DOI: 10.28919/ejma.2023.3.15

Eur. J. Math. Appl. (2023)3:15
URL: http://ejma.euap.org

SECOND HANKEL DETERMINANT FOR A CERTAIN FUNCTION CLASS OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH SĂLĂGEAN DERIVATIVE OPERATOR

EZEKIEL ABIODUN OYEKAN[1,*], AYOTUNDE OLAJIDE LASODE[2], OLUWASEGUN ADESHINA OLUKOYA[1], PETER OLUWAFEMI ADEPOJU[1]

Abstract. In this work, we investigate the bounds on the fourth initial coefficient $|a_4|$ and the second Hankel determinant $|a_2a_4 - a_3^2|$ for a class of analytic and bi-univalent functions defined in the open unit disk. The class is connected with the Sălăgean differential operator and involves a linear combination of the classes of strongly bi-starlike functions and strongly bi-convex functions. It was also shown that the investigated class generalized many known classes by varying the parameters in the definition of the class.

1. Introduction and Preliminaries

For the unit disk defined by $\Omega = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, let $\mathfrak{A}$ denote the class of regular functions defined in Ω. Also let $\mathfrak{S}$ denote the subclass of $\mathfrak{A}$ of regular and univalent functions in Ω such that the functions in it are normalized by the conditions: $f(0) = f'(0) - 1 = 0$. Functions in $\mathfrak{S}$ are therefore expressible as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \Omega. \tag{1}$$

The well-known Koebe $\frac{1}{4}$-theorem, see [1], states that the image domain of every function $f \in \mathfrak{S}$ encloses a disk of radius $\frac{1}{4}$. The consequence of this theorem is that every single function $f \in \mathfrak{S}$ has an inverse function f^{-1} defined by

$$f^{-1}(f(z)) = z, \quad z \in \Omega$$

and

$$f(f^{-1}(w)) = w, \quad w : |w| < r_0(f),\ r_0(f) \geqslant \frac{1}{4}$$

hence

$$\mathfrak{F}(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{2}$$

[1]Department of Mathematical Sciences, Olusegun Agagu University of Science and Technology, P.M.B. 353, Okitipupa, Nigeria

[2]Department of Mathematics, University of Ilorin, P.M.B. 1515, Ilorin, Nigeria

[*]Corresponding author

E-mail address: ea.oyekan@oaustech.edu.ng, segunolukoya2012@yahoo.com, peter.o.adepoju@gmail.com, lasode_ayo@yahoo.com.

Key words and phrases. analytic function; bi-univalent function; strongly bi-starlike function; strongly bi-convex function; coefficient estimate; Hankel determinant; Sălăgean differential operator.

Received 30/06/2023.

A function $f \in \mathfrak{A}$ is said to be *bi-univalent* in Ω if both f and $\mathfrak{F}$ are univalent in Ω. Let Σ denote the class of analytic and bi-univalent functions in Ω. History has it that Lewin [7] was the first author to define and study the class of bi-univalent functions. The author reported that every bi-univalent function has upper bound $|a_2| < 1.51$. Since its introduction, many authors have studied several properties of some subclasses of Σ. One can explore the works in [3–5,10,15,18,19,24] for more properties and definitions of some subclasses of Σ. We observe that class Σ is non-empty since we have some examples like $f(z) = z$, $f(z) = -\log(1-z)$, $f(z) = z/(1-z)$ and $f(z) = \log[(1+z)/(1-z)]^{1/2}$ in Σ.

Denote by $\mathfrak{H}_{j,n}(f)$ is the j^{th} Hankel determinant whose elements are the coefficients of function $f \in \mathfrak{A}$. In 1966, Pommerenke [22] reported the Hankel determinant $\mathfrak{H}_{j,n}(f)$ as

$$\mathfrak{H}_{j,n}(f) = \begin{vmatrix} 1 & a_{n+1} & \dots & a_{n+j-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+j} \\ \vdots & \vdots & & \vdots \\ a_{n+j-1} & a_{n+j} & \dots & a_{n+2j-2} \end{vmatrix}$$

for $j, n \in \mathbb{N}$. In fact, the Hankel determinants

$$\mathfrak{H}_{2,1}(f) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = (a_3 - a_2^2) \quad \text{and} \quad \mathfrak{H}_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = (a_2 a_4 - a_3^2) \tag{3}$$

have been studied by several authors for various subclasses of Σ. Observe that $|\mathfrak{H}_{2,1}(f)|$ is a special case of the well-known Fekete-Szegö functional while $|\mathfrak{H}_{2,2}(f)|$ is called the second Hankel determinant. For instance see [6,11,12,14,17] for some results and applications.

Definition 1.1. *A function $f \in \Sigma$ is valid to be in the class $\Sigma(k, \lambda, \alpha)$ if it satisfies the conditions*

$$\left|\arg\left(\frac{\mathfrak{D}^{k+1} f(z)}{(1-\lambda)\mathfrak{D}^k f(z) + \lambda \mathfrak{D}^{k+1} f(z)}\right)\right| < \frac{\alpha\pi}{2} \tag{4}$$

and

$$\left|\arg\left(\frac{\mathfrak{D}^{k+1} \mathfrak{F}(w)}{(1-\lambda)\mathfrak{D}^k \mathfrak{F}(w) + \lambda \mathfrak{D}^{k+1} \mathfrak{F}(w)}\right)\right| < \frac{\alpha\pi}{2} \tag{5}$$

for $0 < \alpha \leqslant 1$, $0 \leqslant \lambda < 1$; $z, w \in \Omega$, $\mathfrak{F}(w)$ is define by (2) and $\mathfrak{D}^k f(z)$ is from the Sălăgean differential operator $\mathfrak{D}^k : \mathfrak{A} \to \mathfrak{A}$ (see [13, 16, 20, 21, 23]) defined by

$$\mathfrak{D}^0 f(z) = f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

$$\mathfrak{D}^1 f(z) = z f'(z) = z + \sum_{n=2}^{\infty} n a_n z^n$$

$$\mathfrak{D}^2 f(z) = z(\mathfrak{D}^1 f(z))' = z + \sum_{n=2}^{\infty} n^2 a_n z^n$$

which is generally expressed as

$$\mathfrak{D}^k f(z) = z(\mathfrak{D}^{k-1} f(z))' = z + \sum_{n=2}^{\infty} n^k a_n z^n$$

$z \in \Omega$ and $k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$.

Remark 1.2. Some subclasses of the class $\Sigma(k,\lambda,\alpha)$ are given as follows.

