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A NEW CHARACTERIZATION OF MATHIEU SIMPLE GROUPS BY THE NUMBER OF SINGULAR ELEMENTS

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ABSTRACT. Given a finite group G, let $\pi(G)$ denote the set of all primes that divide the order of G. For a prime $p \in \pi(G)$, we define p-singular elements as those elements of G whose order is divisible by p. We denote the proportion of p-singular elements in G by $\mu_p(G)$. Let $\mu(G) := \{\mu_p(G) | p \in \pi(G)\}$ be the set of all proportions of p-singular elements for each prime p that divides $|G|$. In this paper we prove if a finite group G has the same set of proportions as a Mathieu simple group M, then G is isomorphic to M.

1. Introduction

Given a finite group G, let $\pi(G)$ be the set of all primes that divide the order of G. For each prime $p \in \pi(G)$, we define p-singular elements as those elements of G whose order is divisible by p, while p-regular elements are those whose order is not divisible by p. The number of p-singular and p-regular elements plays a crucial role in understanding the structure of the group, particularly in finite simple group theory.

Several researchers have investigated the properties of singular elements in different types of groups. In 1995, Isaacs, Kantor, and Spaltenstein [1] studied the probability of an element in a group being p-singular. In 1999, Guralnick and Libeck [3] studied p-singular elements in Chevalley groups in characteristic p. In 2013, Babai, Guest, Praeger, and others [4] investigated the proportions of r-regular elements in finite classical groups. In 2017, He and Chen [5] give a new characterization of Mathieu simple Groups.

We denote the proportion of p-singular elements in G by $\mu_p(G)$. Let $\mu(G) := \{\mu_p(G) | p \in \pi(G)\}$ the set of all proportions of p-singular elements for each prime p that divides $|G|$. In this paper, we use this set to characterize Mathieu simple Groups. Our main result is the following theorem:

Theorem 1.1. *Let G be a finite group and M a Mathieu simple group. If $\mu(G) = \mu(M)$, then $G \cong M$.*

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2. Some Lemmas

In this section, we will give some useful lemmas. The notation p means a prime always, and p' means a set of primes that does not contain the prime p.

Lemma 2.1. *[1, Lemma 2.2] If $N \trianglelefteq G$, then $\mu_p(G) \geq \mu_p(G/N) + \mu_p(N)/|G:N|$.*

Lemma 2.2. *Let G be a finite group and $N \trianglelefteq G$, X the set of G-conjugate class representives of non-trivial p-elements in N. Then $\mu_p(G) = \mu_p(G/N) + \sum\limits_{x\in X}(1-\mu_p(C_G(x)))$.*

Proof. The number of singular elements in G is

$$S_p(G) = S_p(G/N)\cdot|N| + \sum_{x\in X}|G:C_G(x)|\cdot R_p(C_G(x)),$$

which implies that the ratio of p-singular elements in G is

$$\begin{aligned}\mu_p(G) = \frac{S_p(G)}{|G|} &= \frac{S_p(G/N)\cdot|N| + \sum\limits_{i=1}^{k}|G:C_G(x)|\cdot R_p(C_G(x))}{|G|}\\ &= \mu_p(G/N) + \frac{\sum\limits_{x\in X} R_p(C_G(x_i))}{|C_G(x_i)|}\\ &= \mu_p(G/N) + \sum_{x\in X}(1-\mu_p(C_G(x))).\end{aligned}$$

Thus $\mu_p(G) = \mu_p(G/N) + \sum\limits_{x\in X}(1-\mu_p(C_G(x)))$. □

Lemma 2.3. *[2, Lemma 1] Let G be a finite group and P a Sylow p-subgroup of G. Then $\mu_p(G) = \frac{t}{|P|}$ with $(t,p) = 1$.*

Lemma 2.4. *Let N be the largest p'-normal subgroup of G. Then $\mu_p(G) = \mu_p(G/N)$.*

Proof. According to Lemma 2.2, we have

$$\mu_p(G) = \mu_p(G/N) + \sum_{x\in X}(1-\mu_p(C_G(x))),$$

where X the set of G-conjugate class representives of non-trivial p-elements in N. Since $p \nmid |N|$, it is clear that $\sum\limits_{x\in X}(1-\mu_p(C_G(x))) = 0$. Thus $\mu_p(G) = \mu_p(G/N)$. □

