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ROUGH CONVERGENCE IN A-METRIC SPACES

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ABSTRACT. The purpose of this study is to define rough convergence of sequences in an A-metric space, discuss its fundamental properties, and examine the relationships between rough convergence and rough Cauchy sequence in A-metric spaces.

1. Introduction

Fréchet [7] first introduced the idea of metric space in 1906. Many researchers have been interested in the concept of metric spaces and worked on generalizations of metric spaces in the years that followed. For those interested in studying the generalization of metric spaces, see the research papers in [5, 8, 10, 11, 15]. The idea of A-metric spaces, which resulted from these investigations, was initially presented by Abbas et al. [1] in 2015 as a generalization of the S-metric spaces.

Phu [12] presented the notions of rough limit points and roughness degree in addition to the concept of rough convergence. In finite dimensional normed linear spaces, Phu [12, 13] examined the basic properties of this novel notion, and [14] later extended the findings to infinite dimensional spaces. Many researchers who studied it in the years that followed were interested in the concept of rough convergence, and this idea was expanded to new areas. (See [, 3,]).

Recently, Banerjee and Mondal [4] examined some fundamental properties of sequences regarding rough convergence in a cone metric space. Later, Mondal and Khatun [9] investigated the notion of rough convergence of sequences in an S-metric space.

In this paper, a similar approach, we present the idea of rough convergence of sequences in A-metric spaces using the ideas of rough convergence and A-metric spaces. We explore the basic properties of rough convergence of sequence and rough Cauchy sequence in A-metric spaces. Also, we examine the relationships between rough Cauchy sequences and rough convergence in A-metric spaces. Furthermore, we prove some results using a method similar to [4, 9, 12].

2. Preliminaries

In this section, we recall the concept of A-metric space, rough convergence and some fundamental definitions and notations (See [1, 12, 13]).

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Definition 2.1. *[1] Let X be a nonempty set. A function $A : X^n \to [0,\infty)$ is called an A-metric on X if for any $x_i, a \in X, i = 1,2,...,n$ the following conditions hold;*

(A1) $A(x_1, x_2, \ldots, x_{n-1}, x_n) \geq 0$,

(A2) $A(x_1, x_2, \ldots, x_{n-1}, x_n) = 0 \Leftrightarrow x_1 = x_2 = ... = x_n$,

(A3) $$\begin{aligned} A(x_1, x_2, ..., x_{n-1}, x_n) &\leq A(\underbrace{x_1, x_1, ..., x_1}_{n-1}, a) + A(\underbrace{x_2, x_2, ..., x_2}_{n-1}, a) \\ &+ \cdots + A(\underbrace{x_n, x_n, ..., x_n}_{n-1}, a). \end{aligned}$$

Also, the pair (X, A) is called an A-metric space.

Example 2.1. *[1] Let $X = \mathbb{R}$. Define a function $A : X^n \to [0,\infty)$ by*

$$A(x_1, x_2, ..., x_{n-1}, x_n) = \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|.$$

Then (X, A) is an A-metric space.

Proposition 2.1. *[1] Let (X, A) be an A-metric space and for all $x, y, z \in X$. Then the following is satisfied:*

(i) $A(x, x, ..., x, y) = A(y, y, ..., y, x)$.

(ii) $A(x, x, ..., x, y) \leq (n-1)A(x, x, ..., x, z) + A(y, y, ..., y, z)$.

(iii) $A(x, x, ..., x, z) \leq (n-1)A(x, x, ..., x, y) + A(z, z, ..., z, y)$.

Definition 2.2. *[1] Let (X, A) be an A- metric space. For given $r > 0$ and $x \in X$ the open ball $B_A(x, r)$ and the closed ball $\overline{B}_A(x, r)$ are defined as follows:*

$$B_A(x, r) = \{y \in X : A(y, y, ..., y, x) < r\}$$
$$\overline{B}_A(x, r) = \{y \in X : A(y, y, ..., y, x) \leq r\}.$$

Definition 2.3. *[1] Let (X, A) be an A- metric space. A subset B of X is said to be an open set if for an $r > 0$ such that $B_A(x, r) \subset B$. A subset $F \subset X$ is called closed, if $X \setminus F$ is open.*

Definition 2.4. *[1] Let (X, A) be an A- metric space. (X, A) is said to be bounded if there exists an $r > 0$ such that $A(y, y, ..., y, x) \leq r$ for every $x, y \in X$. Otherwise, X is unbounded.*

Definition 2.5. *[1] Let (X, A) be an A- metric space and let (x_t) be a sequence in X:*

(i) The sequence (x_t) is said to be convergent to x, if for each $\varepsilon > 0$ there exists a natural number t_0 such that $A(x_t, x_t, ..., x_t, x) < \varepsilon$ for every $t \geq t_0$.