(1) $\Sigma(k,0,\alpha)=\mathfrak{S}^{\star}_{\Sigma}(k,\alpha)$, the class of strongly bi-starlike functions of order α generalized by the Sălăgean differential operator that satisfy the conditions

$$\left|\arg\left(\frac{\mathfrak{D}^{k+1}f(z)}{\mathfrak{D}^k f(z)}\right)\right|<\frac{\alpha\pi}{2}\qquad\text{and}\qquad\left|\arg\left(\frac{\mathfrak{D}^{k+1}\mathfrak{F}(w)}{\mathfrak{D}^k\mathfrak{F}(w)}\right)\right|<\frac{\alpha\pi}{2}.$$

(2) $\Sigma(0,0,\alpha)=\mathfrak{S}^{\star}_{\Sigma}(\alpha)$, the class of strongly bi-starlike functions of order α that satisfy the conditions

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\alpha\pi}{2}\qquad\text{and}\qquad\left|\arg\left(\frac{w\mathfrak{F}'(w)}{\mathfrak{F}(w)}\right)\right|<\frac{\alpha\pi}{2}.$$

(3) $\Sigma(1,0,\alpha)=\mathfrak{C}_{\Sigma}(\alpha)$, the class of strongly bi-convex functions of order α that satisfy the conditions

$$\left|\arg\left(\frac{(zf'(z))'}{f'(z)}\right)\right|<\frac{\alpha\pi}{2}\qquad\text{and}\qquad\left|\arg\left(\frac{(w\mathfrak{F}'(w))'}{\mathfrak{F}'(w)}\right)\right|<\frac{\alpha\pi}{2}.$$

(4) $\Sigma(0,\lambda,\alpha)=\Sigma(\lambda,\alpha)$, the class of functions studied in [9] that satisfy the conditions

$$\left|\arg\left(\frac{zf'(z)}{(1-\lambda)f(z)+\lambda f'(z)}\right)\right|<\frac{\alpha\pi}{2}\qquad\text{and}\qquad\left|\arg\left(\frac{w\mathfrak{F}'(w)}{(1-\lambda)\mathfrak{F}(w)+\lambda\mathfrak{F}'(w)}\right)\right|<\frac{\alpha\pi}{2}.$$

2. Needed Lemmas

We shall need the following lemmas to establish our result. Firstly, let $\mathfrak{P}$ be the class of analytic functions with positive real parts in Ω, then the following lemmas hold for

$$p(z)=1+p_1z+p_2z^2+\cdots,\ q(z)=1+q_1w+q_2w^2+\cdots\in\mathfrak{P}. \tag{6}$$

Lemma 2.1 ([1]). *If $p\in\mathcal{P}$, then $|p_n|\leqslant 2,\ \forall n\in\mathbb{N}$.*

Lemma 2.2 ([8]). *If $p\in\mathcal{P}$, then*

$$2p_2=p_1^2+(4-p_1^2)x \tag{7}$$

and

$$4p_3=p_1^3+2(4-p_1^2)p_1x-(4-p_1^2)p_1x^2+2(4-p_1^2)\left(1-|x|^2\right)z \tag{8}$$

for some x, z with $|x|,|z|\leqslant 1$. In view of (6) and (7), it is easily seen that

$$\left.\begin{array}{l}2p_2=p_1^2+x(4-p_1^2)\\ 2q_2=q_1^2+y(4-q_1^2)\end{array}\right\}\implies\left\{\begin{array}{l}2(p_2-q_2)=(4-p_1^2)(x-y)\\ 2(p_2+q_2)=2p_1^2+(4-p_1^2)(x+y)\end{array}\right. \tag{9}$$

and from (6) and (8) we have

$$\left.\begin{array}{l}4p_3=p_1^3+2(4-p_1^2)p_1x-(4-p_1^2)p_1x^2+2(4-p_1^2)\left(1-|x|^2\right)z\\ 4q_3=q_1^3+2(4-q_1^2)q_1y-(4-q_1^2)q_1y^2+2(4-q_1^2)\left(1-|y|^2\right)w\end{array}\right\} \tag{10}$$
$$\implies 4(p_3-q_3)=2p_1^3+2p_1(4-p_1^2)(x+y)-4p_1(4-p_1^2)(x^2+y^2)$$
$$+2(4-p_1^2)[(1-|x|^2)z-(1-|y|^2)w]$$

where $p_1=-q_1$, for some w,x,y,z with $|w|,|x|,|y|,|z|\leqslant 1$ and $|p_1|,|q_1|\in[0,2]$.

Lemma 2.3. *If $f \in \Sigma(k,\lambda,\alpha)$, then*

$$a_2^2 = \frac{\alpha^2(p_2+q_2)}{3^k 4\alpha(1-\lambda)+2^{2k}[2\alpha(\lambda^2-1)-(\alpha-1)(1-\lambda)^2]} \tag{11}$$

$$a_3 = \frac{\alpha(p_2-q_2)}{3^k 4(1-\lambda)} + \frac{\alpha^2(p_1^2+q_1^2)}{2^{2k+1}(1-\lambda)^2} \tag{12}$$

$$\begin{aligned} a_4 = &\left\{\frac{\alpha^2(1+\alpha)(1+\lambda)}{2^{2k+1}\cdot 3(1-\lambda)^2} + \frac{3^{k-1}\alpha^3(1+2\lambda)}{2^{4k+1}(1+\lambda)^2} + \frac{7\alpha(1-\alpha)(2-\alpha)}{9\cdot 2^{2k+2}(1-\lambda)}\right\} p_1^3 \\ &+ \left\{\frac{\alpha^2(1+\lambda)}{2^{2k+1}\cdot 3(1-\lambda)^2} + \frac{5\cdot 2^{k+3}\alpha^2}{3^k(1-\lambda)^2} + \frac{\alpha^2(1+2\lambda)}{2^{2k+3}\cdot 3(1-\lambda)^2} + \frac{3^{k-1}\alpha(1+2\lambda)}{2^{2k}(1-\lambda)}\right\} p_1(p_2-q_2) \\ &- \frac{\alpha(1-\alpha)}{2^{2k+1}\cdot 3(1-\lambda)} p_1(p_2+q_2) + \frac{\alpha}{2^{2k+1}\cdot 3(1-\lambda)}(p_3-q_3) \end{aligned} \tag{13}$$

The results in (11) and (12) were earlier given by Jothibasu [2] while we proof for (13).