Lemma 2.5. *Let N_1 and N_2 be two distinct normal subgroups of G. Suppose that $G/N_1 \cong S_1$ and $G/N_2 \cong S_2$ are simple. Then $G/(N_1\cap N_2) \cong S_1 \times S_2$.*

Proof. Since N_1 and N_2 are normal subgroups of G, we have $N_1N_2 \trianglelefteq G$. This leads to $G/N_1N_2 \cong G/N_1/N_1N_2/N_1$, thus $N_1N_2/N_1 \trianglelefteq G/N_1$. Moreover, G/N_1 is a simple group, then $N_1N_2/N_1 = N_1$ or $N_1N_2/N_1 = G/N_1$. If $N_1N_2/N_1 = N_1$, it is clear that $N_1 = N_2$, which contradicts the assumption that $N_1 \neq N_2$. Therefore, $N_1N_2/N_1 = G/N_1$, it implies $N_1N_2 = G$. Since $G/N_1 = N_1N_2/N_1 \cong N_2/(N_1\cap N_2)$, it follows that $G/N_2 \cong N_1/(N_1\cap N_2)$. And then S_1 and S_2 are minimal normal subgroups of $G/(N_1\cap N_2)$, we obtain that $G/(N_1\cap N_2) \cong S_1\times S_2$. □

Lemma 2.6. *Let P be a p-subgroup of G and $P \leq Z(G)$. Then*

$$\mu_p(G) = \frac{\mu_p(G/P)}{|P|} + 1 - \frac{1}{|P|}.$$

Proof. We use the same notations as Lemma 2.2, then

$$\mu_p(G) = \mu_r(G/P) + \sum_{X \in P-\{1\}} (1 - \mu_p(C_G(x)))$$
$$= \mu_r(G/P) + (|P| - 1)(1 - \mu_p(G)).$$

Thus $\mu_p(G) = \frac{\mu_p(G/P)}{|P|} + 1 - \frac{1}{|P|}$. □

3. Proof of Theorem

In the following table we give the values of $\mu_p(M)$ in all Mathieu simple groups, which can be obtained by the Atlas [6] easily.

Table 1. **The values of $\mu_p(M)$ in Mathieu simple groups**

M	$\lvert M\rvert$	$\mu_2(M)$	$\mu_3(M)$	$\mu_5(M)$	$\mu_7(M)$	$\mu_{11}(M)$	$\mu_{23}(M)$
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	$\frac{9}{16}$	$\frac{2}{9}$	$\frac{1}{5}$		$\frac{2}{11}$	
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	$\frac{43}{64}$	$\frac{8}{27}$	$\frac{1}{5}$		$\frac{2}{11}$	
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	$\frac{39}{128}$	$\frac{1}{9}$	$\frac{1}{5}$	$\frac{2}{7}$	$\frac{2}{11}$	
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$\frac{49}{128}$	$\frac{2}{9}$	$\frac{1}{5}$	$\frac{2}{7}$	$\frac{2}{11}$	$\frac{2}{23}$
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$\frac{539}{1024}$	$\frac{13}{27}$	$\frac{1}{5}$	$\frac{2}{7}$	$\frac{1}{11}$	$\frac{2}{23}$

Claim 1. *G is perfect.*

Assuming that G is not perfect, and let p be a prime divisor of $|G/G'|$. By applying Lemma 2.1, we have $\mu_p(G) \geq \mu_p(G/G') \geq \mu_p(Z_p) = 1 - \frac{1}{p} \geq \frac{1}{2}$. In view to Table 1, if $M = M_{11}$, we observe that $\mu_p(G) < \frac{1}{2}$ for $p \neq 2$, a contradiction. Now if $p = 2$, there exists a maximal normal subgroup of G, say N, such that $G/N \cong Z_2$, and then $\mu_2(N) \leq \frac{1}{8}$. By Lemma 2.2, we have $\frac{9}{16} = \frac{1}{2} + \sum_{x \in X} (1 - \mu_2(C_G(x)))$, that is $\frac{1}{16} = \sum_{x \in X} (1 - \mu_2(C_G(x)))$. So, G has one conjugacy class of non-trivial 2-elements. Since $\mu_2(N) \leq \frac{1}{8}$, which implies $\mu_2(N) = \frac{1}{8}$ by Lemma 2.1 and 2.3. In light of Table 1, it is evident that $\mu_3(N) \leq \frac{4}{9}, \mu_5(N) \leq \frac{2}{5}, \mu_{11}(N) \leq \frac{4}{11}$, these contradict Lemma 2.1. Therefore, N is perfect. Since G/N is abelian and $N \trianglelefteq G$, it follows that $G' \trianglelefteq N$, and then $N/G' \leq G/G'$. Let L be a maximal normal subgroup of N. This implies that N/L is a non-abelian simple group, hence $N = G'$.

