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THE THEOREM OF BOCHNER FOR ADJOINTABLE OPERATORS VALUED MAPS

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Abstract. In this paper, we obtain a generalisation of Bochner's theorem to positive definite functions defined on a locally compact abelian group with values in the space of adjointable operators on a Hilbert C^*-module.

1. Introduction

Initially, Bochner's theorem gives a characterization of the Fourier transform of a positive finite Borel measure on the real line. In its general form, the Bochner's theorem links positive definite functions on a locally compact abelian group to a finite positive Borel measure on the dual group via the Fourier-Stieltjes transform. Bochner's theorem has many generalizations. Examples include references [5–8, 12–14], this list is of course non exhaustive.

In [7] the author extends the Bochner's theorem to the case of positive definite maps from a locally compact abelian group G into $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. In this paper, we extend the results in [7] to positive definite functions from a locally compact abelian group G into the space $End^*_{\mathcal{A}}(\mathcal{M})$ of adjointable operators on a self-dual Hilbert C^*-module $\mathcal{M}$.

The rest of the paper is organized as follows. In Section 2, we provide basic informations about C^*-algebras and Hilbert C^*-modules that we may need. In Section 3, we obtain some results about positive definite functions from G into $End^*_{\mathcal{A}}(\mathcal{M})$. Finally, in Section 4, we state the theorem of Bochner in the framework that we have considered.

2. C^*-algebras and Hilbert C^*-modules

We recall here basic informations about C^*-algebras and Hilbert C^*-modules that we may need in this article. Interested readers are referred to [4, 9] for more details on C^*-algebras and [10, 11] for more details on Hilbert C^*-modules.

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Let $\mathcal{A}$ be an algebra over $\mathbb{C}$. A map $\mathcal{A} \to \mathcal{A}$, $a \mapsto a^*$ is called an *involution* if

$$
\begin{aligned}
a^{**} &= a, \\
(a+b)^* &= a^* + b^*, \\
(\alpha a)^* &= \overline{\alpha} a^*, \\
(ab)^* &= b^* a^*.
\end{aligned}
$$

$\forall a, b \in \mathcal{A}$ and $\forall \alpha \in \mathbb{C}$. A $*$-*algebra* is an algebra with involution.
A *Banach algebra* is a Banach space $\mathcal{A}$ which is also an algebra such that

$$\|ab\| \leq \|a\|\|b\|, \ \forall a, b \in \mathcal{A}.$$

A C^*-*algebra* is a $*$-Banach algebra $\mathcal{A}$ such that

$$\forall a \in \mathcal{A}, \ \|a^*a\| = \|a\|^2.$$

By the Gelfand-Naimark theorem, any C^*-algebra can be realized as a concrete norm-closed subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of the bounded linear operators on some Hilbert space $\mathcal{H}$ [9, page 2]. A C^*-algebra $\mathcal{A}$ is called *unital* if it has a unit (denoted by $1_{\mathcal{A}}$). An element a of the C^*-algebra $\mathcal{A}$ is called *self-adjoint* if $a^* = a$ and it is called *positive* if there exists $b \in \mathcal{A}$ such that $a = b^*b$. Positive elements are automatically self-adjoint. We denote by $\mathcal{A}^+$ the set of all positive elements of $\mathcal{A}$. The space $\mathcal{A}^+$ is a convex cone. Also we will use the following result: If $a, b \in \mathcal{A}^+$ are such that $b \leq a$, then $\|b\| \leq \|a\|$.

A pre-Hilbert module over a C^*-algebra $\mathcal{A}$ is a complex vector space $\mathcal{M}$ which is also a right $\mathcal{A}$-module such that there is a map

$$\mathcal{M} \times \mathcal{M} \to \mathcal{A}, \ (x, y) \mapsto \langle x, y \rangle$$

with the following properties. For $x, y, z \in \mathcal{M}$, $\lambda \in \mathbb{C}$, $a \in \mathcal{A}$,

$$
\begin{aligned}
\langle x, \lambda y + z \rangle &= \lambda \langle x, y \rangle + \langle x, z \rangle, \\
\langle x, ya \rangle &= \langle x, y \rangle a, \\
\langle y, x \rangle &= \langle x, y \rangle^*, \\
\langle x, x \rangle &\in \mathcal{A}^+, \\
\langle x, x \rangle = 0 &\Rightarrow x = 0.
\end{aligned}
$$

The equality

$$\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$$

defines a norm on $\mathcal{M}$. Moreover, if $\mathcal{M}$ is complete with respect to this norm, then $\mathcal{M}$ is called a *Hilbert $\mathcal{A}$-module* or a *Hilbert C^*-module over $\mathcal{A}$*.