Proof. If $f \in \Sigma$ and $\mathfrak{F}(w) = f^{-1}(w)$, then there are analytic functions $p(z)$ and $q(w)$ ($p(0) = 0 = q(0)$, $|p(z)|, |q(w)| < 1$) with $w, z \in \Omega$, so that (4) and (5) transforms into

$$\mathfrak{D}^{k+1} f(z) = [(1-\lambda)\mathfrak{D}^k f(z) + \lambda \mathfrak{D}^{k+1} f(z)][p(z)]^\alpha, \tag{14}$$

and

$$\mathfrak{D}^{k+1}\mathfrak{F}(w) = [(1-\lambda)\mathfrak{D}^k \mathfrak{F}(w) + \lambda \mathfrak{D}^{k+1}\mathfrak{F}(w)][q(w)]^\alpha. \tag{15}$$

If we equate the coefficients in (14) and (15), then we get

$$2^k(1-\lambda)a_2 = \alpha p_1, \tag{16}$$

$$2^{2k}(\lambda^2-1)a_2^2 + 3^k(2-2\lambda)a_3 = \frac{1}{2}[\alpha(\alpha-1)p_1^2 + 2\alpha p_2], \tag{17}$$

$$\begin{aligned} 2^k\alpha(1+\lambda)\left[\frac{1}{2}p_1^2(1+\alpha) - p_2\right] a_2 - 3^k\alpha(1+2\lambda)p_1 a_3 + 3\cdot 4^k(1-\lambda)a_4 \\ = \frac{1}{6}[6\alpha p_3 + 6\alpha(\alpha-1)p_1p_2 + \alpha(\alpha-1)(\alpha-2)p_1^3], \end{aligned} \tag{18}$$

$$-2^k(1-\lambda)a_2 = \alpha q_1,$$

$$2(1-\lambda)(2a_2^2 - a_3)3^k + (\lambda^2-1)2^{2k}a_2^2 = \frac{1}{2}[\alpha(\alpha-1)q_1^2 + 2\alpha q_2],$$

$$\begin{aligned} &-(1-\lambda)3\cdot 4^k a_4 + (1-\lambda)15\cdot 4^k a_2a_3 - (1-\lambda)15\cdot 4^k a_2^3 \\ &+ \frac{1}{2}[2\alpha q_2 - \alpha q_1^2 + (\alpha q_1)^2](1+\lambda)2^k a_2 + (1+2\lambda)3^k\alpha q_1q_3 - (1+2\lambda)2\cdot 3^k\alpha q_1q_3 - (1+2\lambda)2\cdot 3^k\alpha q_1q_2 \\ &= \frac{1}{6}[6\alpha q_3 + 6\alpha(\alpha-1)q_1q_2 + \alpha(\alpha-1)(\alpha-2)q_1^3]. \end{aligned} \tag{19}$$

Subtracting (18) from (19), using (16), (12) and the fact that $p_1 = -q_1$ give (13). □

3. Main Result

Theorem 3.1. *If $f \in \Sigma(k,\lambda,\alpha)$, then*

$$|\mathfrak{H}_{2,2}(f)| \leqslant \begin{cases} \varphi(2) & \textit{for} \quad \mu_1(k,\alpha,\lambda) \geqslant 0 \textit{ and } \mu_2(k,\alpha,\lambda) \geqslant 0, \\ \max\left\{\frac{2^3\alpha^2}{3^{2k}(1-\lambda)^2}\ ,\varphi(2)\right\} & \textit{for} \quad \mu_1(k,\alpha,\lambda) > 0 \textit{ and } \mu_2(k,\alpha,\lambda) < 0, \\ \frac{2^3\alpha^2}{3^{2k}(1-\lambda)^2} & \textit{for} \quad \mu_1(k,\alpha,\lambda) \leqslant 0 \textit{ and } \mu_2(k,\alpha,\lambda) \leqslant 0, \\ \max\{\frac{2^3\alpha^2}{3^{2k}(1-\lambda)^2},\varphi(2)\} & \textit{for} \quad \mu_1(k,\alpha,\lambda) < 0 \textit{ and } \mu_2(k,\alpha,\lambda) > 0, \end{cases}$$

where

$$\mu_1(k,\alpha,\lambda) = \left[\frac{\alpha^3(1+\alpha)(1+\lambda)}{2^{3k+1}\cdot 3(1-\lambda)^3} + \frac{3^{k-1}\alpha^4(1+2\lambda)}{2^{5k+1}(1+\lambda)^2(1-\lambda)} + \frac{7\alpha^2(1-\alpha)(2-\alpha)}{3^2\cdot 2^{3k+2}(1-\lambda)^2} - \frac{\alpha^4}{2^{4k}(1-\lambda)^4} - \frac{\alpha^2(1-\alpha)}{2^{3k+1}\cdot 3(1-\lambda)^2} - \frac{\alpha^3(1-\lambda)}{2^{3k+1}\cdot 3(1-\lambda)^3} - \frac{5\cdot 2^3\alpha^3}{3(1-\lambda)^3} - \frac{\alpha^3(1+2\lambda)}{2^{3k+3}\cdot 3(1-\lambda)^3} + \frac{3^{k-1}\alpha^2(1+2\lambda)}{2^{3k}(1-\lambda)^2} - \frac{\alpha^2(1-\alpha)}{2^{3k+2}\cdot 3(1-\lambda)^2} - \frac{\alpha^2}{2^{3k+3}\cdot 3(1-\lambda)^2} + \frac{\alpha^2}{2\cdot 3^{2k}(1-\lambda)^2}\right],$$

$$\mu_2(k,\alpha,\lambda) = \left[\frac{\alpha^3(1+\lambda)}{2^{3k-1}\cdot 3(1-\lambda)^3} + \frac{5\cdot 2^{k+5}\alpha^3}{2^k\cdot 3^k(1-\lambda)^3} + \frac{\alpha^3(1+2\lambda)}{2^{3k+1}\cdot 3(1-\lambda)^3} + \frac{3^{k-1}\alpha^2(1+2\lambda)}{2^{3k-2}(1-\lambda)^2} - \frac{\alpha^3}{2^{2k}\cdot 3^k(1-\lambda)^3} + \frac{\alpha^2(\alpha-1)}{2^{3k}\cdot 3(1-\lambda)^2} + \frac{\alpha^2}{2^{3k}\cdot 3(1-\lambda)^2} + \frac{\alpha^2}{2^{3k+1}\cdot 3(1-\lambda)^2} - \frac{\alpha^2}{2^{3k}\cdot 3(1-\lambda)^2} - \frac{2^2\alpha^2}{3^{2k}(1-\lambda)^2} + \frac{\alpha^2}{2^{3k}\cdot 3(1-\lambda)^2}\right]$$

and

$$\varphi(2) = \frac{2^3\alpha^2}{3^{2k}(1-\lambda)^2} + 16\mu_1(k,\alpha,\lambda) + 4\mu_2(k,\alpha,\lambda).$$

