DOI: 10.28919/ejma.2023.3.7

Eur. J. Math. Appl. (2023)3:7
URL: http://ejma.euap.org

BEHAVIOR OF A THIRD ORDER NONLINEAR GENERALIZED RATIONAL RECURSIVE SEQUENCE

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Abstract. In this paper, we investigate the global stability, boundedness of solutions of the recursive sequence

(1) $$J_{n+1} = a_0 J_n + a_1 J_{n-1} + a_2 J_{n-2} + \frac{J_n \quad J_{n-2}}{b_0 J_n + b_1 J_{n-1} + b_2 J_{n-2}}, \quad n = 0, 1, ...$$

where a_i and $b_i \in (0, \infty)$, $i = 0, 1, 2$ with the initial conditions J_{-2}, J_{-1}, and $J_0 \in (0, \infty)$.

1. Introduction

Difference equations or discrete dynamical systems is a diverse field which impact almost every branch of pure and applied mathematics.Every dynamical system $a_{n+1} = f(a_n)$ determines a difference equation and vise versa. Recently, there has been great interest in studying the difference equations. One of the reasons for this is a necessity for some techniques whose can be used in investigating equations arising in mathematical models describing real-life situations in population biology, economic, probability theory, genetics, psychology, etc.

Recently, there has been a lot of interest in studying the boundedness character and the periodic nature of nonlinear difference equations. For some results in this area, see for example [18–22]. Difference equations have been studied in various branches of mathematics for a long time. First results in the qualitative theory of such systems were obtained by Poincaré and Perron at the end of the nineteenth and the beginning of the twentieth centuries.
Many researchers have investigated the behavior of the solution of difference equations, for example:

Camouzis et al. [4] investigated the behavior of solutions of the rational recursive sequence

$$J_{n+1} = \frac{\beta J_n^2}{1 + J_{n-1}^2}.$$

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Key words and phrases. local and global stability; boundedness; difference equations; equilibrium point.
Received 30/09/2022.

Elabbasy et al. [8] investigated the global stability, boundedness, periodicity character and gave the solution of some special cases of the difference equation

$$J_{n+1}=\frac{\alpha J_{n-k}}{\beta+\gamma\prod_{i=0}^{k}J_{n-i}}.$$

Grove, Kulenovic and Ladas [11] presented a summary of a recent work and a large of open problems and conjectures on the third order rational recursive sequence of the form

$$J_{n+1}=\frac{\alpha+\beta J_n+\gamma J_{n-1}+\delta J_{n-2}}{A+BJ_n+CJ_{n-1}+DJ_{n-2}}.$$

In [23] Kulenovic, G. Ladas and W. Sizer studied the global stability character and the periodic nature of the recursive sequence

$$J_{n+1}=\frac{\alpha J_n+\beta J_{n-1}}{\gamma J_n+\delta J_{n-1}}.$$

Kulenovic and Ladas [22] studied the second-order rational difference equation

$$J_{n+1}=\frac{\alpha+\beta J_n+\gamma J_{n-1}}{A+BJ_n+CJ_{n-1}}.$$

Ibrahim et al. [12] studied the third order rational difference Equation

$$J_{n+1}=\frac{J_nJ_{n-2}}{J_{n-1}(\alpha+\beta J_nJ_{n-2})}$$

Agarwal et al. [2] studied the solution of fourth-order rational recursive sequence

$$J_{n+1}=a\ J_n+\frac{b\ J_n\ J_{n-3}}{c\ J_{n-2}+d\ J_{n-3}}$$

For other important references , we refer the reader to [1], [3], [5], [6], [7], [9], [10], [13], [14], [15], [16], [17], [23], [24], [25], [26], [27 – 48]

Our goal in this paper is to investigate the global stability and boundedness of solutions of the recursive sequence

$$J_{n+1}=a_0J_n+a_1J_{n-1}+a_2J_{n-2}+\frac{J_n\quad J_{n-2}}{b_0J_n+b_1J_{n-1}+b_2J_{n-2}},\qquad n=0,1,...,$$

where a_i and $b_i\in(0,\infty)$, with the initial conditions $J_{-k},J_{-k+1},...,J_{-1}$ and $J_0\in(0,\infty)$. Here, we recall some notations and results which will be helpful in our investigation.

