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THREE-POINT BOUNDARY VALUE PROBLEMS FOR IMPLICIT CAPUTO TEMPERED FRACTIONAL DIFFERENTIAL EQUATIONS IN b-METRIC SPACES

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Abstract. This paper is concerned with some existence and uniqueness results for a class of problems for nonlinear Caputo tempered implicit fractional differential equations in b-Metric spaces with three-point boundary conditions. The results are based on the $\omega - \varpi$-Geraghty type contraction, the $\digamma$-contraction and the fixed point theory. In addition, two illustrations are provided to highlight the plausibility of our findings.

1. Introduction

In recent years, fractional calculus has shown to be a highly effective tool for dealing with the complexity structures observed in a variety of academic areas. Its theory and application are extensive, and it is concerned with the expansion of integer order differentiation and integration of a function to non-integer order. We refer the reader to the monographs [1–3, 35] and the papers [29, 31]. Many articles and monographs have recently been published in which the authors investigated a wide variety of outcomes for different kinds of fractional differential equations, including with various sorts of conditions. One may see the papers [8, 9, 16, 23, 30], and the references therein.

In [14,15], Czerwik introduced the notion of b-metric. Following these early investigations, the existence of fixed points for various kinds of operators in b-metric spaces has been extensively investigated; see [5, 13, 17, 21, 28] for more details on the concept of b-metric and contractions.

Wardowski [34] established a new inequality employing auxiliary functions to prove the existence and uniqueness of a certain mapping in the context of standard metric space. This inequality is referred to as $\digamma$-contraction. For more details on the $\omega - \varpi$-Geraghty type contraction and the $\digamma$-contraction, we refer the reader to the recent papers [7, 10, 18, 22].

Tempered fractional calculus emerges as an extension of traditional fractional calculus. In prior research by Buschman [12], the initial definitions of fractional integration involving weak singular and exponential kernels were unveiled. See the papers [6, 24–27, 32, 33] and references therein for more details and results on the tempered fractional calculus.

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In [20], the authors considered the following conformable impulsive problem:

$$\begin{cases} \mathcal{T}^{\vartheta}_{\zeta_\jmath}\chi(\zeta) = \aleph\left(\zeta, \chi_\zeta, \mathcal{T}^{\vartheta}_\jmath \chi(\zeta)\right), \quad \zeta \in \Omega_\jmath; \jmath = 0, 1, \dots, \beta, \\ \Delta\chi|_{\zeta=\zeta_\jmath} = \Upsilon_\jmath(\chi_{\zeta_\jmath^-}), \quad \jmath = 1, 2, \dots, \beta, \\ \chi(\zeta) = \mu(\zeta), \quad \zeta \in (-\infty, \varkappa], \end{cases}$$

where $0 \leq \varkappa = \zeta_0 < \zeta_1 < \cdots < \zeta_\beta < \zeta_{\beta+1} = \bar{\varkappa} < \infty$, $\mathcal{T}^{\vartheta}_{\zeta_\jmath}\chi(\zeta)$ is the conformable fractional derivative of order $0 < \vartheta < 1$, $\aleph : \Omega \times \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $\Omega := [\varkappa, \bar{\varkappa}]$, $\Omega_0 := [\varkappa, \zeta_1]$, $\Omega_\jmath := (\zeta_\jmath, \zeta_{\jmath+1}]; \jmath = 1, 2, \dots, \beta$, $\mu : (-\infty, \varkappa] \to \mathbb{R}$ and $\Upsilon_\jmath : \mathcal{Q} \to \mathbb{R}$ are given continuous functions, and $\mathcal{Q}$ is called a phase space.

In [11], the authors investigated the existence of solutions to the boundary value problem for fractional-order differential equations:

$${}^cD^r x(\xi) = f(\xi, x(\xi)),\ \xi \in J := [1, T],\ 0 < r \leq 1,$$

with fractional boundary condition:

$$\sigma x(1) + \beta x(T) = \hbar I^q x(\eta) + \delta, \quad q \in (0, 1].$$

where D^r is the Caputo-Hadamard fractional derivative, $0 < r < 1, 0 < q \leq 1$, and E is a Banach space space with norm $\|\cdot\|$, $f : J \times E \to E$ is given continuous function, $\sigma, \beta, \hbar$ are real constants, and $\eta \in (1, T), \delta \in E$. Existence results are established by applying the Mönch's fixed point theorem and the technique of measures of noncompactness.

In this paper, first we investigate the following class of Caputo tempered fractional differential equation:

$$(1) \qquad \begin{cases} \left({}^C_{\kappa_1}\mathfrak{D}^{\sigma,\hbar}_\xi \chi\right)(\xi) = \widetilde{\mho}\left(\xi, \chi(\xi), \left({}^C_{\kappa_1}\mathfrak{D}^{\sigma,\hbar}_\xi \chi\right)(\xi)\right);\ \xi \in \Xi := [\kappa_1, \kappa_2], \\ \imath\chi(\kappa_1) + \jmath\chi(\kappa_2) = \varsigma\chi(\eta) + \varrho, \end{cases}$$

where $0 < \sigma < 1$, $\hbar \geq 0$, ${}^C_{\kappa_1}\mathfrak{D}^{\sigma,\hbar}_\xi$ is the Caputo tempered fractional derivative, $\widetilde{\mho} : \Xi \times \mathbb{R} \times \mathbb{R}$ is a continuous function, $\kappa_1 < \eta < \kappa_2 < +\infty$, $\imath, \jmath, \varrho, \varsigma$ are real constants.

The study of implicit differential equations using the Caputo tempered fractional derivative in b-metric spaces is initiated in this paper. It is organized as follows: Section 2 introduces some preliminaries, definitions, lemmas and auxiliary results about the tempered fractional derivative. In section 3, we give some existence and uniqueness results for the problem (1) that are based on the $\omega - \varpi$-Geraghty type contraction, $\digamma$-contraction and the fixed point theory. Finally we present some examples to show the validity of our results.