(ii) The sequence (x_t) is said to be a Cauchy sequence if for each $\varepsilon > 0$ there exists a $t_0 \in \mathbb{N}$ such that for all $t, m \geq t_0$ we have $A(x_t, x_t, ..., x_t, x_m) < \varepsilon$.

Lemma 2.1. *Let (X, A) be an A- metric space. Every closed set in (X, A) contains all its limit points.*

Proof. The proof is simple and similar to an ordinary metric space case. □

Definition 2.6. *[12] Let $(X, \|.\|)$ be a normed space and (x_t) be a sequence in X:*

(i) *The sequence (x_t) is said to be r-convergent to x if for any $\varepsilon > 0$, there exists a natural number t_0 such that for all $t \geq t_0$ we have $\|x_t - x\| < r + \varepsilon$, for $r > 0$.*

(ii) *The sequence (x_t) is said to be a rough Cauchy sequence if for each $\varepsilon > 0$, there exists a natural number t_0 such that for all $t, m \geq t_0$ we have $\|x_m - x_t\| < \rho + \varepsilon$, for $\rho > 0$. ρ is roughness degree of (x_t).*

Lemma 2.2. *[14] Let a sequence (x_t) be r-convergent and $\mathrm{LIM}^r x_t \neq \emptyset$. In this case the sequence (x_t) is a ρ-Cauchy sequence for every $\rho \geq 2r$. This bound for the Cauchy degree cannot be generally decreased.*

3. Main Results

In this section, we introduce the notions of rough convergence and rough Cauchy sequences in an A-metric space. Also we discuss fundumental properties of this concepts. Later, we study the relations between rough convergence and rough Cauchy sequences in an A-metric space.

Definition 3.1. *Let (x_t) be a sequence in an A-metric space (X, A) and r be a non-negative real number. (x_t) is said to be rough convergent to x if for any $\varepsilon > 0$, there exists a natural number t_0 such that for all $t \geq t_0$ we have*

$$A(x_t, x_t, ..., x_t, x) < r + \varepsilon \tag{3.1}$$

or equivalently, if

$$\limsup A(x_t, x_t, ..., x_t, x) < r.$$

We denote it

$$x_t \xrightarrow{(X,A)}_r x.$$

The set

$$\mathrm{LIM}^r_A x_t := \{x \in X : x_t \xrightarrow{r} x\}$$

is called the r-limit set of the sequence (x_t). If $\mathrm{LIM}^r_A x_t \neq \emptyset$, then we say that (x_t) is r-convergent. Also, r is called the degree of convergence of the sequence (x_t). If $r = 0$, then we get the ordinary convergence in A-metric space again.

Remark 3.1. *Every convergent sequence in A-metric space is rough convergent.*

But the reverse of this statement may not be true. The following example shows that a rough convergent sequence in an A-metric space may not be convergent in that space.

Example 3.1. *Let $X = \mathbb{R}$. Define a function $A : X^n \to [0, \infty)$ by*

$$A(x_1, x_2, ..., x_{n-1}, x_n) = \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|.$$

It is easy to see that $(\mathbb{R}, A)$ be an A-metric space. Let (x_t) be a sequence in X defined by $(x_t) = (-1)^t$ for all $t \in \mathbb{N}$. It is obvious that (x_t) is not a convergent sequence in X. Because if $x \in \mathbb{R}$, then $A(x_t, x_t, ..., x_t, x) = (n-1) \mid (-1)^t - x \mid$. So $A(x_t, x_t, ..., x_t, x)$ equals to either $(n-1) \mid 1 + x \mid$ or $(n-1) \mid 1 - x \mid$. If $K = \min\{(t-1) \mid 1 + x \mid, (t-1) \mid 1 - x \mid\}$, then for $\varepsilon < K$, there exists infinitely many t for which $A(x_t, x_t, ..., x_t, x) < \varepsilon$ does not hold. In this case, (x_t)

is not a convergent sequence in X. However $r = \max\{(t-1) \mid 1+x \mid, (t-1) \mid 1-x \mid\}$, then for any $\varepsilon > 0$ we have $A(x_t, x_t, ..., x_t, x) < r + \varepsilon$ for all $t \in \mathbb{N}$. So (x_t) is an r-convergent sequence in an A-metric space.