Let I be some interval of real numbers and let

$$F:I^{k+1}\to I$$

be a continuously differentiable function. Then for every set of initial conditions $J_{-k},J_{-k+1},...,J_0\in I$, the difference equation

(2) $$J_{n+1}=F(J_n,J_{n-1},...,J_{n-k}),\quad n=0,1,...,$$

has a unique solution $\{J_n\}_{n=-k}^{\infty}$ [19].

A point $\overline{x}\in I$ is called an equilibrium point of equation(2) if

$$\overline{x}=F(\overline{x},\overline{x},...,\overline{x}).$$

That is, $J_n = \overline{x}$ for $n \geq 0$, is a solution of equation(2), or equivalently, $\overline{x}$ is a fixed point of F.

Definition 1. *The difference equation (2) is said to be persistence if there exist numbers m and M with $0 < m \leq M < \infty$ such that for any initial conditions $J_{-k}, J_{-k+1}, ..., J_{-1}, J_0 \in (0, \infty)$ there exists a positive integer N which depends on the initial conditions such that*

$$m \leq J_n \leq M \qquad \text{for all} \quad n \geq N.$$

Definition 2. *(Stability) Let I be some interval of real numbers.*
(i) The equilibrium point $\overline{x}$ of equation(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $J_{-k}, J_{-k+1}, ..., J_{-1}, J_0 \in I$ with

$$|J_{-k} - \overline{x}| + |J_{-k+1} - \overline{x}| + ... + |J_0 - \overline{x}| < \delta,$$

we have

$$|J_n - \overline{x}| < \epsilon \qquad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of equation(2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of equation(2) and there exists $\gamma > 0$, such that for all $J_{-k}, J_{-k+1}, ..., J_{-1}, J_0 \in I$ with

$$|J_{-k} - \overline{x}| + |J_{-k+1} - \overline{x}| + ... + |J_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n \to \infty} J_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of equation(2) is global attractor if for all $J_{-k}, J_{-k+1}, ..., J_{-1}, J_0 \in I$, we have

$$\lim_{n \to \infty} J_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of equation(2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of equation(2).
(v) The equilibrium point $\overline{x}$ of equation(2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of equation(2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial J_{n-i}} y_{n-i}.$$

Theorem A [18] Assume that $p, q \in \mathbb{R}$ and $k \in \{0, 1, 2, ...\}$. Then

$$|p| + |q| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$J_{n+1} + pJ_n + qJ_{n-k} = 0, \quad n = 0, 1, ... \, .$$

Theorem A can be easily extended to a general linear equations of the form

$$J_{n+k} + p_1 J_{n+k-1} + ... + p_k J_n = 0, \quad n = 0, 1, ...$$

where $p_1, p_2, ..., p_k \in \mathbb{R}$ and $k \in \{1, 2, ...\}$. Then equation(3) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

2. Local Stability of the Equilibrium Point

In this section, we investigate the local stability character of the solutions of Eq.(1). Eq.(1) has a unique equilibrium point and is given by

$$\overline{x} = 0$$

such that $(1 - a_0 - a_1 - a_2)(b_0 + b_1 + b_2) \neq 1$.

$$\overline{x} = a_0\overline{x} + a_1\overline{x} + a_2\overline{x} + \frac{\overline{x}^2}{b_0\overline{x} + b_1\overline{x} + b_2\overline{x}},$$

or,

$$(\overline{x} - a_0\overline{x} - a_1\overline{x} - a_2\overline{x})(b_0\overline{x} + b_1\overline{x} + b_2\overline{x}) = \overline{x}^2,$$

Thus

$$\overline{x}^2\{(1 - a_0 - a_1 - a_2)(b_0 + b_1 + b_2) - 1)\} = 0$$

if $(1 - a_0 - a_1 - a_2)(b_0 + b_1 + b_2)) \neq 1$, then the unique equilibrium point is $\overline{x} = 0$.