Let us give some examples.

(1) Every complex Hilbert space $\mathcal{H}$ is a Hilbert module over $\mathbb{C}$. The inner product is the usual scalar product on $\mathcal{H}$.
(2) Every C^*-algebra $\mathcal{A}$ is a Hilbert module over $\mathcal{A}$. The inner product is defined by

$$\langle x, y \rangle = x^*y, \quad x, y \in \mathcal{A}.$$

(3) Let E_k, $k = 1, \cdots, n$ be Hilbert C^*-modules over the same C^*-algebra $\mathcal{A}$. The direct sum $\overset{n}{\underset{k=1}{\oplus}} E_k$ equipped with the module action and the inner product given by the formula

$$\langle x, y\rangle = \sum_{i=1}^{n} \langle x_i, y_i\rangle$$

where $x = (x_1, x_2 \dots x_n)$ and $y = (y_1, y_2 \dots y_n)$.

From the above first two examples, one notices that Hilbert C^*-module generalizes C^*-algebra and Hilbert space.

Let us denote by $\mathcal{M}'$ the set of all bounded $\mathcal{A}$-linear maps from $\mathcal{M}$ into $\mathcal{A}$. The relations

$$(\lambda f)(x) = \overline{\lambda} f(x) \text{ and } (fa)(x) = a^* f(x),$$

where $\lambda \in \mathbb{C}, f \in \mathcal{M}', x \in \mathcal{M}, a \in \mathcal{A}$, introduce respectively on $\mathcal{M}'$ a structure of vector space over $\mathbb{C}$ and a structure of right $\mathcal{A}$-module. Also, $\mathcal{M}'$ is complete under the norm

$$\|f\| = \sup_{\|x\| \leq 1} \|f(x)\|.$$

There is a natural isometric embedding of $\mathcal{M}$ into $\mathcal{M}'$ via the map $x \mapsto \langle x, \cdot\rangle$. The Hilbert C^*-module $\mathcal{M}$ is called *self-dual* if $\mathcal{M}' = \mathcal{M}$.

We denote by $End_{\mathcal{A}}(\mathcal{M})$, the set of bounded $\mathbb{C}$-linear and $\mathcal{A}$-linear endomorphisms of $\mathcal{M}$. Such maps will be called *operators* on $\mathcal{M}$. We say that an operator T on $\mathcal{M}$ is *adjointable* if there exists an operator T^* on $\mathcal{M}$, called the *adjoint* of T, such that

$$\forall x, y \in \mathcal{M},\ \langle Tx, y\rangle = \langle x, T^* y\rangle. \tag{1}$$

It is known that an element of $End_{\mathcal{A}}(\mathcal{M})$ may not have necessarily an adjoint. An operator that has an adjoint is called *adjointable*. In the sequel, we denote by $End^*_{\mathcal{A}}(\mathcal{M})$ the set of all adjointable operators on $\mathcal{M}$.

3. Adjointable operators valued positive definite functions

Let G be a locally compact abelian group. We denote by $\widehat{G}$ the dual group of G.