Proof. Putting (16), (12) and (13) into (3) gives

$$\begin{aligned} (20)\quad & a_2a_4 - a_3^2 \\ &= \left\{\frac{\alpha^3(1+\alpha)(1+\lambda)}{2^{3k+1}\cdot 3(1-\lambda)^3} + \frac{3^{k-1}\alpha^4(1+2\lambda)}{2^{5k+1}(1+\lambda)^2(1-\lambda)} + \frac{7\alpha^2(1-\alpha)(2-\alpha)}{3^2\cdot 2^{3k+2}(1-\lambda)^2} - \frac{\alpha^4}{2^{4k}(1-\lambda)^4}\right\}p_1^4 \\ &+ \left\{\frac{\alpha^3(1+\lambda)}{2^{3k+1}\cdot 3(1-\lambda)^3} + \frac{5\cdot 2^{k+3}\alpha^3}{2^k\cdot 3^k(1-\lambda)^3} + \frac{\alpha^3(1+2\lambda)}{2^{3k+3}\cdot 3(1-\lambda)^3} + \frac{3^{k-1}\alpha^2(1+2\lambda)}{2^{3k}(1-\lambda)^2} - \frac{\alpha^3}{2^{2k+2}\cdot 3^k(1-\lambda)^3}\right\}p_1^2(p_2-q_2) - \frac{\alpha^2(1-\alpha)p_1^2(p_2+q_2)}{2^{3k+1}\cdot 3(1-\lambda)^2} \\ &- \frac{\alpha^2(p_2-q_2)^2}{2^4\cdot 3^{2k}(1-\lambda)^2} + \frac{\alpha^2(p_3-q_3)}{2^{3k+1}\cdot 3(1-\lambda)^2} \end{aligned}$$

and using (9) and (10) gives

$$\begin{aligned}
(21)\quad & |a_2a_4-a_3^2| \\
&= \Bigg| \left\{ \frac{\alpha^3(1+\alpha)(1+\lambda)}{2^{3k+1}\cdot 3(1-\lambda)^3} + \frac{3^{k-1}\alpha^4(1+2\lambda)}{2^{5k+1}(1+\lambda)^2(1-\lambda)} + \frac{7\alpha^2(1-\alpha)(2-\alpha)}{3^2\cdot 2^{3k+2}(1-\lambda)^2} - \frac{\alpha^4}{2^{4k}(1-\lambda)^4} \right\} p_1^4 \\
&\quad + \left\{ \frac{\alpha^3(1+\lambda)}{2^{3k+1}\cdot 3(1-\lambda)^3} + \frac{5\cdot 2^{k+3}\alpha^3}{2^k\cdot 3^k(1-\lambda)^3} + \frac{\alpha^3(1+2\lambda)}{2^{3k+3}\cdot 3(1-\lambda)^3} + \frac{3^{k-1}\alpha^2(1+2\lambda)}{2^{3k}(1-\lambda)^2} \right. \\
&\qquad \left. - \frac{\alpha^3}{2^{2k+2}\cdot 3^k(1-\lambda)^3} \right\} \frac{p_1^2(4-p_1^2)(x-y)}{2} \\
&\quad - \frac{\alpha^2(1-\alpha)}{2^{3k+1}\cdot 3(1-\lambda)^2} p_1^4 - \frac{\alpha^2(1-\alpha)p_1^2(4-p_1^2)(x+y)}{2^{3k+2}\cdot 3(1-\lambda)^2} - \frac{\alpha^2(4-p_1^2)^2(x-y)^2}{2^5\cdot 3^{2k}(1-\lambda)^2} \\
&\quad + \frac{\alpha^2 p_1^4}{2^{3k+2}\cdot 3(1-\lambda)^2} + \frac{\alpha^2 p_1^2(4-p_1^2)(x+y)}{2^{3k+2}\cdot 3(1-\lambda)^2} \\
&\quad - \frac{\alpha^2 p_1^2(4-p_1^2)(x^2+y^2)}{2^{3k+3}\cdot 3(1-\lambda)^2} + \frac{\alpha^2(4-p_1^2)[(1-|x|^2)z-(1-|y|^2)w]}{2^{3k+2}\cdot 3(1-\lambda)^2} \Bigg|.
\end{aligned}$$

We, without restriction, let $p=p_1$, which implies that $p\in[0,2]$ from Lemma 2.1. So, using triangle inequality with $X=|x|\leqslant 1$ and $Y=|y|\leqslant 1$, then we get

$$\begin{aligned}
(22)\quad & |a_2a_4-a_3^2| \leqslant \left\{ \frac{\alpha^3(1+\alpha)(1+\lambda)}{2^{3k+1}\cdot 3(1-\lambda)^3} + \frac{3^{k-1}\alpha^4(1+2\lambda)}{2^{5k+1}(1+\lambda)^2(1-\lambda)} + \frac{7\alpha^2(1-\alpha)(2-\alpha)}{3^2\cdot 2^{3k+2}(1-\lambda)^2} \right. \\
& \left. + \frac{\alpha^4}{2^{4k}(1-\lambda)^4} \right\} p^4 + \left\{ \frac{\alpha^3(1+\lambda)}{2^{3k+1}\cdot 3(1-\lambda)^3} + \frac{5\cdot 2^{k+3}\alpha^3}{2^k\cdot 3^k(1-\lambda)^3} + \frac{\alpha^3(1+2\lambda)}{2^{3k+3}\cdot 3(1-\lambda)^3} + \frac{3^{k-1}\alpha^2(1+2\lambda)}{2^{3k}(1-\lambda)^2} \right. \\
& \left. + \frac{\alpha^3}{2^{2k+2}\cdot 3^k(1-\lambda)^3} \right\} \frac{p^2(4-p^2)(X+Y)}{2} + \frac{\alpha^2(1-\alpha)}{2^{3k+1}\cdot 3(1-\lambda)^2} p^4 \\
& + \frac{\alpha^2(1-\alpha)p^2(4-p^2)(X+Y)}{2^{3k+2}\cdot 3(1-\lambda)^2} + \frac{\alpha^2(4-p^2)^2(X+Y)^2}{2^5\cdot 3^{2k}(1-\lambda)^2} + \frac{\alpha^2p^4}{2^{3k+2}\cdot 3(1-\lambda)^2} \\
& + \frac{\alpha^2p^2(4-p^2)(X+Y)}{2^{3k+2}\cdot 3(1-\lambda)^2} + \frac{\alpha^2p^2(4-p^2)(X^2+Y^2)}{2^{3k+3}\cdot 3(1-\lambda)^2} + \frac{\alpha^2p^2(4-p^2)}{2^{3k+2}\cdot 3(1-\lambda)^2} - \frac{\alpha^2(4-p^2)(X^2+Y^2)}{2^{3k+2}\cdot 3(1-\lambda)^2}
\end{aligned}$$