Let's begin by converting some of the properties of classical convergence into rough convergence in an A-metric space. A sequence's limit is unique if it converges. This characteristic is not upheld for rough convergence with roughness degree $r > 0$ and only has the following analogy:

Theorem 3.2. *Let (X, A) be an A-metric space and let (x_t) be a sequence in X. We get $diam(\mathrm{LIM}^r_A x_t) \leq nr$.*

Proof. We have to demonstrate that

$$diam(\mathrm{LIM}^r_A x_t) = sup\left\{A(y, y, ..., y, z) : y, z \in \mathrm{LIM}^r_A x_t \leq nr\right\},$$

where (X, A) is an A-metric space. Assume the contrary to be true:

$$diam(\mathrm{LIM}^r_A x_t) > nr$$

then, there exist $y, z \in \mathrm{LIM}^r_A x_t$ satisfying

$$d := \ A(y, y, ..., y, z) > nr.$$

For any $\varepsilon \in (0, \frac{d}{n} - r)$, there is a $t_\varepsilon = \max\{t_1, t_2\} \in \mathbb{N}$ such that for $t \geq t_\varepsilon$, $A(x_t, x_t, ..., x_t, y) < r + \varepsilon$ for all $t \geq t_1$ and $A(x_t, x_t, ..., x_t, z) < r + \varepsilon$ for all $t \geq t_2$. This implies from $(A3)$ and Propositon 2.1

$$\begin{aligned} A(y, y, ..., y, z) &\leq (n-1)A(y, y, ..., y, x_t) + A(z, z, ..., z, x_t) \\ &= (n-1)A(x_t, x_t, ..., x_t, y) + A(x_t, x_t, ..., x_t, z) \\ &< n(r + \varepsilon) \end{aligned}$$

we condiser $\varepsilon = \frac{d}{n} - r$, then we get

$$A(y, y, ..., y, z) \ < \ d,$$

which conflicts with $d = A(y, y, ..., y, z)$. Hence $diam(\mathrm{LIM}^r_A x_t) \leq nr$. $\square$

Theorem 3.3. *Let (X, A) be an A-metric space and let (x_t) be a sequence in X. If $x_t \xrightarrow{(X,A)}_r x$, then it implies that $\overline{B}_A(x, r) = \mathrm{LIM}^r_A x_t$ as (x_t) converges to x.*

Proof. Assume that (x_t) is convergent to x. Let $y \in \overline{B}_A(x, r)$ and let any given $\varepsilon > 0$. So there exists a natural number t such that $A(x_t, x_t, ..., x_t, x) < \frac{\varepsilon}{n-1}$ for all $t \geq t_\varepsilon$ and we also have $A(y, y, ..., y, x) \leq r$. Hence for $t \geq t_\varepsilon$ we have

$$\begin{aligned} A(x_t, x_t, ..., x_t, y) &\leq (n-1)A(x_t, x_t, ..., x_t, x) + A(y, y, ..., y, x) \\ &< (n-1)\frac{\varepsilon}{n-1} + r \\ &= \varepsilon + r. \end{aligned}$$

Therefore $y \in \mathrm{LIM}^r_A x_t$. So,

$$\overline{B}_A(x, r) \subset \mathrm{LIM}^r_A x_t. \tag{3.2}$$

Let $z \in \mathrm{LIM}_A^r x_t$ and any given $\varepsilon > 0$. Now we can choose $t_1, t_2 \in \mathbb{N}$ such that

$$A(x_t, x_t, ..., x_t, z) < r + \frac{\varepsilon}{n} \text{ for all } t \geq t_1 \text{ and } A(x_t, x_t, ..., x_t, x) < \frac{\varepsilon}{n}$$

holds for every $t \geq t_2$. If $t = max\{t_1, t_2\}$, then

$$A(x_t, x_t, ..., x_t, z) < r + \frac{\varepsilon}{n} \quad \text{and} \quad A(x_t, x_t, ..., x_t, x) < \frac{\varepsilon}{n},$$