2. Preliminaries

First, we give the definitions and the notations that we will use throughout this paper. We denote by $C(\Xi, \mathbb{R})$ the Banach space of all continuous functions from Ξ into $\mathbb{R}$, with the following norm

$$\|\chi\|_\infty = \sup_{\xi\in\Xi}\{|\chi(\xi)|\}.$$

As usual, $AC(\Xi,\mathbb{R})$ denotes the space of absolutely continuous functions from Ξ into $\mathbb{R}$. For any $n\in\mathbb{N}^*$, we denote by $AC^n(\Xi)$ the space defined by

$$AC^n(\Xi):=\left\{\widetilde{\mho}:\Xi\to\mathbb{R}:\frac{d^n}{d\xi^n}\widetilde{\mho}(\xi)\in AC(\Xi,\mathbb{R})\right\}.$$

Consider the space $X_b^p(\kappa_1,\kappa_2)$, $(b\in\mathbb{R},\ 1\le p\le\infty)$ of those real-valued Lebesgue measurable functions f on $[\kappa_1,\kappa_2]$ for which $\|\widetilde{\mho}\|_{X_b^p}<\infty$, where the norm is defined by:

$$\|\widetilde{\mho}\|_{X_b^p}=\left(\int_{\kappa_1}^{\kappa_2}|\xi^b\widetilde{\mho}(\xi)|^p\frac{d\xi}{\xi}\right)^{\frac{1}{p}},\quad (1\le p<\infty).$$

Definition 2.1 (The Riemann-Liouville tempered fractional integral [24,32,33]). *Suppose that the real function f is piecewise continuous on $[\kappa_1,\kappa_2]$ and $f\in X_b^p(\kappa_1,\kappa_2)$, $\hbar\ge 0$. Then, the Riemann-Liouville tempered fractional integral of order σ is defined by*

$$ {}_{\kappa_1}I_\xi^{\sigma,\hbar}\widetilde{\mho}(\xi)=e^{-\hbar\xi}\ {}_{\kappa_1}I_\xi^{\sigma}\left(e^{\hbar\xi}\widetilde{\mho}(\xi)\right)=\frac{1}{\Gamma(\sigma)}\int_{\kappa_1}^{\xi}\frac{e^{-\hbar(\xi-\nu)}\widetilde{\mho}(\nu)}{(\xi-\nu)^{1-\sigma}}d\nu, \tag{2}$$

where ${}_{\kappa_1}\mathcal{I}_\xi^{\sigma}$ denotes the Riemann-Liouville fractional integral [19], defined by

$$ {}_{\kappa_1}I_\xi^{\sigma}\widetilde{\mho}(\xi)=\frac{1}{\Gamma(\sigma)}\int_{\kappa_1}^{\xi}\frac{\widetilde{\mho}(\nu)}{(\xi-\nu)^{1-\sigma}}d\nu. \tag{3}$$

Obviously, the tempered fractional integral (2) reduces to the Riemann-Liouville fractional integral (3) if $\hbar=0$.

Definition 2.2 (The Riemann-Liouville tempered fractional derivative [24,32]). *For $n-1<\sigma<n$; $n\in\mathbb{N}$, $\hbar\ge 0$. The Riemann-Liouville tempered fractional derivative is defined by*

$$ {}_{\kappa_1}\mathfrak{D}_\xi^{\sigma,\hbar}\widetilde{\mho}(\xi)=e^{-\hbar\xi}{}_{\kappa_1}\mathfrak{D}_\xi^{\sigma}\left(e^{\hbar\xi}\widetilde{\mho}(\xi)\right)=\frac{e^{-\hbar\xi}}{\Gamma(n-\sigma)}\frac{d^n}{d\xi^n}\int_{\kappa_1}^{\xi}\frac{e^{\hbar\nu}f(\nu)}{(\xi-\nu)^{\sigma-n+1}}d\xi,$$

where ${}_{\kappa_1}\mathfrak{D}_\xi^{\sigma}\left(e^{\hbar\xi}\widetilde{\mho}(\xi)\right)$ denotes the Riemann-Liouville fractional derivative [19], given by

$$ {}_{\kappa_1}\mathfrak{D}_\xi^{\sigma}\left(e^{\hbar\xi}\widetilde{\mho}(\xi)\right)=\frac{d^n}{d\xi^n}\left({}_{\kappa_1}I_\xi^{n-\sigma}\left(e^{\hbar\xi}\widetilde{\mho}(\xi)\right)\right)=\frac{1}{\Gamma(n-\sigma)}\frac{d^n}{d\xi^n}\int_{\kappa_1}^{\xi}\frac{\left(e^{\hbar\nu}\widetilde{\mho}(\nu)\right)}{(\xi-\nu)^{\sigma-n+1}}d\nu.$$

Definition 2.3 (The Caputo tempered fractional derivative [24, 33]). *For $n-1<\sigma<n$; $n\in\mathbb{N}$, $\hbar\ge 0$. The Caputo tempered fractional derivative is defined as*

$$ {}_{\kappa_1}^{C}\mathfrak{D}_\xi^{\sigma,\hbar}\widetilde{\mho}(\xi)=e^{-\hbar\xi}\ {}_{\kappa_1}^{C}\mathfrak{D}_\xi^{\sigma}\left(e^{\hbar\xi}\widetilde{\mho}(\xi)\right)=\frac{e^{-\hbar\xi}}{\Gamma(n-\sigma)}\int_{\kappa_1}^{\xi}\frac{1}{(\xi-\nu)^{\sigma-n+1}}\frac{d^n\left(e^{\hbar\nu}\widetilde{\mho}(\nu)\right)}{d\nu^n}d\nu,$$

where ${}_{\kappa_1}^{C}\mathfrak{D}_\xi^{\sigma,\hbar}\left(e^{\hbar\xi}\widetilde{\mho}(\xi)\right)$ denotes the Caputo fractional derivative [19], given by

$$ {}_{\kappa_1}^{C}\mathfrak{D}_\xi^{\sigma}\left(e^{\hbar\xi}\widetilde{\mho}(\xi)\right)=\frac{1}{\Gamma(n-\sigma)}\int_{\kappa_1}^{\xi}\frac{1}{(\xi-\nu)^{\sigma-n+1}}\frac{d^n\left(e^{\hbar\nu}\widetilde{\mho}(\nu)\right)}{d\nu^n}d\nu.$$

Lemma 2.4 ([24]). *For a constant C,*

$$ {}_{\kappa_1}\mathfrak{D}_\xi^{\sigma,\hbar}C=Ce^{-\hbar\xi}{}_{\kappa_1}\mathfrak{D}_\xi^{\sigma}e^{\hbar\xi},\quad {}_{\kappa_1}^{C}\mathfrak{D}_\xi^{\sigma,\hbar}C=Ce^{-\hbar\xi}\ {}_{\kappa_1}^{C}\mathfrak{D}_\xi^{\sigma}e^{\hbar\xi}.$$

Obviously, ${}_{\kappa_1}\mathfrak{D}_\xi^{\sigma,\hbar}(C)\neq{}_{\kappa_1}^{C}\mathfrak{D}_\xi^{\sigma,\hbar}(C)$. And, ${}_{\kappa_1}^{C}\mathfrak{D}_\xi^{\sigma,\hbar}(C)$ is no longer equal to zero, being different from ${}_{\kappa_1}^{C}\mathfrak{D}_\xi^{\sigma}(C)\equiv 0$.