Definition 3.1. *A function $f : G \to \mathcal{A}$ is said to be positive definite if*

$$\forall N \in \mathbb{N}, \forall a_1, \cdots, a_N \in \mathcal{A}, \forall g_1, \cdots, g_N \in G, \sum_{i=1}^{N} \sum_{j=1}^{N} a_i f(g_i - g_j) a_j^* \in \mathcal{A}^+.$$

Definition 3.2. *An element $T \in End^*_{\mathcal{A}}(\mathcal{M})$ is said to be positive if $\forall x \in \mathcal{M}, \langle Tx, x\rangle$ is a positive element of the C^*-algebra $\mathcal{A}$.*

Definition 3.3. *A function $\varphi : G \to End^*_{\mathcal{A}}(\mathcal{M})$ is said to be positive definite for any $x \in \mathcal{M}$, the function $\varphi_x : G \to \mathcal{A}$ defined by*

$$\varphi_x(g) = \langle \varphi(g)x, x\rangle \tag{2}$$

is positive definite; in other words if

$$\forall x \in \mathcal{M}, \forall N \in \mathbb{N}, \forall a_1, \cdots, a_N \in \mathcal{A}, \forall g_1, \cdots, g_N \in G, \sum_{i=1}^{N} \sum_{j=1}^{N} a_i \langle \varphi(g_i - g_j)x, x\rangle a_j^* \in \mathcal{A}^+.$$

Lemma 3.4. *Let $a, b \in \mathcal{A}$. If $a \in \mathcal{A}^+$ and $a + b \in \mathcal{A}^+$, then $b^* = b$.*

Proof. Set $c = a + b$. There exists $a_1, c_1 \in \mathcal{A}$ such that $a = a_1^* a_1$ and $c = c_1^* c_1$. Then, $b = c - a = c_1^* c_1 - a_1^* a_1$. Thus, $b^* = (c_1^* c_1 - a_1^* a_1)^* = c_1^* c_1 - a_1^* a_1 = b$. □

Theoreme 3.5. *Let $\varphi : G \to End_{\mathcal{A}}^*(\mathcal{M})$ be a positive definite function. Then, $\varphi(0)$ is a positive element of $End_{\mathcal{A}}^*(\mathcal{M})$.*

Proof. Assume that $\varphi : G \to End_{\mathcal{A}}^*(\mathcal{M})$ is a positive definite function. For $x \in \mathcal{M}$, $\varphi_x : G \to \mathcal{A}$ is positive definite. Then, $\forall N \in \mathbb{N}$, $\forall a_1, \cdots, a_N \in \mathcal{A}$, $\forall g_1, \cdots, g_N \in G$,

$$\sum_{i=1}^{N} \sum_{j=1}^{N} a_i \varphi_x(g_i - g_j) a_j^* \in \mathcal{A}^+. \tag{3}$$

Now, if we take $N = 1$, $a_1 = 1_{\mathcal{A}}$, $g_1 = 0$, then we obtain from (3) $\varphi_x(0) \in \mathcal{A}^+$, in other words $\langle \varphi(0)x, x \rangle \in \mathcal{A}^+$. Thus $\varphi(0)$ is a positive element of $End_{\mathcal{A}}^*(\mathcal{M})$. □

Theoreme 3.6. *Let $\varphi : G \to End_{\mathcal{A}}^*(\mathcal{M})$ be a positive definite function. Then, for any element g in G, one has $\varphi(-g) = \varphi(g)^*$.*

Proof. Assume that $\varphi : G \to End_{\mathcal{A}}^*(\mathcal{M})$ is a positive definite function. For $x \in \mathcal{M}$, $\varphi_x : G \to \mathcal{A}$ is positive definite. If we take $N = 2$, $a_1 = 1_{\mathcal{A}}$, $a_2 = a, g_1 = g$, $g_2 = 0$, then we obtain from (3)

$$\langle \varphi(0)x, x \rangle + \langle \varphi(g)x, x \rangle a^* + a\langle \varphi(-g)x, x \rangle + a\langle \varphi(0)x, x \rangle a^* \in \mathcal{A}^+. \tag{4}$$

Since $\langle \varphi(0)x, x \rangle$ and $a\langle \varphi(0)x, x \rangle a^*$ belong to $\mathcal{A}^+$, then we deduce from Lemma 3.4 that $\langle \varphi(g)x, x \rangle a^* + a\langle \varphi(-g)x, x \rangle$ is a self-adjoint element of $\mathcal{A}$, that is

$$(\langle \varphi(g)x, x \rangle a^* + a\langle \varphi(-g)x, x \rangle)^* = \langle \varphi(g)x, x \rangle a^* + a\langle \varphi(-g)x, x \rangle. \tag{5}$$

