geq \overline{A}_1\Big[\mathcal{F}_1(u)\Big]+\overline{A}_2\Big[\mathcal{F}_2(v)\Big]-k_4|\Omega|-2||b_1(x)||_{q(x)}||u||_{\overline{p}(x)}-2||b_2(x)||_{q(x)}||v||_{\overline{p}(x)}$$

Using (L), Lemma 3.2, Theorem 3.4 and Lemma 3.5 we obtain

$$\begin{aligned}E(u,v)&\geq k_1\int_0^{\mathcal{F}_1(u)}\tau^{\theta-1}d\tau+\int_0^{\mathcal{F}_2(v)}\tau^{\theta-1}d\tau\\ &=\frac{k_1}{\theta(p^+)^\theta}\Big(\big(\kappa^{s(.)}_{p(.)}(u)\big)^\theta+\big(\kappa^{s(.)}_{p(.)}(v)\big)^\theta\Big)-k_5||u||-k_6||v||-k_4|\Omega|\\ &\geq\frac{k_1}{\theta(p^+)^\theta}\Big(\min\{||u||^{\theta p^-},||u||^{\theta p^+}\}+\min\{||v||^{\theta p^-},||v||^{\theta p^+}\}\Big)\\ &-\max\{k_5,k_6\}(||u||+||v||)-k_4|\Omega|\end{aligned}\tag{14}$$

Since $\theta p^+>\theta p^->1$, when $||(u,v)||\to+\infty$, i.e $||u||\to+\infty$ or $||v||\to+\infty$. So E is coercive and bounded in W. ∎

Proof of Theorem 3.7

We already know that $E \in C^1(W, R)$ is weakly lower semicontinuous and bounded according to Lemma 3.8, by way of the Ekeland variational principle we have $(u_j, v_j) \subset W$ such that,

$$E(u_j, v_j) \to \inf\ E \quad and \quad E'(u_j, v_j) \to 0. \tag{15}$$

According to (15), we have $|E(u_j, v_j)| \leq k_7$. Thus, it follows from (14) that

$$k_8 \leq |E(u_j, v_j)| \leq k_7$$

then the sequences $\{u_j\}$ and $\{v_j\}$ are bounded in X. So, without loss of generality, there exist subsequences still denoting by $\{u_j\}$ and $\{v_j\}$ such that $u_j \rightharpoonup u_0$ and $v_j \rightharpoonup v_0$ in X. Furthermore, applying Lemma 2.1 and Lebesgue dominated convergence theorem, one can check that

$$\int_\Omega b_1(x) u_j dx \to \int_\Omega b_1(x) u_0 dx \quad \text{and} \quad \int_\Omega b_2(x) v_j dx \to \int_\Omega b_2(x) v_0 dx.$$

According to Lemma 3.5, we obtain

$$u_j \to u_0 \quad and \quad v_j \to v_0 \quad a.e\ x \in \Omega.$$

Moreover, by continuity of F, we get

$$F(u_j(x), v_j(x)) \to F(u_0(x), v_0(x)) \quad a.e\ x \in \Omega.$$

Due to (12) and Lebesgue dominated convergence theorem, we get

$$\int_\Omega F(u_j(x), v_j(x)) dx \to \int_\Omega F(u_0(x), v_0(x)) dx.$$

By (15), we have

$$\begin{aligned}\inf_W\ E &= \lim_{j\to+\infty} E(u_j, v_j) \\ &= \lim_{j\to+\infty} \Big(\overline{A}_1 \Big[\mathcal{F}_1(u_j) \Big] + \overline{A}_2 \Big[\mathcal{F}_2(v_j) \Big] - \int_\Omega b_1(x) u_j dx - \int_\Omega b_2(x) v_j dx \\ &\quad - \int_\Omega F(u_j, v_j) dx \Big).\end{aligned}$$

In view of Brezis-Lieb lemma (see [6]), we have

$$\kappa^{s(.)}_{p(.),q(.)}(u_0) \leq \lim_{j\to+\infty} \inf \kappa^{s(.)}_{p(.),q(.)}(u_j) \quad \text{and} \quad \kappa^{s(.)}_{p(.),q(.)}(v_0) \leq \lim_{j\to+\infty} \inf \kappa^{s(.)}_{p(.),q(.)}(v_j)$$

Due to the continuous monotone increasing property of $\overline{A}_1$ and $\overline{A}_2$, we get

$$\overline{A}_1 \Big(\kappa^{s(.)}_{p(.),q(.)}(u_0) \Big) \leq \lim_{j\to+\infty} \overline{A}_1 \Big(\kappa^{s(.)}_{p(.),q(.)}(u_j) \Big)$$

and

$$\overline{A}_2 \Big(\kappa^{s(.)}_{p(.),q(.)}(v_0) \Big) \leq \lim_{j\to+\infty} \overline{A}_2 \Big(\kappa^{s(.)}_{p(.),q(.)}(v_j) \Big)$$

In conclusion,

$$\begin{aligned}\inf_W E &\geq \overline{A}_1 \Big(\kappa^{s(.)}_{p(.),q(.)}(u_0) \Big) + \overline{A}_2 \Big(\kappa^{s(.)}_{p(.),q(.)}(v_0) \Big) - \int_\Omega b_1(x) u_0 dx - \int_\Omega b_2(x) v_0 dx \\ &- \int_\Omega F(u_0, v_0) dx \\ &= E(u_0, v_0),\end{aligned}$$

which implies $E(u_0, v_0) = \inf_W E$. Thus, $(u_0, v_0) \in W$ is a weak solution of problem (1) if E is differentiable at (u_0, v_0). The proof is complete.

Data Availability Statement: No availability.

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