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CONSTRUCTIVE COUNTERPARTS OF SOME INTERIOR IDEALS IN A CO-QUASIORDERED SEMIGROUP

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Abstract. In this paper, as a further generalization of co-ideals in semigroup with apartness ordered under a co-quasiorder relation, we introduce the notions of interior co-ideals, quasi-interior co-ideals and weak-interior co-ideals. Additionally, we study the properties of these co-ideals of co-quasiordered semigroup with apartness and their interrelationships.

1. Introduction

This report is within the framework of Bishop's constructive mathematics [2, 12, 13, 22, 28]. The 'principle of excluding the third' is not a logical axiom in this principled orientation. This makes it possible to observe sets as relational systems $(S, =, \neq)$ where '$\neq$' is an apartness relation on S compatible with the equality. Therefore, any formula that directly or indirectly contains the equality, has its dual built by the apartness. Thus, for example, the internal binary operation $w : S \times S \longrightarrow S$ should also satisfy the condition

$$(\forall x, y, u, v \in S)(w(x, y) \neq w(u, v) \implies (x \neq u \vee y \neq v)).$$

We call these structures 'algebraic structures with apartness'. They have been the focus of interest of a number of mathematicians for more than a hundred years. The specificity of commutative rings with apartness as well as the properties of their substructures have been studied for over forty years. Semigroups with apartness and semigroups with apartness ordered under co-quasiorder relation have been a subject of interest to a number of researchers for more than twenty years (for example, [3, 5–7, 19–23]). In addition to the above, in such algebraic structures, in addition to recognizable classical substructures, dual substructures can be observed, which are the peculiarity of this logical environment. Thus, for example, in addition to the substructure of ideals in the semigroup with apartness, the substructure of the 'co-ideal' in it can be identified as a constructive dual of the concept of ideals.

In this paper, as a further generalization of co-ideals in a semigroup with apartness ordered under a co-quasiorder relation, we introduce the notions of: interior co-ideals, (left, right) quasi-interior co-ideals and (left, right) weak-interior co-ideals. Additionally, we study the properties of these co-ideals of co-quasiordered semigroup with apartness and their interrelationships.

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2. Preliminaries

2.1. **Logical framework.** The logical and working framework of this report is Bishop's Constitutive Mathematics (Bish) (in sense of books [2, 12, 13]), which includes Intuitionist logic (IL) (for example [28]). Since in IL, the principle of excluding the third is not a valid formula, in the constructive algebra of Bishop's orientation many concepts (including some structures) have their own constructive dual. In that sense, the dual of the equality relation is the apartness relation. It is any consistent, symmetric and co-transitive relation extensive with respect to a given equality ([2, 13, 22, 28]):

(1) $(\forall x \in S)\neg(x \neq x)$ (consistency),

(2) $(\forall x, y \in S)(x \neq y \implies y \neq x)$ (symmetry),

(3) $(\forall x, y, z \in S)(x \neq z \implies (x \neq y \vee y \neq z))$ (co-transitivity) ,

(4) $(\forall x, y, z \in S)((x = y \wedge y \neq z) \implies x \neq z)$ (compatibility with the equality).

For example, in the field $\mathbb{R}$ of real numbers an apartness is determined as follows

$$(\forall a, b \in \mathbb{R})(a \neq b \iff (\exists k \in \mathbb{N})(|a - b| > \frac{1}{k})).$$

In the field $\mathbb{Q}$ of rational numbers an apartness is determined as

$$(\forall a, b \in \mathbb{Q})(a \neq b \iff \neg(a = b)).$$

If A and B are sets with apartness, then in Descartes' product $A \times B$ an apartness is introduced as follows

$$(\forall x, y \in A)(\forall u, v \in B)((x, u) \neq (y, v) \iff (x \neq y \vee u \neq v)).$$

Let $M_{2\times 2}$ be the semigroup of matrices of type 2×2 over $\mathbb{R}$. An apartness in $M_{2\times 2}$ can be introduced as follows

$$(\forall A, B \in M_{2\times 2})(A \neq B \iff (\exists\, i, j \in \{1, 2\})(A_{ij} \neq B_{ij})).$$

The following terms are specific to this principled-logical orientation:

- A subset A of a set with apartness S is strongly extensional in S if holds

(5) $(\forall x, y \in S)(y \in A \implies (x \neq y \vee x \in A))$;

- Let x be an element of S and A be a subset of S. The element x is said to be separate from A, and is written $x \lhd A$, if $(\forall a \in A)(a \neq x)$. The subset $A^{\lhd} =: \{x \in S : (x \lhd A)\}$ is the strong complement of A.