Let $f : (0,\infty)^3 \longrightarrow (0,\infty)$ be a function defined by

$$f(u,v,w) = a_0u + a_1v + a_2w + \frac{u\ w}{b_0u + b_1v + b_2w}.$$

Therefore it follows that

$$\begin{aligned}
f_u(u,v,w) &= a_0 + \frac{(b_0u + b_1v + b_2w)\ w - uw(b_0)}{(b_0u + b_1v + b_2w)^2} = a_0 + \frac{(b_1v + b_2w)\ w}{(b_0u + b_1v + b_2w)^2},\\
f_v(u,v,w) &= a_1 + \frac{(b_0u + b_1v + b_2w)\,0 - u\ w(b_1)}{(b_0u + b_1v + b_2w)^2} = a_1 - \frac{u\ w(b_1)}{(b_0u + b_1v + b_2w)^2},\\
f_w(u,v,w) &= a_2 + \frac{(b_0u + b_1v + b_2w)\,u\ - uw\,(b_2)}{(b_0u + b_1v + b_2w)^2} = a_2 + \frac{(b_0u + b_1v)\,u}{(b_0u + b_1v + b_2w)^2}
\end{aligned}$$

we see that

$$\begin{aligned}
f_u(\overline{x},\overline{x},\overline{x}) &= a_0 + \frac{(b_1\overline{x} + b_2\overline{x})\ \overline{x}}{(b_0\overline{x} + b_1\overline{x} + b_2\overline{x})^2} = a_0 + \frac{(b_1 + b_2)}{(b_0 + b_1 + b_2)^2},\\
f_v(\overline{x},\overline{x},\overline{x}) &= a_1 - \frac{(b_1)}{(b_0 + b_1 + b_2)^2},\\
f_w(\overline{x},\overline{x},\overline{x}) &= a_2 + \frac{(b_0 + b_1)}{(b_0 + b_1 + b_2)^2}
\end{aligned}$$

The linearized equation of Eq.(1) about $\overline{x}$ is

(3)

$$y_{n+1} - \left(a_0 + \frac{(b_1 + b_2)}{(b_0 + b_1 + b_2)^2}\right)y_n - \left(a_1 - \frac{(b_1)}{(b_0 + b_1 + b_2)^2}\right)y_{n-1} + \left(a_2 + \frac{(b_0 + b_1)}{(b_0 + b_1 + b_2)^2}\right)y_{n-2} = 0.$$

Theorem 1. If the following condition satisfies

$$(a_0 + a_1 + a_2 - 1)(b_0 + b_1 + b_2) + 1 < 0$$

Then the equilibrium point of Eq.(1) is locally asymptotically stable.

Proof: It is follows by Theorem A that, Eq.(3) is asymptotically stable if

$$\left|a_0 + \frac{(b_1 + b_2)}{(b_0 + b_1 + b_2)^2}\right| + \left|a_1 - \frac{(b_1)}{(b_0 + b_1 + b_2)^2}\right| + \left|a_2 + \frac{(b_0 + b_1)}{(b_0 + b_1 + b_2)^2}\right| < 1,$$

$$a_0 + \frac{(b_1+b_2)}{(b_0+b_1+b_2)^2} + a_1 - \frac{(b_1)}{(b_0+b_1+b_2)^2} + a_2 + \frac{(b_0+b_1)}{(b_0+b_1+b_2)^2} < 1,$$

$$a_0 + a_1 + a_2 + \frac{(b_1+b_2) - (b_1) + (b_0+b_1)}{(b_0+b_1+b_2)^2} < 1,$$

$$a_0 + a_1 + a_2 + \frac{(b_0+b_1+b_2)}{(b_0+b_1+b_2)^2} < 1,$$

$$(a_0 + a_1 + a_2) + \frac{1}{(b_0+b_1+b_2)} < 1,$$

$$(a_0 + a_1 + a_2 - 1)(b_0+b_1+b_2) + 1 < 0,$$

The proof is complete.

3. Boundedness of Solutions of Eq.(1)

In this section, we study the boundedness of solutions of Eq.(1).

Theorem2. Every solution of Eq.(1) is bounded from above if $a_0 + \frac{1}{b_2} < 1$ & $a_1 < 1$ and $a_2 < 1$.

Proof: Let $\{J_n\}_{n=-3}^{\infty}$ be a solution of Eq.(1). It follows from Eq.(1) that

$$\begin{aligned} J_{n+1} &= a_0 J_n + a_1 J_{n-1} + a_2 J_{n-2} + \frac{J_n \ J_{n-2}}{b_0 J_n + b_1 J_{n-1} + b_2 J_{n-2}} \\ &< a_0 J_n + a_1 J_{n-1} + a_2 J_{n-2} + \frac{J_n}{b_2} = \left(a_0 + \frac{1}{b_2}\right) J_n + a_1 J_{n-1} + a_2 J_{n-2} \\ &< J_n + J_{n-1} + J_{n-2} \end{aligned}$$

Then

$$J_{n+1} \leq J_n + J_{n-1} + J_{n-2} \quad \text{for all} \quad n \geq 0.$$

So every solution of Eq.(1) is bounded from above by $M = J_0 + J_{-1} + J_{-2}$.