for all $t \geq t_\varepsilon$. From (A3) and Proposition 2.1, we can write

$$\begin{aligned} A(x, x, ..., x, z) &\leq (n-1)A(x, x, ..., x, x_t) + A(z, z, ..., z, x_t) \\ &= (n-1)A(x_t, x_t, ..., x_t, x) + A(x_t, x_t, ..., x_t, z) \\ &< n\frac{\varepsilon}{n} + r \\ &= \varepsilon + r, \end{aligned}$$

for all $t \geq t_\varepsilon$. Then, we have $A(x, x, ..., x, z) < r + \varepsilon$ for any $\varepsilon > 0$. Hence $A(x, x, ..., x, z) \leq r$ holds. Therefore $z \in \overline{B}_A(x, r)$. So,

$$\mathrm{LIM}_A^r x_t \subset \overline{B}_A(x, r). \tag{3.3}$$

Consequently, by (3.2) and (3.3), we write

$$\mathrm{LIM}_A^r x_t = \overline{B}_A(x, r).$$

□

The uniqueness of limit (of classical convergence) is obviously a special case of the latter property, because if $r = 0$, then $diam(\mathrm{LIM}_A^r x_t) = nr = 0$, i.e. $\mathrm{LIM}_A^r x_t$ is either empty or a singleton. Now, recall the bounded sequence in an A-metric space. In (X, A) a sequence (x_t) is said to be bounded if and only if there exists a $K \in \mathbb{R}^+$ such that $A(x_t, x_t, ..., x_t, x_m) \leq K$ for all $t, m \in \mathbb{N}$.

Theorem 3.4. *Let (X, A) be an A-metric space and let (x_t) be a sequence in X. If the sequence (x_t) is an r-convergent sequence, then it is bounded.*

Proof. Let (x_t) be an r-convergent sequence in an A-metric space (X, A). Let (x_t) r-converges to x. We show that (x_t) is bounded in X. Then for any $\varepsilon > 0$, there exists natural number a t_0 such that $A(x_t, x_t, ..., x_t, x) < r + \varepsilon$ for all $t \geq t_0$. Let

$$L := \max_{1 \leq i,j \leq t_0} A(x_i, x_i, ..., x_i, x_j).$$

Let $i \leq t_0$ and $j \geq t_0$. From (A3) and Proposition 2.1, we can write

$$\begin{aligned} A(x_j, x_j, ..., x_j, x_k) &\leq (n-1)A(x_j, x_j, ..., x_j, x) + A(x_{t_0}, x_{t_0}, ..., x_{t_0}, x) \\ &< n(r + \varepsilon). \end{aligned}$$

Also

$$\begin{aligned} A(x_i, x_i, ..., x_i, x_j) &\leq (n-1)A(x_i, x_i, ..., x_i, x_k) + A(x_j, x_j, ..., x_j, x_{t_0}) \\ &< (n-1)L + n(r + \varepsilon). \end{aligned}$$

If $i \geq t_0$ and $j \leq t_0$ similarly we can write

$$A(x_i, x_i, ..., x_i, x_j) < n(n-1)(r + \varepsilon) + L.$$

The case for $i \geq t_0$ and $j \geq t_0$ we get

$$\begin{aligned} A(x_i, x_i, ..., x_i, x_j) &\leq (n-1)A(x_i, x_i, ..., x_i, x) + A(x_j, x_j, ..., x_j, x) \\ &< n(r+\varepsilon). \end{aligned}$$

If

$$T > \max\{L, n(r+\varepsilon), (n-1)L + n(r+\varepsilon), n(n-1)(r+\varepsilon) + L\},$$

then $A(x_i, x_i, ..., x_i, x_j) < T$ for all $i, j \in \mathbb{N}$. Therefore (x_t) is bounded in X. □

Theorem 3.5. *Let (X, A) be an A-metric space and let (x_t) be a sequence in X. If the sequence (x_t) is bounded, then it is r-convergent for some degree of roughness r.*

Proof. Let (x_t) be a bounded sequence in an A-metric space (X, A). So there exists a $K > 0$ such that $A(x_i, x_i, ..., x_i, x_j) \leq K$ for all $i, j \in \mathbb{N}$. Hence for any $\varepsilon > 0$ we have $A(x_i, x_i, ..., x_i, x_j) \leq K < K + \varepsilon$ for all $i, j \in \mathbb{N}$. Hence (x_t) rough converges to x_p for every $p \in \mathbb{N}$ for degree of roughness K and hence the result follows. □