or

$$\begin{aligned}
(23)\quad & |a_2a_4-a_3^2| \leqslant \\
& \left[\left\{ \frac{\alpha^3(1+\alpha)(1+\lambda)}{2^{3k+1}\cdot 3(1-\lambda)^3} + \frac{3^{k-1}\alpha^4(1+2\lambda)}{2^{5k+1}(1+\lambda)^2(1-\lambda)} + \frac{7\alpha^2(1-\alpha)(2-\alpha)}{3^2\cdot 2^{3k+2}(1-\lambda)^2} + \frac{\alpha^4}{2^{4k}(1-\lambda)^4} \right\} p^4 \right. \\
& \left. + \frac{\alpha^2(1-\alpha)p^4}{2^{3k+1}\cdot 3(1-\lambda)^2} + \frac{\alpha^2p^4}{2^{3k+2}\cdot 3(1-\lambda)^2} + \frac{\alpha^2p^2(4-p^2)}{2^{3k+2}\cdot 3(1-\lambda)^2} \right] \\
& + \left[\left\{ \frac{\alpha^3(1+\lambda)}{2^{3k+1}\cdot 3(1-\lambda)^3} + \frac{5\cdot 2^{k+3}\alpha^3}{2^k\cdot 3^k(1-\lambda)^3} + \frac{\alpha^3(1+2\lambda)}{2^{3k+3}\cdot 3(1-\lambda)^3} + \frac{3^{k-1}\alpha^2(1+2\lambda)}{2^{3k}(1-\lambda)^2} \right. \right. \\
& \left. \left. + \frac{\alpha^3}{2^{2k+2}\cdot 3^k(1-\lambda)^3} \right\} \frac{p^2(4-p^2)}{2} + \frac{\alpha^2(1-\alpha)p^2(4-p^2)}{2^{3k+2}\cdot 3(1-\lambda)^2} + \frac{\alpha^2p^2(4-p^2)}{2^{3k+2}\cdot 3(1-\lambda)^2} \right] (X+Y) \\
& + \left[\frac{\alpha^2p^2(4-p^2)}{2^{3k+3}\cdot 3(1-\lambda)^2} - \frac{\alpha^2(4-p^2)}{2^{3k+2}\cdot 3(1-\lambda)^2} \right] (X^2+Y^2) + \frac{\alpha^2(4-p^2)^2}{2^5\cdot 3^{2k}(1-\lambda)^2}(X+Y)^2.
\end{aligned}$$

For brevity, let (23) be in the equivalent form

$$(24)\quad |a_2a_4 - a_3^2| \leqslant \mathfrak{U}_1(p) + \mathfrak{U}_2(p)(X+Y) + \mathfrak{U}_3(p)(X^2+Y^2) + \mathfrak{U}_4(p)(X+Y)^2 = \mathfrak{J}_1(X,Y)$$

where

$$(25)\quad \mathfrak{U}_1(p) = \Bigg[\Bigg\{\frac{\alpha^3(1+\alpha)(1+\lambda)}{2^{3k+1}\cdot 3(1-\lambda)^3} + \frac{3^{k-1}\alpha^4(1+2\lambda)}{2^{5k+1}(1+\lambda)^2(1-\lambda)} + \frac{7\alpha^2(1-\alpha)(2-\alpha)}{3^2\cdot 2^{3k+2}(1-\lambda)^2} + \frac{\alpha^4}{2^{4k}(1-\lambda)^4}\Bigg\}p^4 + \frac{\alpha^2(1-\alpha)p^4}{2^{3k+1}\cdot 3(1-\lambda)^2} + \frac{\alpha^2p^4}{2^{3k+2}\cdot 3(1-\lambda)^2} + \frac{\alpha^2(4-p^2)}{2^{3k+2}\cdot 3(1-\lambda)^2}\Bigg] \geqslant 0$$

$$(26)\quad \mathfrak{U}_2(p) = \Bigg[\Bigg\{\frac{\alpha^3(1+\lambda)}{2^{3k+1}\cdot 3(1-\lambda)^3} + \frac{5\cdot 2^{k+3}\alpha^3}{2^k\cdot 3^k(1-\lambda)^3} + \frac{\alpha^3(1+2\lambda)}{2^{3k+3}\cdot 3(1-\lambda)^3} + \frac{3^{k-1}\alpha^2(1+2\lambda)}{2^{3k}(1-\lambda)^2} + \frac{\alpha^3}{2^{2k+2}\cdot 3^k(1-\lambda)^3}\Bigg\}\frac{p^2(4-p^2)}{2} + \frac{\alpha^2(1-\alpha)p^2(4-p^2)}{2^{3k+2}\cdot 3(1-\lambda)^2} + \frac{\alpha^2p^2(4-p^2)}{2^{3k+2}\cdot 3(1-\lambda)^2}\Bigg] \geqslant 0$$

$$(27)\quad \mathfrak{U}_3(p) = \left[\frac{\alpha^2p^2(4-p^2)}{2^{3k+3}\cdot 3(1-\lambda)^2} - \frac{\alpha^2(4-p^2)}{2^{3k+2}\cdot 3(1-\lambda)^2}\right] \leqslant 0$$

$$(28)\quad \mathfrak{U}_4(p) = \frac{\alpha^2(4-p^2)^2}{2^5\cdot 3^{2k}(1-\lambda)^2} \geqslant 0.$$

We now need to maximize the function $\mathfrak{J}_1(X,Y)$ in the closed square

$$S := \{(X,Y) : (X,Y) \in [0,1]\times[0,1]\}$$

for the cases $p=0$, $p=2$ and $p\in(0,2)$.