On the principled-philosophical aspects of this orientation it can be found in the paper [4].

2.2. **Co-quasiordered semigroup with apartness.** If the set on which equality and co-equality are observed is an algebraic structure, then we say that it is an algebraic structure with apartness. In addition to the above, constructive duals of classical substructures can be created in algebraic structures with apartness. For example, in a semigroup with apartness, the dual of the concept of ideals and its characteristics can be observed. Srvenković, Mitrović and Romano in [5,6] discussed semigroups with apartness. In this case, the internal binary operation in a semigroup with apartness is compatible with the apartness relation in the following sense

(6) $(\forall x, y, u, v \in S)(xu \neq yv \implies (x \neq y \vee u \neq v))$.

The co-quasiorder and co-order relations in semigroups with apartness was introduced and analyzed by D. A. Romano (see, for example [19–23]): A relation $\not\leq$ on a semigroup with apartness S is a co-quasiorder on S if holds

(7) $(\forall x, y \in S)(x \not\preceq y \implies x \neq y)$,
(8) $(\forall x, y, z \in S)(x \not\preceq z \implies (x \not\preceq y \vee y \not\preceq z))$ and
(9) $(\forall x, y, u, v \in s \in S)(xu \not\preceq yv \implies (x \not\preceq y \vee u \not\preceq v))$.

A co-quasiorder $\nleqslant$ on a semigroup with apartness S is a co-order on S if the following holds
(10) $(\forall x, y \in S)(x \neq y \implies (x \nleqslant y \vee y \nleqslant x))$.

If a semigroup with apartness is ordered by a co-quasiorder relation (by a co-order relation), then it is said to be co-quasiordered semigroup with apartness (res. co-ordered semigroup with apartness). It can be shown ([21], Lemma 2.2) that if $\not\preceq$ is a co-quasiorder relation on a semigroup with apartness S, then the relation

$$\not\preceq^{\lhd} =: \{(u, v) \in S \times S : (\forall x, y \in S)((x \not\preceq y \wedge (u, v) \neq (x, y))\}$$

is a quasi-order on S. M. A. Baroni also wrote in [1] about this type of relations.

In a semigroup with apartness S ordered under a co-quasiorder relation $\not\preceq$, three classes of substructures can be identified that do not exist in classical theory:

- A subset A od a semigroup with apartness S is a co-subsemugroup of S if the following holds
(11) $(\forall x, y \in S)(xy \in A \implies (x \in A \vee y \in A))$;

- A subset K of a semigroup with apartness ordered under a co-quasiorder $\not\preceq$ is a left co-ideal of S if holds
(12) $(\forall x, y \in S)(xy \in K \implies y \in K)$ and
(13) $(\forall x, y \in S)(x \in K \implies (x \not\preceq y \vee y \in K))$;

- A subset K of a semigroup with apartness ordered under a co-quasiorder $\not\preceq$ is a right co-ideal of S if holds
(14) $(\forall x, y \in S)(xy \in K \implies x \in K)$ and
(13) $(\forall x, y \in S)(x \in K \implies (x \not\preceq y \vee y \in K))$;

- A subset K of a semigroup with apartness ordered under a co-quasiorder $\not\preceq$ is a co-ideal of S if holds
(15) $(\forall x, y \in S)(xy \in K \implies (x \in K \wedge y \in K))$ and
(13) $(\forall x, y \in S)(x \in K \implies (x \not\preceq y \vee y \in K))$.