If we take successively $a = 1_{\mathcal{A}}$ and $a = i1_{\mathcal{A}}$ we obtain

$$\langle \varphi(g)x, x \rangle^* + \langle \varphi(-g)x, x \rangle^* = \langle \varphi(g)x, x \rangle + \langle \varphi(-g)x, x \rangle \tag{6}$$

and

$$\langle \varphi(g)x, x \rangle^* - \langle \varphi(-g)x, x \rangle^* = -\langle \varphi(g)x, x \rangle + \langle \varphi(-g)x, x \rangle. \tag{7}$$

Now, we add equalities (6) and (7) to obtain $\langle \varphi(g)x, x \rangle^* = \langle \varphi(-g)x, x \rangle$. So,

$$\langle \varphi(-g)x, x \rangle = \langle x, \varphi(g)x \rangle = \langle \varphi(g)^* x, x \rangle.$$

Thus, $\varphi(-g) = \varphi(g)^*$.

□

Theoreme 3.7. *Let $\varphi : G \to End_{\mathcal{A}}^*(\mathcal{M})$ be a positive definite function. Assume that for all $x \in \mathcal{M}$, $\varphi_x(0)$ is 0 or $\varphi_x(0)$ is an invertible element of $\mathcal{Z}(\mathcal{A})$ (the center of $\mathcal{A}$). Then, $\forall g \in G$, $\|\varphi(g)\| \leq 2\|\varphi(0)\|$.*

Proof. Let us assume that $\varphi : G \to End_{\mathcal{A}}^*(\mathcal{M})$ is a positive definite function. We recall (4):

$$\langle \varphi(0)x, x \rangle + \langle \varphi(g)x, x \rangle a^* + a\langle \varphi(-g)x, x \rangle + a\langle \varphi(0)x, x \rangle a^* \in \mathcal{A}^+. \tag{8}$$

Now, if $\varphi_x(0) = 0$ we have $\langle \varphi(g)x, x \rangle a^* + a\langle \varphi(-g)x, x \rangle \in \mathcal{A}^+$. Taking $a = -\varphi_x(g)$ and using the fact that $\varphi(-g) = \varphi(g)^*$, we have

$$-\left[\varphi_x(g)\varphi_x(g)^* + \varphi_x(g)\varphi_x(g)^*\right] \in \mathcal{A}^+,$$
$$\Rightarrow \varphi_x(g)^*\varphi_x(g) = \varphi_x(g)\varphi_x(g)^* = 0$$

because $\varphi_x(g)^*\varphi_x(g)$ and $\varphi_x(g)\varphi_x(g)^*$ are positive elements in $\mathcal{A}$. Then, we have

$$\|\varphi_x(g)\|^2 = \|\varphi_x(g)^*\varphi_x(g)\| = 0.$$

This implies $\varphi_x(g) = 0$, $\forall x \in \mathcal{M}$. Then, $\varphi(g) \equiv 0$. Therefore, $\|\varphi(g)\| \leq 2\|\varphi(0)\|$. Now, if $\varphi_x(0) \neq 0$ is invertible, then we set $a = -\varphi_x(g)\varphi_x(0)^{-1}$ in (8) to obtain $\varphi_x(0) - \varphi_x(g)\varphi_x(0)^{*-1}\varphi_x(g)^* - \varphi_x(g)\varphi_x(0)^{-1}\varphi_x(g)^* +$ $\varphi_x(g)\varphi_x(0)^{-1}\varphi_x(0)\varphi_x(g)^{*-1}\varphi_x(g) \in \mathcal{A}^+$. Taking into account the fact that $\varphi_x(0)$ is a positive element of $\mathcal{A}$, hence self-adjoint, we have

$$\varphi_x(0) - \varphi_x(g)\varphi_x(0)^{-1}\varphi_x(g)^* \in \mathcal{A}^+.$$
$$\Rightarrow \varphi_x(0) - \varphi_x(0)^{-1}\varphi_x(g)\varphi_x(g)^* \in \mathcal{A}^+ \text{ (because } \varphi_x(0)^{-1} \in \mathcal{Z}(\mathcal{A})),$$
$$\Rightarrow \varphi_x(0)^2 - \varphi_x(g)\varphi_x(g)^* \in \mathcal{A}^+ \text{ (multiplication by } \varphi_x(0)).$$