Lemma 2.5 ([24, 33]). *Let $f(\xi) \in AC^n[\kappa_1, \kappa_2]$, $\hbar \geq 0$ and $n-1 < \sigma < n$. Then the Caputo tempered fractional derivative and the Riemann-Liouville tempered fractional integral have the following composite properties:*

$$ {}_{\kappa_1}I_{\xi}^{\sigma,\hbar}\left[{}_{\kappa_1}^{C}\mathfrak{D}_{\xi}^{\sigma,\hbar}f(\xi)\right] = f(\xi) - \sum_{k=0}^{n-1} e^{-\hbar\xi}\frac{(\xi-\kappa_1)^k}{k!}\left[\left.\frac{d^k\left(e^{\hbar\xi}f(\xi)\right)}{d\xi^k}\right|_{\xi=\kappa_1}\right], $$

and

$$ {}_{\kappa_1}^{C}\mathfrak{D}_{\xi}^{\sigma,\hbar}\left[{}_{a}I_{\xi}^{\sigma,\hbar}f(\xi)\right] = f(\xi), \text{ for } \sigma \in (0,1). $$

Lemma 2.6. *Let $\mho \in L^1(\Xi)$ and $0 < \sigma \leq 1$, $\hbar \geq 0$, $\kappa_1 < \eta < \kappa_2 < +\infty$, $\imath, \jmath, \varrho, \varsigma$ are real constants such that $\imath + \jmath e^{-\hbar\kappa_2} + \varsigma e^{-\hbar\eta} \neq 0$. Then, the boundary value problem*

$$ \begin{cases} \left({}_{\kappa_1}^{C}\mathfrak{D}_{\xi}^{\sigma,\hbar}y\right)(\xi) = \mho(\xi);\ \xi \in \Xi := [\kappa_1, \kappa_2], \\ \imath\chi(\kappa_1) + \jmath\chi(\kappa_2) = \varsigma\chi(\eta) + \varrho, \end{cases} \tag{4} $$

has a unique solution given by

$$ \begin{aligned} \chi(\xi) &= c_1 + c_2\int_{\kappa_1}^{\eta} e^{-\hbar(\eta-\nu)}(\eta-\nu)^{(\sigma-1)}\mho(\nu)d\nu - c_3\int_{\kappa_1}^{\kappa_2} e^{-\hbar(\kappa_2-\nu)}(\kappa_2-\nu)^{(\sigma-1)}\mho(\nu)d\nu \\ &\quad + \frac{1}{\Gamma(\sigma)}\int_{\kappa_1}^{\xi} e^{-\hbar(\xi-\nu)}(\xi-\nu)^{(\sigma-1)}\mho(\nu)d\nu, \end{aligned} \tag{5} $$

where

$$ c_1 = \frac{\varrho e^{-\hbar\xi}}{\imath + \jmath e^{-\hbar\kappa_2} + \varsigma e^{-\hbar\eta}}, $$

$$ c_2 = \frac{\varsigma e^{-\hbar\xi}}{\Gamma(\sigma)(\imath + \jmath e^{-\hbar\kappa_2} + \varsigma e^{-\hbar\eta})}, $$

and

$$ c_3 = \frac{\jmath e^{-\hbar\xi}}{\Gamma(\sigma)(\imath + \jmath e^{-\hbar\kappa_2} + \varsigma e^{-\hbar\eta})}. $$

Proof. Applying the Riemann-Liouville tempered fractional integral of order σ to

$$ \left({}_{0}^{C}\mathfrak{D}_{\xi}^{\sigma,\hbar}y\right)(\xi) = \mho(\xi), $$

and by Lemma 2.5 and if $\xi \in \Xi$, we have

$$ \chi(\xi) - \chi(\kappa_1)e^{-\hbar\xi} = \frac{1}{\Gamma(\sigma)}\int_{\kappa_1}^{\xi} e^{-\hbar(\xi-\nu)}(\xi-\nu)^{(\sigma-1)}\mho(\nu)d\nu. $$

Hence, we get

$$ \chi(\xi) = \chi(\kappa_1)e^{-\hbar\xi} + \frac{1}{\Gamma(\sigma)}\int_{\kappa_1}^{\xi} e^{-\hbar(\xi-\nu)}(\xi-\nu)^{(\sigma-1)}\mho(\nu)d\nu. $$

Thus,

$$ \chi(\kappa_2) = \chi(\kappa_1)e^{-\hbar\kappa_2} + \frac{1}{\Gamma(\sigma)}\int_{\kappa_1}^{\kappa_2} e^{-\hbar(\kappa_2-\nu)}(\kappa_2-\nu)^{(\sigma-1)}\mho(\nu)d\nu, $$

and

$$ \chi(\eta) = \chi(\kappa_1)e^{-\hbar\eta} + \frac{1}{\Gamma(\sigma)}\int_{\kappa_1}^{\eta} e^{-\hbar(\eta-\nu)}(\eta-\nu)^{(\sigma-1)}\mho(\nu)d\nu. $$

From the mixed boundary condition $\imath\chi(\kappa_1) + \jmath\chi(\kappa_2) = \varsigma\chi(\eta) + \varrho$, we get

$$\imath\chi(\kappa_1) + \jmath\left(\chi(\kappa_1)e^{-\hbar\kappa_2} + \frac{1}{\Gamma(\sigma)}\int_{\kappa_1}^{\kappa_2} e^{-\hbar(\kappa_2-\nu)}(\kappa_2-\nu)^{(\sigma-1)}\mho(\nu)d\nu\right)$$
$$= \varsigma\left(\chi(\kappa_1)e^{-\hbar\eta} + \frac{1}{\Gamma(\sigma)}\int_{\kappa_1}^{\eta} e^{-\hbar(\eta-\nu)}(\eta-\nu)^{(\sigma-1)}\mho(\nu)d\nu\right) + \varrho.$$

Thus,

$$\chi(\kappa_1) = \frac{\frac{\varsigma}{\Gamma(\sigma)}\int_{\kappa_1}^{\eta} e^{-\hbar(\eta-\nu)}(\eta-\nu)^{(\sigma-1)}\mho(\nu)d\nu + \varrho - \frac{\jmath}{\Gamma(\sigma)}\int_{\kappa_1}^{\kappa_2} e^{-\hbar(\kappa_2-\nu)}(\kappa_2-\nu)^{(\sigma-1)}\mho(\nu)d\nu}{\imath + \jmath e^{-\hbar\kappa_2} + \varsigma e^{-\hbar\eta}}.$$

Hence, we obtain (5).