It is obvious that a (left, right) co-ideal K in a co-quasiordered semigroup with apartness S is a strongly extensional subset in S. In addition, if K is a left co-ideal in S, then the set $K^{\lhd}$ is a left ideal in S. Indeed:

(i) Let $x, y, t \in S$ such that $y \in K^{\lhd}$ and $t \in K$. Then $xy \in K$ ot $xy \neq t \in K$ by strongly extensionality of K in S. The first option would give $y \in K$ because K is a left co-ideal in S which is impossible due to $y \lhd K$. Therefore, it must be $xy \neq t \in K$. This means $xy \lhd K$.

(ii) Let $x, y, t \in S$ be such that $x \not\preceq^{\lhd} y$, $y \in K^{\lhd}$ and $t \in K$. Then $t \neq x \in x$ or $x \in K$ by co-transitivity of K in S. The second option would give $x \not\preceq y$ or $y \in K$ which contradicts the hypotheses. Therefore, it must be $x \neq t \in K$. This means $x \lhd K$.

Analogous to the previous one, it can be shown that the set $K^{\lhd}$ is a (right) ideal in S if K is a (right) co-ideal in S.

In addition to the above, it is not difficult to conclude that:

Lemma 2.1. *Every (left, right) co-ideal of a co-quasiordered semigroup with apartness S is a co-subsemigroup of S.*

Example 2.1. Let $S = \{a, b, c, d\}$ be a set with apartness. The multiplication '·' is given by

·	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

and a co-quasiorder is given as

$$\not\preceq = \{(a, c), (a, d), (b, a), (b, c), (b, d), (c, a), (c, b), (c, d), (d, a), (d, b), (d, c)\}.$$

Thus, S is a semigroup with apartness ordered under $\not\preceq$.

An interested reader can find more details about semi-groups with apartness in our review paper [22].

3. Objects

The concept of interior ideals of a semigroup S has been introduced by S. Lajos in [11] as a subsemigroup J of S such that $SJS \subseteq J$. The interior ideals of semigroups have been also studied by G. Szász in [26, 27]. In [9, 10] N. Kehayopulu and M. Tsingelis introduced the concepts of interior ideals in ordered semigroups. W. Jantanan, O. Johdee and N. Praththong in [8] and M. M. Krishna Rao in [14] also wrote about interior ideals in ordered semigroup. The concepts of weak-interior ideals and quasi-interior ideals were introduced in articles [15, 16] by M. M. Krishna Rao. D. A. Romano has analyzed the concepts of weak-interior ideals and quasi-interior ideals in ordered semigroups ([24]).

The concept of co-ideals in rings was first introduced by W. Ruitenberg in his dissertation [25]. D. A. Romano then analyzed this concept in commutative rings with apartness in his dissertation [17] which enabled him to design the concepts of co-equality and co-congruence relations (for example, [18, 20–23]). These substructures in algebraic structures were also the subject of interest of A. S. Troelstra and D. van Dalen in Chapret 8 of the book [28].

In addition to the above, this author has designed two specific order relations in semigroups with apartness (see, for example [19, 20, 22]): the concept of co-quasiordered relations and the concept of co-ordered relations. This enabled the observation of semigroups with apartness ordered under a co-quasiorder relation (under a co-order relation) compatible with semigroup operation in them.

In this paper, as a further generalization of co-ideals in a semigroup with apartness ordered under a co-quasiorder relation, we introduce the notions of:

- interior co-ideals (Section 4),
- (left, right) quasi-interior co-ideals (Section 5) and
- (left, right) weak-interior co-ideals (Section 6).

Additionally, we study the properties of these co-ideals of co-ordered semigroup with apartness and their interrelationships as constructive counterparts of the mentioned classic concepts of interior ideals, weak-interior ideals and quasi-interior ideals of a semigroup.

4. The main result: Interior co-ideals

Lajos [11] defined the concept of an interior ideal in a semigroup. Interior ideal in a semigroup was studied by Szasz [26, 27]: A non-empty subset J of a semigroup S is an interior ideal in S if iot is a subsemigroup of S and holds $SJS \subseteq J$. This means

(16) $J \neq \emptyset$,

(17) $(\forall x, y \in S)((x \in J \wedge y \in J) \implies xy \in J)$,

(18) $(\forall x, u, v \in S)(x \in J \implies uxv \in J)$.