Remark 3.6. *Let (x_t) be a sequence in X and (X, A) be an A-metric space. The sequence (x_t) is bounded if and only if there exists an $r \geq 0$ such that $\mathrm{LIM}_A^r x_t \neq \emptyset$. For all $r > 0$, a bounded sequence (x_t) is always contains a subsequence (x_{t_k}) with*

$$\mathrm{LIM}_A^{(x_{t_k}),r} x_{t_k} \neq \emptyset.$$

A subsequence of a convergent sequence also converges to the same limit point as the original sequence. We present the following property of rough convergence in A-metric spaces:

Proposition 3.1. *Let (X, A) be an A-metric space and let (x_t) be a sequence in X. If (x_{t_s}) is a subsequence of (x_t) then,*

$$\mathrm{LIM}_A^r x_t \subseteq \mathrm{LIM}_A^r x_{t_s}.$$

Proof. Let $x \in \mathrm{LIM}_A^r x_t$ and $\varepsilon > 0$. Then there exists a $k \in \mathbb{N}$ such that $A(x_t, x_t, ..., x_t, x) < r + \varepsilon$ for all $t \geq k$. Let $t_m > k$ for some $m \in \mathbb{N}$. Then $t_s > k$ for all $s \geq m$. Therefore $A(x_{t_s}, x_{t_s}, ..., x_{t_s}, x) < r + \varepsilon$ for all $s > m$. Hence $x \in \mathrm{LIM}_A^r x_{t_s}$ and $\mathrm{LIM}_A^r x_t \subseteq \mathrm{LIM}_A^r x_{t_s}$. □

This sequel's important result is the one that was examined in Phu [12] in a normed linear space.

Theorem 3.7. *Let (x_t) be a sequence in an A-metric space (X, A). For all $r \geq 0$, the r-limit set $\mathrm{LIM}_A^r x_t$ of a sequence (x_t) is closed.*

Proof. Let (y_m) be a sequence in $\mathrm{LIM}_A^r x_t$ converges to some y. For each $\varepsilon > 0$, by definition there are $m_{\varepsilon/n}$ and $t_{\varepsilon/n}$ such that

$$A(y_{m_{\varepsilon/n}}, y_{m_{\varepsilon/n}}, ..., y_{m_{\varepsilon/n}}, y) < \frac{\varepsilon}{n} \text{ and } A(x_t, x_t, ..., x_t, y_{m_{\varepsilon/n}}) < r + \frac{\varepsilon}{n}$$

for all $t \geq t_{\varepsilon/n}$. By (A3) and Proposition 2.1

$$\begin{aligned} A(y, y, ..., y, x_t) &\leq (n-1)A(y, y, ..., y, y_{m_{\varepsilon/n}}) + A(x_t, x_t, ..., x_t, y_{m_{\varepsilon/n}}) \\ &= (n-1)A(y_{m_{\varepsilon/n}}, y_{m_{\varepsilon/n}}, ..., y_{m_{\varepsilon/n}}, y) + A(x_t, x_t, ..., x_t, y_{m_{\varepsilon/n}}) \\ &< n\frac{\varepsilon}{n} + r \\ &= \varepsilon + r. \end{aligned}$$

Therefore $y \in \mathrm{LIM}^r_A x_t$. Consequently, $\mathrm{LIM}^r_A x_t$ is a closed set in X. □

Theorem 3.8. *Let (x_t) and (y_t) be two sequences in an A-metric space (X, A) with the property that $A(x_i, x_i, ..., x_i, y_i) \leq \frac{r}{n-1}$ for all $i \geq t_1 \in \mathbb{N}$ and $r > 0$. If (x_t) converges to $x \in X$, then (y_t) is r-convergent to x.*

Proof. Let any given $\varepsilon > 0$. Since (x_t) converges to x, for $\varepsilon > 0$, there exists a natural number t_2 such that $A(x_t, x_t, ..., x_t, x) < \varepsilon$ for all $t \geq t_2$. If we take $t_0 = \max\{t_1, t_2\}$, then for all $t \geq t_0$ from (A3) and Proposition 2.1 we get

$$\begin{aligned} A(y_t, y_t, ..., y_t, x) &\leq (n-1)A(y_t, y_t, ..., y_t, x_t) + A(x, x, ..., x, x_t) \\ &= (n-1)A(y_t, y_t, ..., y_t, x_t) + A(x_t, x_t, ..., x_t, x) \\ &\leq (n-1)\frac{r}{n-1} + \varepsilon \\ &= r + \varepsilon, \end{aligned}$$

for all $t \geq t_0$. In this case (y_t) is r-convergent to x.