In contrast, we can simply demonstrate by Lemma 2.4 and Lemma 2.5 that if χ verifies equation (5), then it satisfied the problem (4). □

Definition 2.7 ([5]). *Let $\mathcal{P}$ be a set and $\varepsilon \geq 1$ be a given real number. A distance function $\gamma : \mathcal{P} \times \mathcal{P} \to [0, \infty)$ is a b-metric if the requirements that follow are met for all $\chi_1, \chi_2, \chi_3 \in \mathcal{P}$:*

(1) $\gamma(\chi_1, \chi_2) = 0$ *if and only if* $\chi_1 = \chi_2$,
(2) $\gamma(\chi_1, \chi_2) = \gamma(\chi_2, \chi_1)$,
(3) $\gamma(\chi_1, \chi_2) \leq \varepsilon[\gamma(\chi_1, \chi_3) + \gamma(\chi_3, \chi_2)]$ *(b-triangular inequality).*

Then, $(\mathcal{P}, \gamma, \varepsilon)$ is called a b-metric space with parameter ε.

Let $\overline{\mho}$ be the set of all increasing and continuous function $\varpi : [0, \infty) \to [0, \infty)$ satisfying the property: $\varpi(\varepsilon\chi) \leq \varepsilon\varpi(\chi) \leq \varepsilon\chi$, for $\varepsilon > 1$ and $\varpi(0) = 0$. We denote by Φ the family of all nondecreasing functions $\eta : [0, \infty) \to [0, \frac{1}{\varepsilon^2})$ for some $\varepsilon \geq 1$.

Definition 2.8 ([5]). *Let $(\mathcal{P}, \gamma, \varepsilon)$ be a b-metric space, $\mathfrak{S} : \mathcal{P} \to \mathcal{P}$ is said to be a generalized ω-ϖ-Geraghty mapping whenever there exists $\omega : \mathcal{P} \times \mathcal{P} \to [0, \infty)$ such that*

$$\omega(\chi_1, \chi_2)\varpi(\varepsilon^3\gamma(\mathfrak{S}(\chi_1), \mathfrak{S}(\chi_2)) \leq \eta(\varpi(\gamma(\chi_1, \chi_2)))\varpi(\gamma(\chi_1, \chi_2)),$$

for $\chi_1, \chi_2 \in \mathcal{P}$, where $\eta \in \Phi$.

Definition 2.9 ([5]). *Let $\mathcal{P}$ be a non empty set, $\mathfrak{S} : \mathcal{P} \to \mathcal{P}$ and $\omega : \mathcal{P} \times \mathcal{P} \to [0, \infty)$ be given mappings. The operator $\mathfrak{S}$ is orbital ω-admissible if for $\chi \in \mathcal{P}$, we have*

$$\omega(\chi, \mathfrak{S}(\chi)) \geq 1 \Rightarrow \omega(\mathfrak{S}(\chi), \mathfrak{S}^2(\chi)) \geq 1.$$

Definition 2.10 ([7]). *A mapping $\widehat{\Phi} : \mathcal{P} \to \mathcal{P}$ is said to be a generalized nonlinear $F-$contraction if there exist the functions $F : [0, \infty) \to \mathbb{R}$ and $\wp : [0, \infty) \to [0, \infty)$ such that for all $\chi, \Im \in \mathcal{P}$ such that $\widehat{\Phi}\chi \neq \widehat{\Phi}\Im$.*

$$\wp(\gamma(\chi, \Im)) + F(\overline{\varpi}\gamma(\widehat{\Phi}\chi, \widehat{\Phi}\Im)) \leq F(A^{\epsilon\gamma}(\chi, \Im)), \tag{6}$$

where $\overline{\varpi} > 1$, and

$$A^{\epsilon\gamma}(\chi, \Im) = \max\left\{\gamma(\chi, \Im), \gamma(\chi, \widehat{\Phi}\chi), \gamma(\Im, \widehat{\Phi}\Im), \frac{\beta}{2\epsilon}\left[\gamma(\Im, \widehat{\Phi}\chi) + \gamma(\chi, \widehat{\Phi}\Im)\right]\right\}, \beta \in [0, 1].$$

Theorem 2.11 ([4], Corollary 3.1). *Let $(\mathcal{P}, \gamma)$ be a complete b-metric space and $\widehat{\Phi} : \mathcal{P} \to \mathcal{P}$ be a generalized ω-ϖ-Geraghty mapping where* (a): $\widehat{\Phi}$ *is* ω*-admissible;*

(b): *there exists* $\chi_0 \in \mathcal{P}$ *where* $\omega(\chi_0, \widehat{\Phi}(\chi_0)) \geq 1$*;*

(c): *If* $(\chi_n)_{n\in N} \subset \mathcal{P}$ *with* $\chi_n \to \chi$ *and* $\omega(\chi_n, \chi_{n+1}) \geq 1$*, then* $\omega(\chi_n, \chi) \geq 1$*,*

Then $\widehat{\Phi}$ *has a fixed point. Moreover, if*

(d): *for all fixed points* χ, λ *of* $\widehat{\Phi}$*, either*

$$\omega(\chi, \lambda) \geq 1 \quad \text{or} \quad \omega(\lambda, \chi) \geq 1,$$

Then $\widehat{\Phi}$ *has a unique fixed point.*

Theorem 2.12 ([7], Theorem 12)**.** *Let* $(\mathcal{P}, \gamma, \epsilon)$ *be a complete b-metric space. A generalized nonlinear* $\digamma$*-contraction* $\widehat{\Phi}$ *has a fixed point if the following statements are true:*

(1): $\widehat{\Phi}$ *is increasing, that is, if* $\kappa_1 < \kappa_2$*, then* $\digamma(\kappa_1) < \digamma(\kappa_2)$*, for all* $\kappa_1, \kappa_2 \in [0, \infty)$*;*

(2): $\beta < 1$*;*

(3): $\frac{\varepsilon}{\sigma} < 1$*;*

(4): $\liminf_{\chi \to \xi^+} \wp(\chi) > 0$*, for any* $\xi \geq 0$.

3. Main Results

Let $(C(\Xi, \mathbb{R}), \gamma, 2)$ be the complete b-metric space with $\varepsilon = 2$, such that $\gamma : C(\Xi, \mathbb{R}) \times C(\Xi, \mathbb{R}) \to [0, \infty)$, is given by:

$$\gamma(\chi, \Im) = \|(\chi - \Im)^2\|_\infty := \sup_{\xi \in \Xi} |\chi(\xi) - \Im(\xi)|^2.$$

In this section, we establish some existence results for problem (1).