Let a semigroup $(S, \cdot)$ is ordered by a order relation $\leqslant$. N. Kehayopulu in [9], Definition 1, in determining interior ideals in such a semigroup adds a requirement

(19) $(\forall x, y \in S)((y \in J \wedge x \leqslant y) \implies x \in J)$.

The following definition introduces the concept of (left, right) interior co-ideals in semigroup with apartness ordered under a co-quasiorder relation in it.

Definition 4.1. Let $(S, =, \neq), \cdot, \not\preceq)$ be a semigroup with apartness ordered under a co-quasiorder relation. A subset K of S is an interior co-ideal of S if holds (11), (13) and the following holds

(20) $(\forall u, v, x \in S)(uxv \in K \implies x \in K)$.

First, we show that an interior co-ideal K in a co-quasiordered semigroup with apartness S is a strongly extensional subset in S.

Lemma 4.1. *Every interior co-ideal of a co-quasiordered semigroup with apartness S is strongly extensional.*

Proof. Let $x, y \in S$ be such that $x \in K$. Then $x \not\preceq y \vee y \in K$ by (13). Thus $x \neq y \vee y \in K$ by (7). This means that the set K is a strongly extensional subset of S. $\square$

Second, let us show that the concept of interior co-ideals in co-quasi ordered semigroups is well defined.

Theorem 4.1. *Let $K\ (\neq S)$ be an interior co-ideal in a co-quasiordered semigroup with apartness S. Then the set $K^{\lhd}$ is an interior ideal in the ordered semigroup $((S, =, \neq), \cdot, \not\preceq^{\lhd})$.*

Proof. As already mentioned, the relation $\not\preceq^{\lhd}$ is a quasi-order on semigroup S. The subset $K^{\lhd}$ is inhabited due to the condition $K \neq S$.

Let $x, y, t \in S$ be arbitrary elements such that $x \lhd K$, $y \lhd K$ and $t \in K$. Then $xy \in K$ or $xy \neq t \in K$ by strongly extensionality of K in S. The first option would give $x \in K$ or $y \in K$ by (11) which is contrary to the hypotheses. Therefore, it must be $xy \neq t \in K$. This means $xy \lhd K$.

Let $x, u, v, t \in S$ be arbitrary elements such that $x \lhd K$ and $t \in K$. Then $uxv \in K \vee uxv \neq t \in K$ by strongly extensionality of K in S. The first option would give $x \in K$ by (20) which is contrary to the hypothesis $x \lhd K$. Therefore, it must be $uxv \neq t \in K$. This means $uxv \in K^{\lhd}$.

Let $x, y, t \in S$ be such that $y \in K^{\lhd}$, $x \not\preceq^{\lhd} y$ and $t \in K$. Then $t \not\preceq y \vee y \in K$. The second option is impossible. So, $t \not\preceq y$. Thus $t \not\preceq x \vee x \not\preceq y$ by co-transitivity of the relation $\not\preceq$. Since the second option is impossible, we conclude $x \neq t \in K$. This means $x \in K^{\lhd}$. $\square$

The following theorem connects the concept of co-ideals and concept of interior co-ideal in a co-quasiordered semigroup with apartness.

Theorem 4.2. *Every co-ideal of a co-quasiordered semigroup with apartness is an interior co-ideal in it.*

Proof. Let K be a co-ideal in a semigroup with apartness S ordered under a co-quasiorder $\not\preceq$. Then K is a co-subsemigroup of S which, moreover, satisfies the condition (13). Let $u, v, x \in S$ be such that $uxv \in K$. Then $ux \in K$ because K is a left co-ideal in S. Thus $x \in K$ since K is a right co-ideal in S. So, the set K is an interior co-ideal in S. □

The reverse of the previous theorem is realized under one special condition.

Theorem 4.3. *Suppose that a co-quasiordered semigroup with apartness S satisfies one additional condition:*

(A) *For every $a \in S$ there exists an element $x_a \in S$ such that $a \not\preceq^{\lhd} ax_aa$.*

Then the co-ideals and the interior co-ideals in S coincide.