Case 1: If $p=0$, then from (24),

$$\mathfrak{J}_1(X,Y) = \frac{\alpha^2(X+Y)^2}{2\cdot 3^{2k}(1-\lambda)^2}$$

and since maximum of $\mathfrak{J}_1(X,Y)$ occurs at $X=1=Y$, then we get

$$\max\{\mathfrak{J}_1(X,Y) : (X,Y)\in[0,1]\times[0,1]\} = \mathfrak{J}_1(1,1) = \frac{2\alpha^2}{3^{2k}(1-\lambda)^2}.$$

Case 2: If $p=2$, then from (24),

$$\mathfrak{J}_1(X,Y) = \frac{\alpha^3(1+\alpha)(1+\lambda)}{2^{3k-3}\cdot 3(1-\lambda)^3} + \frac{3^{k-1}\alpha^4(1+2\lambda)}{2^{5k-3}(1+\lambda)^2(1-\lambda)} + \frac{7\alpha^2(1-\alpha)(2-\alpha)}{2^{3k-2}\cdot 3^2(1-\lambda)^2} + \frac{\alpha^4}{2^{4k-4}(1-\lambda)^4} + \frac{\alpha^2(1-\alpha)}{2^{3k-3}\cdot 3(1-\lambda)^2} + \frac{\alpha^2}{2^{3k-2}\cdot 3(1-\lambda)^2}$$

and observe that this is a constant function.

Case 3: If $p\in(0,2)$, and we let $X+Y=M$ and $XY=N$, then from (24),

$$(29)\quad \mathfrak{J}_1(X,Y) = \mathfrak{U}_1(p) + \mathfrak{U}_2(p)M + [\mathfrak{U}_3(p)+\mathfrak{U}_4(p)]M^2 - 2\mathfrak{U}_3(p)N = \mathfrak{J}_2(M,N), \quad M\in[0,2] \text{ and } N\in[0,1].$$

To determine the maximum of

$$\mathfrak{J}_2(M,N) : (M,N) \in T := [0,2] \times [0,1],$$

consider

$$\frac{\partial \mathfrak{J}_2(M,N)}{\partial M} = \mathfrak{U}_2(p) + 2[\mathfrak{U}_3(p) + \mathfrak{U}_4(p)]M = 0$$

and

$$\frac{\partial \mathfrak{J}_2(M,N)}{\partial N} = -2\mathfrak{U}_2(p) = 0$$

which clearly show that the function $\mathfrak{J}_2(M,N)$ has no critical point inside T, therefore $\mathfrak{J}_1(X,Y)$ has no critical point in the square S. We can then conclude that function $\mathfrak{J}_1(X,Y)$ does not have local maximum value in the inside of square S.

Now we investigate the maximum of $\mathfrak{J}_1(X,Y)$ on the boundary of the square S.

Case 3(a): If $X = 0$, $Y \in [0,1]$ (or $Y = 0$, $X \in [0,1]$), then from (29),

(30) $$\mathfrak{J}_1(0,Y) = \mathfrak{U}_1(p) + \mathfrak{U}_2(p)Y + [\mathfrak{U}_3(p) + \mathfrak{U}_4(p)]Y^2 = \tau_1(Y),$$

and

$$\tau_1'(Y) = \mathfrak{U}_2(p) + 2[\mathfrak{U}_3(p) + \mathfrak{U}_4(p)]Y,$$

hence since $[\mathfrak{U}_3(p) + \mathfrak{U}_4(p)] \geqslant 0$, then

$$\tau_1'(Y) = \mathfrak{U}_2(p) + 2[\mathfrak{U}_3(p) + \mathfrak{U}_4(p)]Y > 0.$$

This means that the function $\tau_1(Y)$ is an increasing function for all $Y \in [0,1]$ and the maximum occurs at $Y = 1$. We can now say that from (30),

(31) $$\max\{\mathfrak{J}_1(0,Y) : Y \in [0,1]\} = \mathfrak{J}_1(0,1) = \mathfrak{U}_1(p) + \mathfrak{U}_2(p) + \mathfrak{U}_3(p) + \mathfrak{U}_4(p).$$

Case 3(b): If $X = 1$, $Y \in [0,1]$ (or $Y = 1$, $X \in [0,1]$), then from (29),

(32) $$\mathfrak{J}_1(1,Y) = \mathfrak{U}_1(p) + \mathfrak{U}_2(p) + \mathfrak{U}_3(p) + \mathfrak{U}_4(p) + [\mathfrak{U}_2(p) + 2\mathfrak{U}_4(p)]Y + [\mathfrak{U}_3(p) + \mathfrak{U}_4(p)]Y^2 = \tau_2(Y),$$

then

$$\tau_2'(Y) = [\mathfrak{U}_2(p) + 2\mathfrak{U}_4(p)] + 2[\mathfrak{U}_3(p) + \mathfrak{U}_4(p)]Y.$$

hence since $[\mathfrak{U}_3(p) + \mathfrak{U}_4(p)] \geqslant 0$, then

$$\tau_2'(Y) = [\mathfrak{U}_2(p) + 2\mathfrak{U}_4(p)] + 2[\mathfrak{U}_3(p) + \mathfrak{U}_4(p)]Y > 0.$$

This means that $\tau_2(Y)$ is an increasing function $\forall Y \in [0,1]$ and the maximum occurs at $Y = 1$. We can now say that from (32),

(33) $$\max\{\mathfrak{J}_1(1,Y) : Y \in [0,1]\} = \mathfrak{J}_1(1,1) = \mathfrak{U}_1(p) + 2[\mathfrak{U}_2(p) + \mathfrak{U}_3(p)] + 4\mathfrak{U}_4(p).$$

It can be concluded from (31) and (33) that for $p \in (0,2)$,

$$\mathfrak{U}_1(p) + 2[\mathfrak{U}_2(p) + \mathfrak{U}_3(p)] + 4\mathfrak{U}_4(p) > \mathfrak{U}_1(p) + \mathfrak{U}_2(p) + \mathfrak{U}_3(p) + \mathfrak{U}_4(p).$$

Hence

(34) $$\max\{\mathfrak{J}_1(X,Y) : X \in [0,1], Y \in [0,1]\} = \mathfrak{U}_1(p) + 2[\mathfrak{U}_2(p) + \mathfrak{U}_3(p)] + 4\mathfrak{U}_4(p)$$

and in conclusion, since $\tau_1(1) \leqslant \tau_2(1)$ for $t \in [0,2]$, then

$$\max\{\mathfrak{J}_1(X,Y)\} = F(1,1)$$

occurs on the boundary of square S. The implication of this is that the maximum of $\mathfrak{J}_1$ occurs at $X = 1 = Y$ on the closed square S