Definition 3.1. *By a solution of problem (1), we mean a function* $\chi \in C(\Xi, \mathbb{R})$ *satisfying the integral equation*

$$\chi(\xi) = c_1 + c_2 \int_{\kappa_1}^{\eta} e^{-\hbar(\eta-\nu)} (\eta - \nu)^{(\sigma-1)} \mho(\nu) d\nu - c_3 \int_{\kappa_1}^{\kappa_2} e^{-\hbar(\kappa_2-\nu)} (\kappa_2 - \nu)^{(\sigma-1)} \mho(\nu) d\nu$$
$$+ \frac{1}{\Gamma(\sigma)} \int_{\kappa_1}^{\xi} e^{-\hbar(\xi-\nu)} (\xi - \nu)^{(\sigma-1)} \mho(\nu) d\nu.$$

where $\mho \in C(\Xi, \mathbb{R})$ *such that* $\mho(\xi) = \widetilde{\mho}(\xi, \chi(\xi), \mho(\xi))$.

The hypotheses:

(H_1): There exist continuous functions $\overline{p} : \Xi \to [0, \infty)$ and $\overline{q} : \Xi \to (0, 1)$ where for $\chi, \Im, \chi_1, \Im_1 \in C(\Xi, \mathbb{R})$ and $\xi \in \Xi$, we have

$$|\widetilde{\mho}(\xi, \chi(\xi), \chi_1(\xi)) - \widetilde{\mho}(\xi, \Im(\xi), \Im_1(\xi))| \leq \overline{p}(\xi)|\chi(\xi) - \Im(\xi)| + \overline{q}(\xi)|\chi_1(\xi) - \Im_1(\xi)|,$$

with

$$\left\| c_2 \int_{\kappa_1}^{\eta} e^{-\hbar(\eta-\nu)} (\eta - \nu)^{(\sigma-1)} \frac{\overline{p}^*}{1 - \overline{q}*} d\nu \right\|_\infty^2$$
$$+ \left\| c_3 \int_{\kappa_1}^{\kappa_2} e^{-\hbar(\kappa_2-\nu)} (\kappa_2 - \nu)^{(\sigma-1)} \frac{\overline{p}^*}{1 - \overline{q}*} d\nu \right\|_\infty^2$$
$$+ \left\| \frac{1}{\Gamma(\sigma)} \int_{\kappa_1}^{\xi} e^{-\hbar(\xi-\nu)} (\xi - \nu)^{(\sigma-1)} \frac{\overline{p}^*}{1 - \overline{q}*} d\nu \right\|_\infty^2 \leq \varpi(\|(\chi - \Im)^2\|_\infty),$$

where $\overline{p}^* = \sup_{\xi \in \Xi} |\overline{p}(\xi)|$ and $\overline{q}^* = \sup_{\xi \in \Xi} |\overline{q}(\xi)|$.

(H_2): There exist $\varpi \in \overline{\mho}$ and $\overline{\hbar}_0 \in C(\Xi, \mathbb{R})$ and a function $\overline{\theta} : C(\Xi, \mathbb{R}) \times C(\Xi, \mathbb{R}) \to \mathbb{R}$, such that

$$\overline{\theta}\Bigg(\overline{\hbar}_0(\xi), c_1 + c_2 \int_{\kappa_1}^{\eta} e^{-\hbar(\eta-\nu)}(\eta-\nu)^{(\sigma-1)}\mho(\nu)d\nu - c_3 \int_{\kappa_1}^{\kappa_2} e^{-\hbar(\kappa_2-\nu)}(\kappa_2-\nu)^{(\sigma-1)}\mho(\nu)d\nu + \frac{1}{\Gamma(\sigma)}\int_{\kappa_1}^{\xi} e^{-\hbar(\xi-\nu)}(\xi-\nu)^{(\sigma-1)}\mho(\nu)d\nu\Bigg) \geq 0,$$

where $\mho \in C(\Xi, \mathbb{R})$ such that $\mho(\xi) = \widetilde{\mho}(\xi, \overline{\hbar}_0(\xi), \mho(\xi))$.

(H_3): For each $\xi \in \Xi$, and $\chi, \Im \in C(\Xi, \mathbb{R})$, we have

$$\overline{\theta}(\chi(\xi), \Im(\xi)) \geq 0$$

implies

$$\overline{\theta}\Bigg(c_1 + c_2 \int_{\kappa_1}^{\eta} e^{-\hbar(\eta-\nu)}(\eta-\nu)^{(\sigma-1)}\mho(\nu)d\nu - c_3 \int_{\kappa_1}^{\kappa_2} e^{-\hbar(\kappa_2-\nu)}(\kappa_2-\nu)^{(\sigma-1)}\mho(\nu)d\nu + \frac{1}{\Gamma(\sigma)}\int_{\kappa_1}^{\xi} e^{-\hbar(\xi-\nu)}(\xi-\nu)^{(\sigma-1)}\mho(\nu)d\nu, c_1 + c_2 \int_{\kappa_1}^{\eta} e^{-\hbar(\eta-\nu)}(\eta-\nu)^{(\sigma-1)}\Delta(\nu)d\nu - c_3 \int_{\kappa_1}^{\kappa_2} e^{-\hbar(\kappa_2-\nu)}(\kappa_2-\nu)^{(\sigma-1)}\Delta(\nu)d\nu + \frac{1}{\Gamma(\sigma)}\int_{\kappa_1}^{\xi} e^{-\hbar(\xi-\nu)}(\xi-\nu)^{(\sigma-1)}\Delta(\nu)d\nu\Bigg) \geq 0,$$

where $\mho, \Delta \in C(\Xi, \mathbb{R})$ such that

$$\mho(\xi) = \widetilde{\mho}(\xi, \chi(\xi), \mho(\xi))$$

and

$$\Delta(\xi) = \widetilde{\mho}(\xi, \Im(\xi), \Delta(\xi)).$$

(H_4): If $(\chi_n)_{n\in N} \subset C(\Xi, \mathbb{R})$ with $\chi_n \to \chi$ and $\overline{\theta}(\chi_n, \chi_{n+1}) \geq 1$, then

$$\overline{\theta}(\chi_n, \chi) \geq 1.$$

(H_5): For all fixed solutions χ, λ of problem (1)), either

$$\overline{\theta}(\chi(\xi), \lambda(\xi)) \geq 0,$$

or

$$\overline{\theta}(\lambda(\xi), \chi(\xi)) \geq 0.$$

Theorem 3.2. *Assume that the hypotheses (H_1)-(H_4) hold. Then, the problem (1) has a least one solution defined on Ξ. Moreover, if (H_5) holds, we get the uniqueness of the solution.*