Taking the norm, we obtain

$$\|\varphi_x(g)\varphi_x(g)^*\| \leq \|\varphi_x(0)\|^2$$
$$\Rightarrow \|\varphi_x(g)\|^2 \leq \|\varphi_x(0)\|^2$$
$$\Rightarrow \|\varphi_x(g)\| \leq \|\varphi_x(0)\|.$$

Therefore, if $\|x\| = \|y\| = 1$, we have $\|\langle\varphi(g)x, y\rangle\| \leq 2\|\varphi(0)\|$. This implies $\|\varphi(g)\| := \sup_{\|x\|=1,\, \|y\|=1} \|\langle\varphi(g)x, y\rangle\| \leq 2\|\varphi(0)\|$. □

4. On a theorem of Bochner

We denote by $\Sigma(\widehat{G})$ the Borel σ-field of $\widehat{G}$, the dual group of the locally compact group G.

Definition 4.1. *A measure $m : \Sigma(\widehat{G}) \to End^*_{\mathcal{A}}(\mathcal{M})$ is called positive regular if*

(1) *$\forall E \in \Sigma(\widehat{G})$, $m(E)$ is a positive element in $End^*_{\mathcal{A}}(\mathcal{M})$.*
(2) *$\forall x \in \mathcal{M}$, the measure $m_x : \Sigma(\widehat{G}) \to \mathcal{A}$ is a regular vector measure.*

Definition 4.2. *A measure $m : \Sigma(\widehat{G}) \to End^*_{\mathcal{A}}(\mathcal{M})$ is said to be bounded if there exists $\alpha > 0$ such that $\|m(E)\| \leq \alpha$, $\forall E \in \Sigma(\widehat{G})$.*

Now, we consider on $End^*_{\mathcal{A}}(\mathcal{M})$ the weakest topology that makes continuous all the functions $\varphi_{x,y} : End^*_{\mathcal{A}}(\mathcal{M}) \to \mathcal{A}$, $T \mapsto \varphi_{x,y}(T) = \langle Tx, y\rangle$. A map $\psi : G \to End^*_{\mathcal{A}}(\mathcal{M})$ is said to be continuous if $\forall \varepsilon > 0$, $\forall x, y \in \mathcal{M}$, $\exists N_{\varepsilon,x,y}$ neighbourhood of 0 in G such that if $g - g' \in N_{\varepsilon,x,y}$, then $\|\langle(\psi(g) - \psi(g'))\, x, y\rangle\| < \varepsilon$.

Let $\varphi : G \to End^*_{\mathcal{A}}(\mathcal{M})$ be a positive definite and continuous function. Let $x \in \mathcal{M}$. Then, the map $\varphi_x : G \to \mathcal{A}$ is positive definite and continuous. Using the Gelfand-Naimark theorem and Theorem 2.8 of [7], we obtain a regular positive bounded measure $\mu_x : \Sigma(\widehat{G}) \to \mathcal{A}$ such that

(9) $$\varphi_x(g) = \int_{\widehat{G}} (\gamma, g) d\mu_x(\gamma).$$

That is

(10) $$\langle \varphi(g)x, x\rangle = \int_{\widehat{G}} (\gamma, g) d\mu_x(\gamma),$$

Consider $E \in \Sigma(\widehat{G})$. Define the map

$$\tau_E : \mathcal{M} \to \mathcal{A},\ x \mapsto \tau_E(x) = \mu_x(E).$$

Consider the map $\Gamma_E : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ defined by

(11) $$\Gamma_E(x, y) = \left[\tau_E(\frac{x+y}{2}) - \tau_E(\frac{x-y}{2})\right] + i\left[\tau_E(\frac{x+iy}{2}) - \tau_E(\frac{x-iy}{2})\right]$$

Theoreme 4.3. *The map $\Gamma_E : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ has the following properties*