Proof. Let K be an interior co-ideal of a co-quasiprdered semigroup S that satisfies additional condition (A). Let $a, b \in S$ be such that $ab \in K$. Then there exist elements $x_a, y_b \in S$ such that $a \not\preceq^{\lhd} ax_aa$ and $b \not\preceq^{\lhd} by_bb$. Thus

$$ab \not\preceq (ax_aa)(by_bb) \vee (ax_aa)(by_bb) \in K$$

by (13). The first option would be $a \not\preceq ax_aa$ or $b \not\preceq by_bb$ by (9). We have obtained a contradiction with hypothesis (A). So it has to be $(ax_aa)(by_bb) \in K$. From here it follows

$$(ax_a)a(by_bb) = (ax_aa)(by_bb) \in K \implies a \in K \text{ and}$$

$$(ax_aa)b(y_bb) = (ax_aa)(by_bb) \in K \implies b \in K$$

because K is an interior co-ideal of S. This proves that K is a co-ideal of S. □

Analogous to the previous theorem, the following theorem can also be proved:

Theorem 4.4. *Suppose that a co-quasiordered semigroup with apartness S satisfies one additional condition:*

(B) *For every $a \in S$ there exist element $x_a, y_a \in S$ such that $a \not\preceq^{\lhd} x_aa^2y_a$.*

Then the co-ideals and the interior co-ideals in S coincide.

Proof. Let K be an interior co-ideal of a co-quasiprdered semigroup S that satisfies additional condition (B). Let $a, b \in S$ be such that $ab \in K$. Then there exist elements $x_a, y_a, x_b, y_b \in S$ such that $a \not\preceq^{\lhd} x_aa^2y_a$ and $b \not\preceq^{\lhd} x_bb^2y_b$. Thus

$$ab \not\preceq (x_aa^2y_a)(x_bb^2y_b) \vee (x_aa^2y_a)(x_bb^2y_b) \in K$$

by (13). The first option would be $a \not\preceq x_aa^2y_a$ or $b \not\preceq x_bb^2y_b$ by (9). We have obtained a contradiction with hypothesis (B). So it has to be $(x_aa^2y_a)(x_bb^2y_b) \in K$. From here it follows

$$x_aa^2(y_ax_bb^2y_b) = (x_aa^2y_a)(x_bb^2y_b) \in K \implies a^2 \in K \text{ and}$$

$$(x_aa^2y_ax_b)b^2y_b = (x_aa^2y_a)(x_bb^2y_b) \in K \implies b^2 \in K$$

because K is an interior co-ideal of S. Therefore, $a \in K \wedge b \in K$ by (11). This proves that K is a co-ideal of S. □

Remark 4.1. The class of ordered semigroups with apartness that satisfies the condition (A) is recognized as a class of regular co-quasiordered semigroups wirh apartness, while the class of ordered semigroups with apartness that satisfy the condition (B) is recognized as a class of co-quasiordered intra-regular semigroups with apartness.

The family $\mathfrak{Intc}(S)$ of all interior co-ideals of a semigroup with apartness S ordered under a co-quasiorder is not empty because $\emptyset, S \in \mathfrak{Intc}(S)$.

Theorem 4.5. *The family $\mathfrak{Intc}(S)$ of all interior co-ideals of a semigroup with apartness S ordered under a co-quasiorder is a complete lattice.*

Proof. Let $\{K_i\}_{i\in I}$ be a family of interior co-ideals of a semigroup with apartness $((S,=,\neq),\cdot,\not\preceq)$ ordered under a co-quasiorder $\not\preceq$.

(a) Let $x,y \in S$ such that $xy \in \bigcup_{i\in I} K_i$. Then there exists an index $k \in I$ such that $xy \in K_k$. Thus $x \in K_k \subseteq \bigcup_{i\in I} K_i$ and $y \in K_k \subseteq \bigcup_{i\in I} K_i$ by (15). This means that the set $\bigcup_{i\in I} K_i$ is a co-ideal of S.

Let $x,u,v \in S$ be such $uxv \in \bigcup_{i\in I} K_i$. Then there exists an index $k \in I$ such that $uxv \in K_k$. Thus $x \in K_k \subseteq \bigcup_{i\in I} K_i$ by (20) because K_k is an interior co-ideal of S.