Proof. Consider the operator $\mathcal{K} : C(\Xi, \mathbb{R}) \to C(\Xi, \mathbb{R})$ defined by:

$$(\mathcal{K}\chi)(\xi) = c_1 + c_2 \int_{\kappa_1}^{\eta} e^{-\hbar(\eta-\nu)}(\eta-\nu)^{(\sigma-1)}\mho(\nu)d\nu - c_3 \int_{\kappa_1}^{\kappa_2} e^{-\hbar(\kappa_2-\nu)}(\kappa_2-\nu)^{(\sigma-1)}\mho(\nu)d\nu + \frac{1}{\Gamma(\sigma)}\int_{\kappa_1}^{\xi} e^{-\hbar(\xi-\nu)}(\xi-\nu)^{(\sigma-1)}\mho(\nu)d\nu, \tag{7}$$

where $\mho \in C(\Xi,\mathbb{R})$ such that $\mho(\xi)=\widetilde{\mho}(\xi,\chi(\xi),\mho(\xi))$.
The function $\omega : C(\Xi,\mathbb{R})\times C(\Xi,\mathbb{R}) \to [0,\infty)$ is given by:

$$\begin{cases} \omega(\chi,\lambda)=1; & if\ \overline{\theta}(\chi(\xi),\lambda(\xi))\geq 0,\ \xi\in\Xi, \\ \omega(\chi,\lambda)=0; & eles. \end{cases}$$

First, we prove that $\mathcal{K}$ is a generalized ω-ϖ-Geraghty operator:
For any $\chi,\lambda\in C(\Xi,\mathbb{R})$.Then, for each $\xi\in\Xi$, we have

$$\begin{aligned} |(\mathcal{K}\chi)(\xi)-(\mathcal{K}\lambda)(\xi)| = {} & c_2\int_{\kappa_1}^{\eta} e^{-\hbar(\eta-\nu)}(\eta-\nu)^{(\sigma-1)}|\mho(\nu)-\Delta(\nu)|d\nu \\ & + c_3\int_{\kappa_1}^{\kappa_2} e^{-\hbar(\kappa_2-\nu)}(\kappa_2-\nu)^{(\sigma-1)}|\mho(\nu)-\Delta(\nu)|d\nu \\ & + \frac{1}{\Gamma(\sigma)}\int_{\kappa_1}^{\xi} e^{-\hbar(\xi-\nu)}(\xi-\nu)^{(\sigma-1)}|\mho(\nu)-\Delta(\nu)|d\nu, \end{aligned}$$

where $\mho,\ \Delta\in C(\Xi,\mathbb{R})$ such that

$$\mho(\xi)=\widetilde{\mho}(\xi,\chi(\xi),\mho(\xi)) \text{ and } \Delta(\xi)=\widetilde{\mho}(\xi,\lambda(\xi),\Delta(\xi)).$$

From (H_1) we have

$$\|\mho-\Delta\|_\infty \leq \frac{\overline{p}^*}{1-\overline{q}*}\|(\chi-\lambda)^2\|_\infty^{\frac{1}{2}}.$$

Next, we have

$$\begin{aligned} |(\mathcal{K}\chi)(\xi)-(\mathcal{K}\lambda)(\xi)| = {} & c_2\int_{\kappa_1}^{\eta} e^{-\hbar(\eta-\nu)}(\eta-\nu)^{(\sigma-1)}\frac{\overline{p}^*}{1-\overline{q}*}\|(\chi-\lambda)^2\|_\infty^{\frac{1}{2}}d\nu \\ & + c_3\int_{\kappa_1}^{\kappa_2} e^{-\hbar(\kappa_2-\nu)}(\kappa_2-\nu)^{(\sigma-1)}\frac{\overline{p}^*}{1-\overline{q}*}\|(\chi-\lambda)^2\|_\infty^{\frac{1}{2}}d\nu \\ & + \frac{1}{\Gamma(\sigma)}\int_{\kappa_1}^{\xi} e^{-\hbar(\xi-\nu)}(\xi-\nu)^{(\sigma-1)}\frac{\overline{p}^*}{1-\overline{q}*}\|(\chi-\lambda)^2\|_\infty^{\frac{1}{2}}d\nu. \end{aligned}$$

Thus,

$$\begin{aligned} \omega(\chi,\lambda)|(\mathcal{K}\chi)(\xi)-(\mathcal{K}\lambda)(\xi)|^2 \leq {} & \|(\chi-\lambda)^2\|_\infty\omega(\chi,\lambda) \\ & \times\left[\left\|c_2\int_{\kappa_1}^{\eta} e^{-\hbar(\eta-\nu)}(\eta-\nu)^{(\sigma-1)}\frac{\overline{p}^*}{1-\overline{q}*}d\nu\right\|_\infty^2\right. \\ & + \left\|c_3\int_{\kappa_1}^{\kappa_2} e^{-\hbar(\kappa_2-\nu)}(\kappa_2-\nu)^{(\sigma-1)}\frac{\overline{p}^*}{1-\overline{q}*}d\nu\right\|_\infty^2 \\ & \left.+ \left\|\frac{1}{\Gamma(\sigma)}\int_{\kappa_1}^{\xi} e^{-\hbar(\xi-\nu)}(\xi-\nu)^{(\sigma-1)}\frac{\overline{p}^*}{1-\overline{q}*}d\nu\right\|_\infty^2\right] \\ \leq {} & \|(\chi-\lambda)^2\|_\infty\varpi(\|(\chi-\lambda)^2\|_\infty). \end{aligned}$$

Hence,

$$\omega(\chi,\lambda)\varpi(2^3\gamma(\mathcal{K}(\chi),\mathcal{K}(\lambda))\leq\eta(\varpi(\gamma(\chi,\lambda))\varpi(\gamma(\chi,\lambda)),$$

where $\eta\in\Phi$, $\varpi\in\overline{\mho}$, with $\eta(\xi)=\frac{1}{8}\xi$, and $\varpi(\xi)=\xi$. So, $\mathcal{K}$ is generalized ω-ϖ-Geraghty operator.

Let $\chi,\lambda\in C(\Xi,\mathbb{R})$ such that

$$\omega(\chi,\lambda)\geq 1.$$

Thus, for each $\xi \in \Xi$, we have

$$\overline{\theta}(\chi(\xi), \lambda(\xi)) \geq 0.$$

This implies from (H_3) that

$$\overline{\theta}(\mathcal{K}\chi(\xi), \mathcal{K}\lambda(\xi)) \geq 0,$$

which gives

$$\omega(\mathcal{K}(\chi), \mathcal{K}(\lambda)) \geq 1.$$

Hence, $\mathcal{K}$ is a ω-admissible.