□

Theorem 3.9. *Let (x_t) be a seqeuence that is r-convergent to x in an A-metric space (X, A). If a sequence $(\xi_t) \in \mathrm{LIM}^r_A x_t$ that converges to ξ, then (x_t) is $(n-1)r$-convergent to ξ.*

Proof. Since (ξ_t) converges to ξ for any given $\varepsilon > 0$, there exists a natural number t_1 such that $A(\xi_t, \xi_t, ..., \xi_t, \xi) < \frac{\varepsilon}{n}$ for all $t \geq t_1$. Also since (x_t) is r-convergent to x there exists a natural number t_2 such that $A(x_t, x_t, ..., x_t, x) < r + \frac{\varepsilon}{n}$ for all $t \geq t_2$. Let $t_0 = max\{t_1, t_2\}$ and take a ξ_m of (ξ_t) where $m > t_0$.
Then for all $t \geq t_0$ by (A3) and Proposition 2.1 we get that

$$\begin{aligned} A(x_t, x_t, ..., x_t, \xi) &\leq (n-1)A(x_t, x_t, ..., x_t, \xi_m) + A(\xi, \xi, ..., \xi, \xi_m) \\ &= (n-1)A(x_t, x_t, ..., x_t, \xi_m) + A(\xi_m, \xi_m, ..., \xi_m, \xi) \\ &< (n-1)(r + \frac{\varepsilon}{n}) + \frac{\varepsilon}{n} \\ &= (n-1)r + \varepsilon. \end{aligned}$$

Hence, $A(x_t, x_t, ..., x_t, \xi) < (n-1)r + \varepsilon$ for all $t \geq t_0$. Consequently, the result follows. □

Definition 3.2. *Let (x_t) be a sequence in an A-metric space (X, A). $\xi \in X$ is said to be a cluster point of (x_t) if for any $\varepsilon > 0$, there exists a natural number m such that $m > p$, $p \in \mathbb{N}$ we have $A(x_m, x_m, ..., x_m, \xi) < \varepsilon$.*

The concept of a closed ball has been discussed using both S-metric and cone metric space ([4, 9]). Here, we used an A-metric to examine the similarity.

Theorem 3.10. *Let (x_t) be an r-convergent sequence in an A-metric space (X, A). Then for any cluster point of (x_t), $\mathrm{LIM}^r_A x_t \subset \overline{B}_A(c, r)$ holds.*

Proof. Let $x \in \mathrm{LIM}^r_A x_t$. Then for any given $\varepsilon > 0$, there exists a natural number t_0 such that $A(x_t, x_t, ..., x_t, x) < r + \frac{\varepsilon}{n}$ for all $t \geq t_0$. Also since c is a cluster point c of (x_t), there exists a natural number m such that $m > t_0$ we have $A(x_m, x_m, ..., x_m, c) < \frac{\varepsilon}{n}$. From (A3) and

Proposition 2.1 we can write the following:

$$\begin{aligned}
A(c,c,...,c,x) &\leq (n-1)A(c,c,...,c,x_m)+A(x,x,...,x,x_m)\\
&= (n-1)A(x_m,x_m,...,x_m,c)+A(x_m,x_m,...,x_m,x)\\
&< (n-1)\frac{\varepsilon}{n}+r+\frac{\varepsilon}{n}\\
&= r+\varepsilon.
\end{aligned}$$

So, $A(c,c,...,c,x)<r+\varepsilon$. Since ε is selected arbitrarily, we have $A(c,c,...,c,x)\leq r$ and hence $A(x,x,...,x,c)\leq r$. Therefore $x\in\overline{B}_A(c,r)$. Consequently, the result follows. □