Also to consider the maximum point at (34), let $\varphi : (0,2) \longrightarrow \mathbb{R}$ be defined as

$$\varphi(p) = \max\{\mathfrak{J}_1(X,Y)\} = \mathfrak{J}_1(1,1) = \mathfrak{U}_1(p) + 2[\mathfrak{U}_2(p) + \mathfrak{U}_3(p)] + 4\mathfrak{U}_4(p). \tag{35}$$

Putting (25), (26), (27) and (28) into (35) gives

$$\begin{aligned} \varphi(p) = \frac{2^3\alpha^2}{3^{2k}(1-\lambda)^2} &+ \Bigg[\frac{\alpha^3(1+\alpha)(1+\lambda)}{2^{3k+1}\cdot 3(1-\lambda)^3} + \frac{3^{k-1}\alpha^4(1+2\lambda)}{2^{5k+1}(1+\lambda)^2(1-\lambda)} + \frac{7\alpha^2(1-\alpha)(2-\alpha)}{3^2\cdot 2^{3k+2}(1-\lambda)^2} \\ &+ \frac{\alpha^4}{2^{4k}(1-\lambda)^4} + \frac{\alpha^2(1-\alpha)}{2^{3k+1}\cdot 3(1-\lambda)^2} - \frac{\alpha^3(1-\lambda)}{2^{3k+1}\cdot 3(1-\lambda)^3} - \frac{5\cdot 2^3\alpha^3}{3(1-\lambda)^3} - \frac{\alpha^3(1+2\lambda)}{2^{3k+3}\cdot 3(1-\lambda)^3} \\ &+ \frac{3^{k-1}\alpha^2(1+2\lambda)}{2^{3k}(1-\lambda)^2} + \frac{\alpha^2(1-\alpha)}{2^{3k+2}\cdot 3(1-\lambda)^2} - \frac{\alpha^2}{2^{3k+3}\cdot 3(1-\lambda)^2} + \frac{\alpha^2}{2\cdot 3^{2k}(1-\lambda)^2}\Bigg]p^4 \\ &+ \Bigg[\frac{\alpha^3(1+\lambda)}{2^{3k-1}\cdot 3(1-\lambda)^3} + \frac{5\cdot 2^{k+5}\alpha^3}{2^k\cdot 3^k(1-\lambda)^3} + \frac{\alpha^3(1+2\lambda)}{2^{3k+1}\cdot 3(1-\lambda)^3} + \frac{3^{k-1}\alpha^2(1+2\lambda)}{2^{3k-2}(1-\lambda)^2} \\ &- \frac{\alpha^3}{2^{2k}\cdot 3^k(1-\lambda)^3} - \frac{\alpha^2(1-\alpha)}{2^{3k}\cdot 3(1-\lambda)^2} + \frac{\alpha^2}{2^{3k}\cdot 3(1-\lambda)^2} \\ &+ \frac{\alpha^2}{2^{3k+1}\cdot 3(1-\lambda)^2} - \frac{\alpha^2}{2^{3k}\cdot 3(1-\lambda)^2} - \frac{2^2\alpha^2}{3^{2k}(1-\lambda)^2} + \frac{\alpha^2}{2^{3k}\cdot 3(1-\lambda)^2}\Bigg]p^2. \end{aligned} \tag{36}$$

and for brevity we have

$$\varphi(p) = \frac{2^3\alpha^2}{3^{2k}(1-\lambda)^2} + \mu_1(k,\alpha,\lambda)p^4 + \mu_2(k,\alpha,\lambda)p^2. \tag{37}$$

Let us investigate the maximum value of $\varphi(p)$ in the interval $(0,2)$, then from (37) we get

$$\varphi'(p) = 4\mu_1(k,\alpha,\lambda)p^3 + 2\mu_2(k,\alpha,\lambda)p.$$

Result 1: If $\mu_1(k,\alpha,\lambda) \geqslant 0$ and $\mu_2(k,\alpha,\lambda) \geqslant 0$, then we observe that $\varphi(p)$ is an increasing function, hence the maximum point has to be on the boundary $p = 2$, so

$$\begin{aligned} \max\{\mathfrak{J}_1(X,Y) : X \in [0,1], Y \in [0,1]\} &= \varphi(2) \\ &= \frac{2^3\alpha^2}{3^{2k}(1-\lambda)^2} + 16\mu_1(k,\alpha,\lambda) + 4\mu_2(k,\alpha,\lambda). \end{aligned} \tag{38}$$

Result 2: If $\mu_1(k,\alpha,\lambda) > 0$ and $\mu_2(k,\alpha,\lambda) < 0$, then we get

$$\varphi'(p) = 4\mu_1(k,\alpha,\lambda)p^3 + 2\mu_2(k,\alpha,\lambda)p = 0$$

which implies that we have a critical point at

$$p_0 = \sqrt{\frac{-\mu_2}{2\mu_1}} \tag{39}$$

and for

$$\varphi''(p_0) = 12\mu_1(k,\alpha,\lambda)p^2 + 2\mu_2(k,\alpha,\lambda) > 0,$$

then the maximum value of function $\varphi(p)$ is at p_0, so from (37) we get

$$\varphi(p_0) = \frac{2^3\alpha^2}{3^{2k}(1-\lambda)^2} - \frac{[\mu_2(k,\alpha,\lambda)]^2}{4\mu_1(k,\alpha,\lambda)},$$

hence

$$\varphi(p_0) < \frac{2^3\alpha^2}{3^{2k}(1-\lambda)^2}$$

and

$$\max\{\mathfrak{J}_1(X,Y) : X \in [0,1], Y \in [0,1]\} = \max\left\{\frac{2^3\alpha^2}{3^{2k}(1-\lambda)^2}, \varphi(2)\right\}.$$

Result 3: If $\mu_1(k,\alpha,\lambda) \leqslant 0$ and $\mu_2(k,\alpha,\lambda) \leqslant 0$, then observe that $\varphi'(p) \leqslant 0$ which shows that function $\varphi(p)$ is a decreasing function, hence the maximum point has to be on the boundary $p = 0$, so

$$\max\{\mathfrak{J}_1(X,Y) : X \in [0,1], Y \in [0,1]\} = \varphi(0) = \frac{2^3\alpha^2}{3^{2k}(1-\lambda)^2}. \tag{40}$$