Now, by (H_2), there exist $\overline{\hbar}_0 \in C(\Xi, \mathbb{R})$ such that

$$\omega(\overline{\hbar}_0, \widehat{\Phi}(\overline{\hbar}_0)) \geq 1.$$

Thus, by (H_4), if $\left(\overline{\hbar}_n\right)_{n\in N} \subset \mathcal{P}$ with $\overline{\hbar}_n \to \overline{\hbar}$ and $\omega(\overline{\hbar}_n, \overline{\hbar}_{n+1}) \geq 1$, then

$$\omega(\overline{\hbar}_n, \overline{\hbar}) \geq 1.$$

From an application of Theorem 2.11, we conclude that $\mathcal{K}$ has a fixed point χ which is a solution of problem (1).

Moreover, (H_5) implies that if χ and λ are fixed points of $\mathcal{K}$, then either

$$\overline{\theta}(\chi, \lambda) \geq 0 \quad or \quad \overline{\theta}(\lambda, \chi) \geq 0.$$

This implies that either

$$\omega(\chi, \lambda) \geq 1 \quad or \quad \omega(\lambda, \chi) \geq 1,$$

Hence, problem (1) has a unique solution. □

Now, we prove an existence and uniqueness result by using the F-contraction fixed point theorem.

Theorem 3.3. *Assume there exist constants $1 > \omega' > 0$, $\omega'' > 0$, where $\overline{\varpi} = \omega''(1 - \omega') > \sqrt{2}$, such that for each $\chi, \Im, \chi_1, \Im_1 \in C(\Xi, \mathbb{R})$ and $\xi \in \Xi$, we have*

$$\begin{aligned} &|\widetilde{\mho}(\xi, \chi, \Im) - \widetilde{\mho}(\xi, \chi_1, \Im_1)| \\ &\leq \frac{\Gamma(\sigma+1)(\imath + \jmath e^{-\hbar\kappa_2} + \varsigma e^{-\hbar\eta})}{2\omega''(\kappa_2 - \kappa_1)^{\sigma}(2 + \imath + \jmath e^{-\hbar\kappa_2} + \varsigma e^{-\hbar\eta})\left[1 + \sup_{\xi\in\Xi}|\chi(\xi)| + \sup_{\xi\in\Xi}|\chi_1(\xi)|\right]}|\chi - \chi_1| \\ &\quad + \omega'|\Im - \Im_1|. \end{aligned} \tag{8}$$

Then, the problem (1) has a unique solution.

Proof. Let $\mathcal{K} : C(\Xi, \mathbb{R}) \to C(\Xi, \mathbb{R})$ defined as in (7), For any $\chi,\ \lambda \in C(\Xi, \mathbb{R})$. For each $\xi \in \Xi$ we have

$$\begin{aligned} |(\mathcal{K}\chi)(\xi) - (\mathcal{K}\lambda)(\xi)|^2 = \Bigg\{ & c_2 \int_{\kappa_1}^{\eta} e^{-\hbar(\eta-\nu)}(\eta - \nu)^{(\sigma-1)}|\mho(\nu) - \Delta(\nu)|d\nu \\ & + c_3 \int_{\kappa_1}^{\kappa_2} e^{-\hbar(\kappa_2-\nu)}(\kappa_2 - \nu)^{(\sigma-1)}|\mho(\nu) - \Delta(\nu)|d\nu \\ & + \frac{1}{\Gamma(\sigma)} \int_{\kappa_1}^{\xi} e^{-\hbar(\xi-\nu)}(\xi - \nu)^{(\sigma-1)}|\mho(\nu) - \Delta(\nu)|d\nu \Bigg\}^2, \end{aligned}$$

where $\mho,\ \Delta \in C(\Xi,\mathbb{R})$ such that

$$\mho(\xi)=\widetilde{\mho}(\xi,\chi(\xi),\mho(\xi)) \text{ and } \Delta(\xi)=\widetilde{\mho}(\xi,\chi(\xi),\Delta(\xi)).$$

Since, for each $\xi\in\Xi$, we have

$$\|\mho-\Delta\|_\infty \leq \frac{\Gamma(\sigma+1)(\imath+\jmath e^{-\hbar\kappa_2}+\varsigma e^{-\hbar\eta})}{2(1-\omega')\omega''(\kappa_2-\kappa_1)^\sigma(2+\imath+\jmath e^{-\hbar\kappa_2}+\varsigma e^{-\hbar\eta})} \times \frac{|\chi(\xi)-\lambda(\xi)|}{1+\sup_{\xi\in\Xi}|\varkappa_2(\xi)|+\sup_{\xi\in\Xi}|\varkappa_2'(\xi)|}.$$

Then, we get

$$\begin{aligned}
&|(\mathcal{K}\chi)(\xi)-(\mathcal{K}\lambda)(\xi)|^2\\
&\leq \left\{\frac{1}{\varpi\left[2+2\sup_{\xi\in\Xi}|\chi(\xi)|+2\sup_{\xi\in\Xi}|\lambda(\xi)|\right]}|\chi(\xi)-\lambda(\xi)|\right\}^2\\
&\leq \frac{1}{\varpi^2\left[2+2\sup_{\xi\in\Xi}|\chi(\xi)|+2\sup_{\xi\in\Xi}|\lambda(\xi)|\right]^2}\left\{\sqrt{|\chi(\xi)-\lambda(\xi)|^2}\right\}^2\\
&\leq \frac{1}{\varpi^2\left[2+2\sup_{\xi\in\Xi}|\chi(\xi)|+2\sup_{\xi\in\Xi}|\lambda(\xi)|\right]^2}\left\{\sqrt{\sup_{\xi\in\Xi}|\chi(\xi)-\lambda(\xi)|^2}\right\}^2\\
&\leq \frac{1}{\varpi^2\left[2+\sup_{\xi\in\Xi}|\chi(\xi)-\lambda(\xi)|^2\right]}\left\{\sqrt{\sup_{\xi\in\Xi}|\chi(\xi)-\lambda(\xi)|^2}\right\}^2.
\end{aligned}$$

Consequently, we get

$$\varpi^2\gamma(\mathcal{K}\chi,\mathcal{K}\lambda)\leq\frac{\gamma(\chi,\lambda)}{2+\gamma(\chi,\lambda)}.$$

Now, applying natural logarithm on the previous inequality, we obtain

$$\ln(2+\gamma(\chi,\lambda))+\ln(\varpi^2\gamma(\mathcal{K}\chi,\mathcal{K}\lambda))\leq\ln(\gamma(\chi,\lambda))\leq\ln(A^{\epsilon\gamma}(\chi,\lambda)),$$

where

$$A^{\epsilon\gamma}(\chi,\lambda)=\max\left\{\gamma(\chi,\lambda),\gamma(\chi,\mathcal{K}\chi),\gamma(\lambda,\mathcal{K}\lambda),\frac{\beta}{2\epsilon}\left[\gamma(\lambda,\mathcal{K}\chi)+\gamma(\chi,\mathcal{K}\lambda)\right]\right\},\beta<\frac{1}{2}.$$