Theorem 3.11. *Let (X,A) be an A-metric space. If there exists sequences (x_t) and (y_t) in X such that $x_t \xrightarrow{(X,A)}_{r_0} x$ and $y_t \xrightarrow{(X,A)}_{r_1} y$, then for degree of roughness $r=(n-1)(r_0+r_1)$*

$$A(x_t,x_t,...,x_t,y_t) \xrightarrow{(X,A)}_{r} A(x,x,...,x,y).$$

Proof. Let (x_t) and (y_t) be two sequences rough convergent to x and y respectively in (X,A) such that $x_t \xrightarrow{(X,A)}_{r_0} x$ and $y_t \xrightarrow{(X,A)}_{r_1} y$. Then for given $\varepsilon>0$, there exist $t_1,t_2\in\mathbb{N}$ such that $t\geq t_1$ we can write $A(x_t,x_t,...,x_t,x)<r_0+\frac{\varepsilon}{2(n-1)}$ and for all $t\geq t_2$ we can write $A(y_t,y_t,...,y_t,y)<r_1+\frac{\varepsilon}{2(n-1)}$. Let $t_0=\max\{t_1,t_2\}$ and $r=(n-1)(r_0+r_1)$. Therefore for every $t\geq t_0$ from (A3) and Proposition 2.1 we get that

$$\begin{aligned}
A(x_t,x_t,...,x_t,y_t) &\leq (n-1)A(x_t,x_t,...,x_t,x)+A(y_t,y_t,...,y_t,x)\\
&\leq (n-1)A(x_t,x_t,...,x_t,x)+(n-1)A(y_t,y_t,...,y_t,y)\\
&\quad +A(y,y,...,y,x)\\
&< (n-1)(\frac{\varepsilon}{2(n-1)}+r_0)+(n-1)(\frac{\varepsilon}{2(n-1)}+r_1)\\
&\quad +A(x,x,...,x,y)\\
&= (n-1)(r_0+r_1)+\varepsilon+A(x,x,...,x,y)\\
&= r+\varepsilon+A(x,x,...,x,y)
\end{aligned}$$

which implies

$$A(x_t,x_t,...,x_t,y_t)-A(x,x,...,x,y)<r+\varepsilon. \tag{3.4}$$

On the other hand, we can write

$$\begin{aligned}
A(x,x,...,x,y) &\leq (n-1)A(x,x,...,x,x_t)+A(y,y,...,y,x_t)\\
&\leq (n-1)A(x,x,...,x,x_t)+(n-1)A(y,y,...,y,y_t)+A(x_t,x_t,...,x_t,y_t)\\
&= (n-1)A(x_t,x_t,...,x_t,x)+(n-1)A(y_t,y_t,...,y_t,y)+A(x_t,x_t,...,x_t,y_t)\\
&< (n-1)(\frac{\varepsilon}{2(n-1)}+r_0)+(n-1)(\frac{\varepsilon}{2(n-1)}+r_1)+A(x_t,x_t,...,x_t,y_t)\\
&= (n-1)(r_0+r_1)+\varepsilon+A(x_t,x_t,...,x_t,y_t)\\
&= r+\varepsilon+A(x_t,x_t,...,x_t,y_t)
\end{aligned}$$

so we have

$$A(x,x,...,x,y)-A(x_t,x_t,...,x_t,y_t)<r+\varepsilon. \tag{3.5}$$

Therefore by (3.4) and (3.5)

$$|A(x_t, x_t, ..., x_t, y_t) - A(x, x, ..., x, y)| < r + \varepsilon,$$

i.e.

$$A(x_t, x_t, ..., x_t, y_t) \xrightarrow{(X,A)}_r A(x, x, ..., x, y).$$

□

Definition 3.3. *Let (X, A) be an A-metric space and let (x_t) be a sequence in X. The sequence (x_t) is said to be a rough Cauchy sequence if for each $\varepsilon > 0$, there exists a $t_0 \in \mathbb{N}$ such that for all $t, m \geq t_0$ we have*

$$A(x_t, x_t, ..., x_t, x_m) < \rho + \varepsilon, \text{ for } \rho > 0.$$

Proposition 3.2. *Let (X, A) be an A-metric spaces and let (x_t) be a sequence in X:*

(i) Let (x_t) be a sequence in an A-metric space (X, A) with a Cauchy degree ρ. If $\rho' > \rho$, then ρ' is also a Cauchy degree of (x_t).

(ii) A sequence (x_t) in an A-metric space (X, A) is bounded if and only if there exists a $\rho \geq 0$ such that (x_t) is a ρ-Cauchy sequence.