(1) *Γ_E is $\mathcal{A}$-linear in the second variable and $\mathcal{A}$-involution linear in the first variable.*
(2) *$\Gamma_E(x, y) = \Gamma_E(y, x)^*$, $\forall x, y \in \mathcal{M}$.*
(3) *Γ_E is bounded apart from E.*

Proof. If we set $\mu_{x,y}(E) = \Gamma_E(x, y)$, then we have

$$\langle \varphi(g)x, y\rangle = \int_{\widehat{G}} (\gamma, g) d\mu_{x,y}(\gamma).$$

Then, using the properties of the $\mathcal{A}$-valued product $\langle \cdot, \cdot\rangle$, we obtain that Γ_E is and $\mathcal{A}$-linear in y and $\mathcal{A}$-involution linear in x. Then, we deduce the property (1). Moreover, the observation that the measure μ_x in the equality (9) is the same for all $g \in G$ and the fact that $\langle \varphi(g)x, y\rangle = \langle \varphi(-g)y, x\rangle^*$ conduct to the property (2). From the property (2) it is sufficient to show that $\Gamma_E(x, x)$ is bounded to obtain the property (3). Note that

$$\|\Gamma_E(x, x)\| = \|\mu_x(E)\| \leqslant \|\mu_x\|$$

where $\|\mu_x\|$ denote the total variation of μ_x given by

$$\|\mu_x\| = \mu_x(\hat{G}) = \int_{\hat{G}} d\mu_x(\gamma) = \int_{\hat{G}} (\gamma, 0) d\mu_x(\gamma) = \langle \varphi(0)x, x\rangle.$$

On the other hand, $\langle \varphi(0)x, x\rangle \leqslant \|\varphi(0)\|\|x\|^2$. Thus, $\|\Gamma_E(x, x)\| \leq \|\varphi(0)\|\|x\|^2$. We conclude that $\Gamma_E(x, y)$ is bounded apart from E. □

From Theorem 4.3 we deduce the existence of a unique self-adjoint element $m_\varphi(E)$ of $End^*_{\mathcal{A}}(\mathcal{M})$ such that

(12) $$\Gamma_E(x, y) = \langle m_\varphi(E)x, y\rangle.$$

Moreover, $m_\varphi(E)$ is a positive element of $End^*_{\mathcal{A}}(\mathcal{M})$ since

$$\langle m_\varphi(E)x, x\rangle = \Gamma(x, x) = \mu_x(E) \in \mathcal{A}^+.$$

Fix $x \in \mathcal{M}$ and consider the map $m_{\varphi,x} : \Sigma(\widehat{G}) \to \mathcal{M}, E \mapsto m_{\varphi,x}(E) = m_\varphi(E)x$.

Theoreme 4.4. *Assume that $\mathcal{M}$ is a self-dual Hilbert $\mathcal{A}$-module. Then, the map $m_{\varphi,x} : \Sigma(\widehat{G}) \to \mathcal{M}, E \mapsto m_{\varphi,x}(E)$ is a regular vector measure.*

Proof. Take $y \in \mathcal{M}' = \mathcal{M}$. Set $E = \bigcup_{n=1}^{\infty} E_n$ where the E_n are pairwise disjoint elements of $\Sigma(\widehat{G})$. Then,

$$\begin{aligned} y(m_{\varphi,x}(E)) =& \langle m_\varphi(E)x, y\rangle = \Gamma_E(x,y) = \Gamma_{\bigcup_{n=1}^{\infty} E_n}(x,y) \\ =& \sum_{n=1}^{\infty} \Gamma_{E_n}(x,y) = \sum_{n=1}^{\infty} y(m_{\varphi,x}(E_n)) = y(\sum_{n=1}^{\infty} m_{\varphi,x}(E_n)) \end{aligned}$$

because $\Gamma_{(\cdot)}(x,y)$ is the linear combinaison of four σ-additive regular vector measures. Thus $m_{\varphi,x}(E) = \sum_{n=1}^{\infty} m_{\varphi,x}(E_n)$. Moreover, $m_{\varphi,x}$ is regular. □