Result 4: If $\mu_1(k,\alpha,\lambda) < 0$ and $\mu_2(k,\alpha,\lambda) > 0$, then observe that

$$\varphi'(p) = 4\mu_1(k,\alpha,\lambda)p^3 + 2\mu_2(k,\alpha,\lambda)p = 0$$

implies that we have a critical point at

$$p_1 = \sqrt{\frac{-\mu_2}{2\mu_1}} \tag{41}$$

and for

$$\varphi''(p_1) = 12\mu_1(k,\alpha,\lambda)p^2 + 2\mu_2(k,\alpha,\lambda) < 0,$$

then the maximum value of function $\varphi(p)$ is at p_1, so from (37) we get

$$\varphi(p_1) = \frac{2^3\alpha^2}{3^{2k}(1-\lambda)^2} - \frac{[\mu_2(k,\alpha,\lambda)]^2}{4\mu_1(k,\alpha,\lambda)},$$

hence

$$\varphi(p_1) > \frac{2^3\alpha^2}{3^{2k}(1-\lambda)^2}$$

and

$$\max\{\mathfrak{J}_1(X,Y) : X \in [0,1], Y \in [0,1]\} = \max\left\{\frac{2^3\alpha^2}{3^{2k}(1-\lambda)^2}, \varphi(2)\right\}.$$

Hence the proof complete from the discussed cases. □

References

[1] A.W. Goodman, Univalent functions vol. II, Mariner Publishing Company Inc., Tampa, Florida, (1983).

[2] J. Jothibasu, Certain subclasses of bi-univalent functions defined by Sălăgean operator, Electron. J. Math. Anal. Appl. 3(1) (2015) 150–157.

[3] A.O. Lasode, Estimates for a generalized class of analytic and bi-univalent functions involving two q-operators, Earthline J. Math. Sci. 10(2) (2022) 211–225. https://doi.org/10.34198/ejms.10222.211225.

[4] A.O. Lasode, T.O. Opoola, On a generalized class of bi-univalent functions defined by subordination and q-derivative operator, Open J. Math. Anal. 5(2) (2021) 46–52. https://doi.org/10.30538/psrp-oma2021.0092.

[5] A.O. Lasode, T.O. Opoola, Hankel determinant of a subclass of analytic and bi-univalent functions defined by means of subordination and q-differentiation, Int. J. Nonlinear Anal. Appl. 13(2) (2022) 3105–3114. https://doi.org/10.22075/IJNAA.2022.24577.2775.

[6] A.O. Lasode, T.O. Opoola, Coefficient problems of a class of q-starlike functions associated with q-analogue of Al-Oboudi-Al-Qahtani integral operator and nephroid domain, J. Class. Anal. 20(1) (2022) 35–47. https://doi.org/10.7153/jca-2022-20-04.

[7] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc. 18 (1967) 63–68. https://doi.org/10.1090/S0002-9939-1967-0206255-1.

[8] R.J. Libera, E.J. Zlotkiewicz, Early coefficients of the inverse of a regular convex function, Proc. Amer. Math. Soc. 85 (1982) 225–230. https://doi.org/10.1090/S0002-9939-1982-0652447-5.

[9] G. Murugusundaramoorthy, N. Magesh, V. Prameela, Coefficient bounds for certain subclasses of bi-univalent functions, Abst. Appl. Anal., 2013 (2013) Article ID 573017. https://doi.org/10.1155/2013/573017.

[10] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Arch. Ration. Mech. Anal. 32 (1969) 100–112. https://doi.org/10.1007/BF00247676.

[11] O.A. Olukoya, E.A. Oyekan, Coefficient bounds for the functions in the class of modified hyperbolic tangent function, Coast J. Fac. Sci., Ondo state Univ. Sci. Technol., Okitipupa, 1(2) (2019) 222–224.

[12] E.A. Oyekan. Certain geometric properties of functions involving Galuê type Struve function, Ann. Math. Computat. Sci. 8 (2022) 43–53.

[13] E.A. Oyekan, Gegenbauer polynomials for certain subclasses of Bazilevič functions associated with a generalized operator defined by convolution, Gulf J. Math. 14 (2023) 77–88. https://doi.org/10.56947/gjom.v14i2.967.

[14] E.A. Oyekan, S. O. Aderibola, New results on the Chebyshev polynomial bounds for classes of univalent functions, Asia Pac. J. Math. 7 (2020) 12–22. https://doi.org/10.28924/APJM/7-24.

[15] E.A. Oyekan, I.T. Awolere, Polynomial bounds for bi-univalent functions associated with the probability of generalized distribution defined by generalized polylogarithms via Chebyshev polynomials, Coast J. Fac. Sci. Technol., Okitipupa 2 (2020) 222–224.

[16] E.A. Oyekan, I.T. Awolere, A new subclass of univalent functions connected with convolution defined via employing a linear combination of two generalized differential operators involving sigmoid function, Maltepe J. Math. 2(2) (2020) 82–96. https://doi.org/10.47087/mjm.791841.

[17] E.A. Oyekan, A.O. Lasode, Estimates for some classes of analytic functions associated with Pascal distribution series, error function, Bell numbers and q-differential operator, Nigerian J. Math. Appl. 32 (2022) 163–173.

[18] E.A. Oyekan, T.A. Olatunji, A.O. Lasode, Application of (p, q)-Gegenbauer polynomials on a family of bi-univalent functions, Earthline J. Math. Sci. 12(2) (2023) 271–284. https://doi.org/10.34198/ejms.12223.271284.

[19] E.A. Oyekan, T.O. Opoola, On subclasses of bi-univalent functions defined by generalized Sălăgean operator related to shell-like curves connected with Fibonacci numbers, Libertas Math. 41 (2021) 1–20.

[20] E.A. Oyekan, S.R. Swamy, P.O. Adepoju, T.A. Olatunji, Quasi-convolution properties of a new family of close-to-convex functions involving a q-p-Opoola differential operator, Intern. J. Math. Trends Technol. 69(5) 70–77.https://doi.org/10.14445/22315373/IJMTT-V69I5P506.

[21] E. A. Oyekan, S. R. Swamy and T. O. Opoola, Ruscheweyh derivative and a new generalized operator involving convolution, Intern. J. Math. Trends Technol. 67(1) (2021), 88–100. https://doi.org/10.14445/22315373/IJMTT-V67I1P513.

[22] C. Pommerenke, On the coefficients and Hankel determinants of univalent functions, Proc. Lond. Math. Soc. 41 (1966) 111–122. https://doi.org/10.1112/jlms/s1-41.1.111.

[23] G.S. Sălăgean, Subclasses of univalent functions, Lect. Notes Math. 1013 (1983) 362–372. https://doi.org/10.1007/BFb0066543.

[24] H.M. Srivastava, A.K. Mishra, P. Gochhayt, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett. 23 (2010) 1188–1192. https://doi.org/10.1016/j.aml.2010.05.009.