Theorem 3.12. *Let (X, A) be an A-metric space and let (x_t) be a sequence in X. The sequence (x_t) is rough convergent if and only if (x_t) is a ρ-Cauchy sequence for every $\rho \geq nr$. This bound for the Cauchy degree cannot be generally decreased.*

Proof. Let (x_t) is rough convergent in A-metric space (X, A), i.e. $\text{LIM}^r_A x_t \neq \emptyset$. Let $x \in \text{LIM}^r_A x_t$. Then, for every $\varepsilon > 0$, there exists a $t_\varepsilon \in \mathbb{N}$ such that $m, t \geq t_\varepsilon$ implies

$$A(x_m, x_m, ..., x_m, x) \leq r + \frac{\varepsilon}{n} \text{ and } A(x_t, x_t, ..., x_t, x) \leq r + \frac{\varepsilon}{n}.$$

Now, for $m, t \geq t_\varepsilon$ from (A3) and Proposition 2.1 we get

$$\begin{aligned} A(x_m, x_m, \cdots, x_m, x_t) &\leq (n-1)A(x_m, x_m, ..., x_m, x) + A(x_t, x_t, ..., x_t, x) \\ &= nr + \varepsilon. \end{aligned}$$

Hence, (x_t) is a ρ-Cauchy sequence for $\rho \geq nr$. By Proposition 3.2, every $\rho \geq nr$ is also a Cauchy degree of (x_t).

Let (x_t) be a rough Cauchy sequence in A-metric space (X, A). Since (x_t) be a rough Cauchy sequence, then it is bounded. Hence, (x_t) is rough convergent for $\rho > 0$. It is clear that this bound nr can not be generally decreased, similar to Lemma 2.2 in [14].

□

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References

[1] M. Abbas, B. Ali, Y.I. Suleiman, Generalized coupled common fixed point results in partially ordered A-metric spaces, Fixed Point Theory Appl. 2015 (2015), 64. https://doi.org/10.1186/s13663-015-0309-2.

[2] M. Arslan, E. Dündar, Rough convergence in 2-normed spaces, Bull. Math. Anal. 10 (2018), 1-9.

[3] S. Aytar, Rough statistical convergence, Numer. Funct. Anal. Optim. 29 (2008), 291-303. https://doi.org/10.1080/01630560802001064.

[4] A.K. Banerjee, R. Mondal, Rough convergence of sequences in a cone metric space, J Anal. 27 (2019), 1179-1188. https://doi.org/10.1007/s41478-019-00168-2.

[5] B.C. Dhage, Generalized metric spaces and mapping with fixed point, Bull. Cal. Math. Soc. 84 (1992), 329-336.

[6] E. Dündar, C. Çakan, Rough convergence of double sequences, Demonstr. Math. 47 (2014), 638-651.

[7] M.M. Fréchet, Sur quelques points du calcul fonctionnel, Rend. Circ. Mat. Palermo. 22 (1906), 1-72.

[8] S. Gähler, 2-metrische Räume und ihre topologische struktur, Math. Nachr. 26 (1963) 115-148. https://doi.org/10.1002/mana.19630260109.

[9] R. Mondal, S. Khatun, Rough convergence of sequences in a S-metric space, (2022). http://arxiv.org/abs/2204.04696.

[10] Z. Mustafa, B. Sims, Some remarks concerninig D-metric spaces, in: Proceedings of the International Conferences on Fixed Point Theory and Applications, Valencia, Spain, (2003), 189-198.

[11] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7 (2006), 289-297.

[12] H.X. Phu, Rough convergence in normed linear spaces, Numer. Funct. Anal. Optim. 22 (2001), 199-222. https://doi.org/10.1081/NFA-100103794.

[13] H.X. Phu, Rough continuity of linear operators, Numer. Funct. Anal. Optim. 23 (2002), 139-146. https://doi.org/10.1081/NFA-120003675.

[14] H.X. Phu, Rough convergence in infinite dimensional normed spaces, Numer. Funct. Anal. Optim. 24 (2003), 285-301. https://doi.org/10.1081/NFA-120022923.

[15] S. Sedghi, N. Shobeand A. Aliouche, A generalization of fixed point theorem in S-metric spaces, Mat. Vesnik. 64 (2012), 258-266.