Let $G > N > N_0 \geq N_1 > 1$ be a normal series of G, where N_1 is a minimal normal subgroup of G. Note that $\mu_2(N) = \frac{1}{8}$, it follows that the order of 2-elements in N are 2 and N has only one conjugacy class of 2-elements. Thus, any $\{2\}$-subgroup of N is a elementary abelian group. If N_0 has an element of order 2, then all 2-elements of N are in N_0. Hence $|N/N_0|$ is a prime. and So N/N_0 is solvable, a contradiction. Now we have $2 \nmid |N_0|$, it implies N_0 is solvable, and yielding N_1 is an elementary abelian group. Moreover, since N/N_0 is a simple group, $|N/N_0|$ has at least three prime divisors. Thus $|N_0|$ is the product of two prime, a prime or 1. Next we divide three cases.

Case I. Assume that $|N_0|$ is the product of two primes. If $N_0 = N_1$, then $N_1 \cong Z_3^2$. Thus $G/C_G(N_1) \lesssim \mathrm{GL}(2, 3)$. Moreover, N_1 may also be isomorphic to $Z_{11} \times Z_3$, $Z_{11} \times Z_5$ or $Z_5 \times Z_3$.

Therefore, we have $G/C_G(N_1) \lesssim Aut(Z_{33})$, $G/C_G(N_1) \lesssim Aut(Z_{11} \times Z_5)$ or $G/C_G(N_1) \lesssim Aut(Z_5 \times Z_3)$. But the fact that $\mathrm{GL}(2,3)$, $Aut(Z_{33})$, $Aut(Z_{11} \times Z_5)$ and $Aut(Z_5 \times Z_3)$ are solvable, it follows that $N \leq C_G(N_1)$, hence $N_0 = Z(N)$. If there exist a 2-element in G that is not centered but normalizes N_1, then only the 2-element in G-N normalizes N_1, while the 2-element in N is both centered and normalizes N_1. Considering the case of $|N_1|$ is the product of primes 5 and 11, then $\mu_5 \geq \frac{2}{5}$, $\mu_{11}(G) \geq \frac{5}{11}$, a contradictory. Now $|N_1| = 9$, then $N_0 = N_1$. Since $(|N_0|, |G/N|) = 1$, $N = N_0 \times N/N_0$ can be obtained. In such case, it is clear that $\mu_3(G) \geq \mu_3(N_0 \rtimes Z_2)$, which implies $\mu_3(G) \leq \frac{4}{9}$, a contradiction.

Case II. $|N_0|$ is a prime. We can get $N_1 \cong Z_3$, Z_5 or Z_{11}. When $|N_1| = 3$, it is obvious that $\mu_3(G) \geq \frac{1}{3}$, contradiction. If $|N_1| = 5$, then $\mu_5(G) \geq \frac{2}{5}$, a contradiction. If $|N_1| = 11$, then $\mu_{11}(G) \geq \frac{5}{11}$, a contradiction.

Case III. $|N_0| = 1$, it is easy to prove that N is a simple group. According to Table of Atlas [6], there does not exist such simple group, also a contradiction.

When $M = M_{22}$ or M_{23}, the fact that $\mu_p(M) < \frac{1}{2}$ for all p, which is a contradiction.