One proves that the function $f : \widehat{G} \to \mathbb{C}$, $\gamma \mapsto (\gamma, g) := \gamma(g)$ is integrable with respect to $m_{\varphi,x}$ by following the similar proof in [7, page 63]. Finally, observe that

$$\begin{aligned} \langle \int_{\widehat{G}} (\gamma,g) dm_{\varphi,x}(\gamma), y\rangle &= \int_{\widehat{G}} (\gamma,g) d\langle m_{\varphi,x}(\gamma), y\rangle \\ &= \int_{\widehat{G}} (\gamma,g) d\langle m_{\varphi}(\gamma)x, y\rangle \\ &= \int_{\widehat{G}} (\gamma,g) d\mu_{x,y} \\ &= \langle \varphi(g)x, y\rangle. \end{aligned}$$

We summarize all the above computations to obtain the following Bochner-like theorem.

Theoreme 4.5. *Let $\mathcal{A}$ be a unital C^*-algebra. Let $\mathcal{M}$ be a self-dual Hilbert $\mathcal{A}$-module. Let $\varphi : G \to End^*_{\mathcal{A}}(\mathcal{M})$ be a positive definite function such that for $x \in \mathcal{M}$, $\varphi_x(0)$ is 0 or is invertible in $\mathcal{A}$ with inverse in the center of $\mathcal{A}$. Then, there exists a positive regular vector measure $m_\varphi : \Sigma(\widehat{G}) \to End^*_{\mathcal{A}}(\mathcal{M})$ such that*

$$\varphi(g)x = \int_{\widehat{G}} (\gamma,g) d(m_\varphi(\gamma)x), \ \forall g \in G. \tag{13}$$

References

[1] Y. Zhang, Bounded gaps between primes, Ann. Math. 179 (2014) 1121–1174. https://doi.org/10.4007/annals.2014.179.3.7.

[2] O. Avayu, E. Almeida, Y. Prior, T. Ellenbogen, Composite functional metasurfaces for multispectral achromatic optics, Nat. Commun. 8 (2017) 14992. https://doi.org/10.1038/ncomms14992.

[3] W. Rudin, Functional analysis, McGraw-Hill, New York, 1973.

[4] W. Averson, An invitation to C^*-algebras, Springer-Verlag, New York, 1976.

[5] W. H. Barker, The spherical Bochner theorem on semisimple Lie groups, J. Funct. Anal. 20 (1975) 179-207. https://doi.org/10.1016/0022-1236(75)90040-3.

[6] K. Chadan, Generalization of Bochner's theorem for functions of the positive type, J. Phys. A: Math. Theor. 40 (2007) 14395. https://doi.org/10.1088/1751-8113/40/48/006.

[7] P. L. Falb, On a theorem of Bochner, Publ. Math. l'I.H.É.S. 36 (1969) 59-67. https://doi.org/10.1007/BF02684597.

[8] Y. Kakihara, On a Hilbert module over an operator algebra and its applications to harmonic analysis, Kodai Math. J. 6 (1983) 289-300. https://doi.org/10.2996/kmj/1138036796

[9] E. Kaniuth, A. T.-M. Lau, Fourier and Fourier-Stieltjes algebras on locally compact groups, Mathematical Surveys and Monographs, 231 Amer. Math. Soc., 2018.

[10] E.C. Lance, Hilbert C^*-modules - a Toolkit for Operator Algebraists, London Math. Soc. Lecture Note Ser. 210, Cambridge University Press, Cambridge, England, 1995.

[11] V. M. Manuilov, E. V. Troitsky, Hilbert C^*-Modules, Translations of Mathematical Monographs, 226, Amer. Math. Soc., 2005.

[12] M. Rabaoui, A Bochner type theorem for inductive limits of Gelfand pairs, Ann. Inst. Fourier. 58 (5) (2008) 1551-1573. https://doi.org/10.5802/aif.2392.

[13] I. Toure, K. Kangni, Harmonic analysis on internally Gelfand pairs associated to groupoids, Int. J. Anal. Appl. 17 (2019) 928-939. https://doi.org/10.28924/2291-8639.

[14] K. D. Wodome, K. Kangni, V. S. K. Assiamoua, On an extension of Bochner's theorem, Far East J. Math. Sci. 67 (2012) 1-19. http://www.pphmj.com/abstract/6859.htm.