4. Application

In this section we will study the solution of Eq(1) when $a_0 = a_1 = a_2 = b_1 = 0$ and $b_0 = b_2 = 1$.

In this case, we have the following difference equation.

$$J_{n+1} = \frac{J_n \ J_{n-2}}{J_n + J_{n-2}} \tag{4}$$

The following theorem gives the solution of Eq(4).

Theorem3 Let $\{J_n\}_{n=-1}^{\infty}$ be a solution of Eq.(4). Then equation (4) has the solutions

$$J_n = \frac{J_0 \ J_{-1} J_{-2}}{(F_{n-1}) J_0 J_{-2} + (F_n) J_0 J_{-1} + (F_{n+1}) J_{-1} J_{-2}} \tag{5}$$

where $F_n = F_{n-1} + F_{n-3}$, $n \geq 3$, $F_0 = 0, F_1 = 1, F_2 = 1$ and the initial conditions J_{-2}, $J_{-1}$and J_0 are arbitrary positive real numbers.

Proof.

For $n = 1$,

$$J_1 = \frac{J_0\ J_{-2}}{J_0 + J_{-2}} = \frac{J_0\ J_{-1}J_{-2}}{J_0J_{-1} + J_{-1}J_{-2}} = \frac{J_0\ J_{-1}J_{-2}}{(F_0)J_0J_{-2} + (F_1)J_0J_{-1} + (F_2)J_{-1}J_{-2}}$$

hence the result holds.

Now suppose that the relation (5) holds for $n = k$. That is;

$$J_k = \frac{J_0\ J_{-1}J_{-2}}{(F_{k-1})J_0J_{-2} + (F_k)J_0J_{-1} + (F_{k+1})J_{-1}J_{-2}}$$

We are going to show that the relation (5) holds for $n = k + 1$

$$\begin{aligned} J_{k+1} &= \frac{J_k\ \ J_{k-2}}{J_k + J_{k-2}} \\ &= \frac{\left(\frac{J_0\ J_{-1}J_{-2}}{(F_{k-1})J_0J_{-2}+(F_k)J_0J_{-1}+(F_{k+1})J_{-1}J_{-2}}\right)\left(\frac{J_0\ J_{-1}J_{-2}}{(F_{k-3})J_0J_{-2}+(F_{k-2})J_0J_{-1}+(F_{k-1})J_{-1}J_{-2}}\right)}{\left(\frac{J_0\ J_{-1}J_{-2}}{(F_{k-1})J_0J_{-2}+(F_k)J_0J_{-1}+(F_{k+1})J_{-1}J_{-2}}\right)+\left(\frac{J_0\ J_{-1}J_{-2}}{(F_{k-3})J_0J_{-2}+(F_{k-2})J_0J_{-1}+(F_{k-1})J_{-1}J_{-2}}\right)} \end{aligned}$$

$$= \frac{(J_0\ J_{-1}J_{-2})}{((F_{k-3})J_0J_{-2} + (F_{k-2})J_0J_{-1} + (F_{k-1})J_{-1}J_{-2}) + ((F_{k-1})J_0J_{-2} + (F_k)J_0J_{-1} + (F_{k+1})J_{-1}J_{-2})}$$

$$= \frac{(J_0\ J_{-1}J_{-2})}{F_{k-3}J_0J_{-2} + F_{k-2}J_0J_{-1} + F_{k-1}J_{-1}J_{-2} + F_{k-1}J_0J_{-2} + F_kJ_0J_{-1} + F_{k+1}J_{-1}J_{-2}}$$

$$= \frac{(J_0\ J_{-1}J_{-2})}{(F_{k-3} + F_{k-1})J_0J_{-2} + (F_{k-2} + F_k)J_0J_{-1} + (F_{k-1} + F_{k+1})J_{-1}J_{-2}}$$

$$= \frac{J_0\ J_{-1}J_{-2}}{(F_k)J_0J_{-2} + (F_{k+1})J_0J_{-1} + (F_{k+2})J_{-1}J_{-2}}$$

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