When $M = M_{12}$, the fact that $\mu_p(M) < \frac{1}{2}$ if $p \neq 2$, which contradicts $\mu_r(G) \geq \frac{1}{2}$. Now if $p = 2$, there exist a maximal normal subgroup of G, say N, then $G/N \cong Z_2$. Furthermore, $N_{5'}$ is a 5'-normal subgroup of G, such that $G/N_{5'} \cong S$, where S is non-abelian simple group. As per Lemma 2.5, we have $G/(N \cap N_{5'}) \cong Z_2 \times S$. Since S is a non-abelian simple group, its prime divisors are at least 3. Referring to Table of Atlas (cf. [6]), S may be isomorphic to A_5 , A_6, $L_2(11)$, M_{11} and M_{12}. However, $\mu_5(A_5) = \mu_5(A_6) = \mu_5(L_2(11)) = \frac{2}{5} > \mu_5(G)$, a contradictory. Therefore, $S \cong M_{11}$ or M_{12}, it follows that $G/(N \cap N_{5'}) \cong Z_2 \times M_{11}$ or $G/(N \cap N_{5'}) \cong Z_2 \times M_{12}$. It is easy to compute $\mu_2(Z_2 \times M_{11}) = \frac{1}{2} + \frac{1}{2} \cdot \frac{9}{16} = \frac{25}{32} > \mu_2(G)$ and $\mu_2(Z_2 \times M_{11}) = \frac{1}{2} + \frac{1}{2} \cdot \frac{43}{64} = \frac{107}{128} > \mu_2(G)$, leading to $G/(N \cap N_{5'}) \ncong Z_2 \times M_{11}$ and $G/(N \cap N_{5'}) \ncong Z_2 \times M_{12}$, yielding $G/N \ncong Z_2$.

In view to Table 1, when $M = M_{24}$, it is evident that $\mu_p(G) < \frac{1}{2}$ for $p \neq 2$, a contradiction. Now if $p = 2$, there exist a maximal normal subgroup of G, say N, which leads to $G/N \cong Z_2$. The following steps are the same as in the previous paragraph, then $G/(N \cap N_{5'}) \cong Z_2 \times S$. We deduce from Table of Atlas [6] that S may be isomorphic to A_5 , A_6, $L_2(11)$, $L_2(23)$, M_{11}, A_8, M_{12}, M_{22}, M_{23} and M_{24}. However, by comparing the value of μ yields $S \cong A_8$ or M_{24}. It follows that $G/(N \cap N_{5'}) \cong Z_2 \times A_8$ or $G/(N \cap N_{5'}) \cong Z_2 \times M_{24}$. It is easy to compute $\mu_2(Z_2 \times A_8) = \frac{93}{128} > \mu_2(G)$ and $\mu_2(Z_2 \times M_{24}) = \frac{1563}{2048} > \mu_2(G)$, leading to $G/(N \cap N_{5'}) \ncong Z_2 \times A_8$ and $G/(N \cap N_{5'}) \ncong Z_2 \times M_{24}$, that is $G/N \ncong Z_2$.

Hence, G is perfect.

Claim 2. *G/N is a simple group.*

By Lemma 2.3, we have if $\mu(G) = \mu(M)$, yielding $|G| = |M|$. Denote by N the largest p'-normal subgroup in G. Since $\mu_p(G) = \frac{1}{p}$ or $\frac{2}{p}$, we deduce from Lemma 2.4 that $\mu_p(G/N) = \mu_p(G)$, hence G/N has minimal normal subgroup N_1/N. Note that $p \parallel |G|$, it implies that N_1/N is simple group. If $N_1/N \cong Z_p$, then G/N has normal Sylow p-subgroup N_1/N, we can set $G/N \cong Z_p \rtimes \overline{H}$. According to the N/C Theorem, we have $\overline{H}/C_{\overline{H}}(Z_p) \lesssim Z_{p-1}$. As G is complete group, which implies that $\overline{H}/C_{\overline{H}}(Z_p) = 1$, yielding $G/N \cong Z_p \times \overline{H}$, a contradiction. If $N_1/N \cong S$ (S is non-abelian simple), note that N_1/N is unique minimal normal of G/N, so $S \lesssim G/N \lesssim Aut(S)$. We know that the outer automorphism groups of finite simple groups are solvable. As G is perfect, we have G/N is simple.

By Table 1 we easily see $\mu_p(G) = \frac{1}{p}$ or $\frac{2}{p}$ for all prime factor except 2 and 3. If G has two different normal subgroups N_1 and N_2, which are the largest p_1' and p_2' normal subgroups respectively, such that $G/N_1 \cong S_1$, $G/N_2 \cong S_2$, we conclude that $G/(N_1 \cap N_2) \cong S_1 \times S_2$. According to the value of $\mu_3(G)$, it follows that

$$\mu_3(G) \geq \mu_3(S_1 \times S_2) = 1 - (1 - \mu_3(S_1))(1 - \mu_3(S_2)). \tag{3.1}$$