If we choose $\digamma(\xi)=\ln(\xi)$ and $\wp(\xi)=\ln(2+\xi)$, we see that all the conditions of Theorem 2.12 are satisfied. So, $\mathcal{K}$ has a unique fixed point which is the solution of our problem. □

4. Some Examples

Example 4.1. *Consider the following problem:*

(9)
$$\begin{cases}\left({}_0^C\mathfrak{D}_\xi^{\frac{1}{2},\hbar}\chi\right)(\xi)=\dfrac{\sin(\xi)}{8(1+|\chi(\xi)|)}+\dfrac{e^{-\xi}}{2\left(1+\left|\left({}_0^C\mathfrak{D}_\xi^{\frac{1}{2},\hbar}\chi\right)(\xi)\right|\right)},\\[2ex] \chi(0)+\chi(1)=\chi(\frac{1}{2}).\end{cases}$$

Set

$$\widetilde{\mho}(\xi,\chi,\Im)=\frac{\sin(\xi)}{8(1+|\chi|)}+\frac{e^{-\xi}}{2(1+|\Im|)},$$

where $\xi\in\Xi:=[0,1],\ \chi,\Im\in\mathbb{R}$.

Let $(C(\Xi,\mathbb{R}),\gamma,2)$ be the complete b-metric space with $\varepsilon=2$, such that $\gamma: C(\Xi,\mathbb{R})\times C(\Xi,\mathbb{R})\to[0,\infty)$, is given by:

$$\gamma(\chi,\Im)=\|(\chi-\Im)^2\|_\infty:=\sup_{\xi\in\Xi}|\chi(\xi)-\Im(\xi)|^2.$$

For any $\chi,\bar{\chi},\Im,\bar{\Im}\in C(\Xi,\mathbb{R})$, and $\xi\in\Xi$. If $|\chi(\xi)|\le|\Im(\xi)|$, then

$$|\widetilde{\mho}(\xi,\chi,\bar{\chi})-\widetilde{\mho}(\xi,\Im,\bar{\Im})|\le\frac{\sin(\xi)\|\chi-\Im\|_\infty}{8}+\frac{e^{-\xi}\|\bar{\chi}-\bar{\Im}\|_\infty}{2},$$

Thus, hypothesis (H_2) is satisfied with

$$\overline{p}(\xi)=\frac{\sin(\xi)}{8},\quad and\quad \overline{q}(\xi)=\frac{e^{-\xi}}{2}.$$

Define the functions $\eta(\xi)=\frac{1}{8}\xi$, $\varpi(\xi)=\xi$, $\sigma: C(\Xi,\mathbb{R})\times C(\Xi,\mathbb{R})\to\mathbb{R}_+^*$ with

$$\begin{cases}\omega(\chi,\Im)=1;\ if\ \overline{\theta}(\chi(\xi),\Im(\xi))\ge 0,\ \xi\in\Xi,\\ \omega(\chi,\Im)=0;\ else,\end{cases}$$

and $\overline{\theta}: C(\Xi,\mathbb{R})\times C(\Xi,\mathbb{R})\to\mathbb{R}$ with $\overline{\theta}(\chi,\Im)=\|\chi-\Im\|_\infty$.
Hypothesis (H_2) is satisfied with $\overline{\hbar}_0(\theta)=\chi(0)$. Also, (H_3) holds from the definition of the function $\overline{\theta}$. calculations indicate that all of the requirements of Theorem 3.2 are met. As a result, we obtain the existence of solutions and the uniqueness for problem (9).

Example 4.2. *Next, consider the following problem:*

(10)
$$\begin{cases}\left({}_0^C\mathfrak{D}_\xi^{\frac{1}{2},\hbar}\chi\right)(\xi)=\frac{\Gamma(\frac{3}{2})(1+\frac{1}{e}+e^{-\frac{1}{4}})}{4(3+\frac{1}{e}+e^{-\frac{1}{4}})\left(1+\|\chi\|_\infty+\left\|\left({}_0^C\mathfrak{D}_\xi^{\frac{1}{2},\hbar}\chi\right)\right\|_\infty\right)}\\ \qquad+\frac{1}{20\left(1+\left|\left({}_0^C\mathfrak{D}_\xi^{\frac{1}{2},\hbar}\chi\right)(\xi)\right|\right)};\ \theta\in\Xi:=[0,1],\\ \chi(0)+\chi(1)=\chi(\frac{1}{4}).\end{cases}$$

Set

$$\widetilde{\mho}(\xi,\chi(\xi),\Im(\xi))=\frac{\Gamma(\frac{3}{2})(1+\frac{1}{e}+e^{-\frac{1}{4}})|\chi(\xi)|}{4(3+\frac{1}{e}+e^{-\frac{1}{4}})(1+\|\chi\|_\infty+\|\Im\|_\infty)}+\frac{1}{20(1+|\Im(\xi)|)},$$

where $\xi\in\Xi$, $\chi,\Im\in C(\Xi,\mathbb{R})$.

Let $(C(\Xi,\mathbb{R}),\gamma,2)$ be the complete b-metric space with $\varepsilon=2$, such that $\gamma: C(\Xi,\mathbb{R})\times C(\Xi,\mathbb{R})\to[0,\infty)$, is given by:

$$\gamma(\chi,\Im)=\|(\chi-\Im)^2\|_\infty:=\sup_{\xi\in\Xi}|\chi(\xi)-\Im(\xi)|^2.$$

For any $\chi,\bar{\chi},\Im,\bar{\Im}\in C(\Xi,\mathbb{R})$, and $\xi\in\Xi$. If $|\chi(\xi)|\le|\Im(\xi)|$, then

$$|\widetilde{\mho}(\xi,\chi(\xi),\bar{\chi}(\xi))-\widetilde{\mho}(\xi,\Im(\xi),\bar{\Im}(\xi))|\le\frac{\Gamma(\frac{3}{2})(1+\frac{1}{e}+e^{-\frac{1}{4}})}{4(3+\frac{1}{e}+e^{-\frac{1}{4}})(1+\|\chi\|_\infty+\|\Im\|_\infty)}|\chi(\xi)-\Im(\xi)|$$
$$+\frac{1}{20}|\bar{\chi}(\xi)-\bar{\Im}(\xi)|.$$

Then, hypothesis (8) is satisfied with

$$\omega''=2,\ \omega'=\frac{1}{20}\ and\ \overline{\varpi}=\frac{19}{10}>\sqrt{2}.$$

Since all requirements of Theorem 3.3 are verified, we conclude the existence the uniqueness of solutions for problem (10).

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