If both S_1 and S_2 have factor 3, the inequality (3.1) does not hold by easy calculation, a contradiction. Thus, there is no 3-factor in one of S_1 and S_2, which can be set as S_1. It follows using classification of simple group that $S_1 \cong S_z(2^{2m+1})$, the power of 2 of G/N is $2^{2m+1} \geq 2^6 (m \geq 1)$. Furthermore, $|M|_2 \leq 2^{10}$, it follows that $m \leq 2$, that is $S_1 \cong S_z(8)$ or $S_z(32)$. It is easy to compute $\mu_7(S_z(8)) = \frac{3}{7} > \mu_7(G)$, a contradiction. Moreover, $|S_z(32)| \nmid |G|$, a contradiction. Therefore, $N_1 = N_2$, which implies that there exist π'-maximal normal subgroup in G, where π' is the prime graph component not containing 2 and 3. This leads to $G/N \cong S$.

Now N is $\{2,3\}$-group. When $M = M_{11}$, M_{22} and M_{23}, if $3 \big| |N|$, then $\mu_3(G) \geq \mu_3(S) \geq \frac{1}{3}$, which contradicts $\mu_3(G) \leq \frac{2}{9}$. Hence N is $\{2\}$-group.

When $M = M_{12}$, let N have a Sylow 3-subgroup P_3, it is obvious that $|P_3| = 3$ or 9. Since $G = NN_G(P_3)$, it follows that $G/N \cong N_G(P_3)/(N \cap N_G(P_3))$, we obtain that the normal series $N_G(P_3) > N \cap N_G(P_3) > P_3 > 1$. Furthermore, we known that $C_G(P_3) \trianglelefteq N_G(P_3)$ and there is no inclusion relationship between $C_G(P_3)$ and $N \cap N_G(P_3)$, and so $N_G(P_3) = C_G(P_3)(N \cap N_G(P_3))$. Thus, we conclude that $N_G(P_3)/(N \cap N_G(P_3)) \cong C_G(P_3)/(N \cap C_G(P_3)) \cong S$. Note that $N \cap C_G(P_3) \cong P_3 \times C_1$, it can be shown that $C_G(P_3)/C_1/(N \cap C_G(P_3))/C_1 \cong S$. The fact that $P_3 \cong Z_3$, Z_9 or Z_3^2, it is clear that $P_3 \leq Z(G)$. Applying Lemma 2.2 and 2.6, we have $\mu_3(G) = \mu_3(G/P_3) + \sum\limits_{x \in X}(1 - \mu_3(C_G(x)))$, which implies

$$\begin{aligned}
\mu_3(G) &= \mu_3(G/N) + \sum_{x \in X}(1 - \mu_3(C_G(P_3)/C_1)) \\
&= \mu_3(G/N) + \sum_{x \in X} \frac{(1 - \mu_3(C_G(P_3)/C_1/(N \cap C_G(P_3))/C_1))}{|P_3|} \\
&= \mu_3(G/N) + \sum_{x \in X} \frac{1}{|P_3|}(1 - \mu_3(S))
\end{aligned}$$

Moreover, since $\mu_3(S) \leq \mu_3(G)$ and $|P_3| = 3$ or 9, it is evident that $\mu_3(S) = \frac{1}{3}, \frac{1}{9}, \frac{2}{9}$ or $\frac{4}{9}$. However, we can obtain the above equation does not hold for all the values of $\mu_3(S)$ by calculation, a contradiction. Hence N is a $\{2\}$-group.

Claim 3. *$G \cong M$, where M is Mathieu simple Groups.*

Note that N is $\{2\}$-group of G if $|G| < 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. Since $|G| = |M|$ and $\mu(G) = \mu(M)$, referring to [6, Page 239-242], it follows that S can be uniquely determined as M_{11}, M_{12}, M_{22} and M_{23}, respectively. It is evident that $G \cong M$ for $|G| < 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. Now if $|G| = |M_{24}|$, we have N is $\{2,3\}$-group in G. Referring to [6, Page 239-242], S is only isomorphic to M_{23} or M_{24}. Furthermore, we conclude from Table 1 that $\mu_{11}(M_{23}) = \frac{2}{11} > \mu_{11}(M_{24}) = \frac{1}{11}$, which contradicts Lemma 2.1. The fact that $|G| = |M_{24}|$ and $\mu(G) = \mu(M_{24})$, it follows that $G \cong M_{24}$.

In summary, G is isomorphic to Mathieu simple Groups. This concludes the proof.

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