Let $x,y \in S$ be arbitrary elements such that $x \in \bigcup_{i\in I} K_i$. Then there exists an index $k \in I$ such that $x \in K_k$. Thus $x \not\preceq y \vee y \in K_k \subseteq \bigcup_{i\in I} K_i$ by (13).

This shows that the set $\bigcup_{i\in I} K_i$ is an interior co-ideal of S.

(b) Let X be the family of all interior co-ideals of S contained in $\bigcap_{i\in I} K_i$. Then $\cup X$ is the maximal interior co-ideal of S contained in $\bigcap_{i\in I} K_i$ according to part (a) of this evidence.

(c) If we put $\sqcup_{i\in I} K_i = \bigcup_i K_i$ and $\sqcap_{i\in I} K_i = \cup X$ then $(\mathfrak{Inct}(S), \sqcup, \sqcap)$ is a complete lattice. □

Corollary 4.1. *For any inhabited subset X of S there is a maximal interior co-ideal contained in X.*

Proof. The proof of this Corollary follows directly from part (b) in the proof of the previous theorem. □

Corollary 4.2. *For any element x of S there is a maximal interior co-ideal K_x of S such that $x \lhd K_x$.*

Proof. Proof of this Corollary is obtained if in the previous Corollary we take $X = \{u \in S : u \neq x\}$. □

Example 4.1. Let $S = \{0,1,2,3,4\}$ and operation '$\cdot$' defined on S as follows:

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	3	2
3	0	3	3	3	3
4	0	4	2	3	4

Then S forms a semigroup with apartness. The co-quasiorder relation on S is given by

$$\not\preceq = \{(1,0),(1,2),(1,3),(1,4),(2,0),(2,3),(3,0),(4,0),(4,3)\}.$$

Then S is a co-quasiordered semigroup with apartness. By direct verification one can establish that the sets $\{1,2,3,4\}$, $\{1,2,4\}$ and $\{1\}$ are interior co-ideals in semigroup S.

Example 4.2. The interval $S = \langle 0, 1\rangle \subseteq \mathbb{R}$ is a semigroup with apartness ordered under the co-order $\nleqslant$ determined as follows

$$(\forall a, b \in \mathbb{R})(a \nleqslant b \iff (\exists k \in \mathbb{N})(a > \frac{1}{k} + b)).$$

Co-ideals in this semigroup have a form of $\langle a, 1\rangle$ for any element $a \in \langle 0, 1\rangle$. So, the co-ideal $\langle a, 1\rangle$ is an interior co-ideal in S by Theorem 4.2.

5. The main result: Weak-interior co-ideals

Let S be a semigroup with apartness. The concept of (left, right) weak-interior co-ideals of a semigroup with apartness is created as the constructive counterpart of the notion of (left, right) weak-interior ideals in the classical case. However, if S is a semigroup with apartness ordered under a co-quasiorder, the determination of the notion of (left, right) weak-interior coideal in S must be adapted to the specific order requirement of this semigroup. These peculiarities motivate us to look at the class of weak-interior co-ideals in a semigroup with apartness and in a co-quasiordered semigroup with apartness separately.

5.1. **Weak-interior co-ideals.** M. M. Krishna Rao introduced in the paper [16] the notion of weak-interior ideal as a generalization of interior ideal of semigroup and he analyzed some features of such a newly introduced type of ideal and its connection with interior ideal:

- A non-empty subset J of a semigroup S is said to be a left weak-interior ideal of S if J is a subsemigroup of S and holds $SJJ \subseteq J$. In other words, J is a left weak-interior ideal of a semigroup S if valid

(16) $J \neq \emptyset$,

(17) $(\forall x, y \in S)((x \in J \wedge y \in J) \implies xy \in J)$,

(21) $(\forall x, u, v \in S)((u \in J \wedge v \in J) \implies xuv \in J)$.

- A non-empty subset J of a semigroup S is said to be a right weak-interior ideal of S if J is a subsemigroup of S and holds $JJS \subseteq J$. In other words, J is a right weak-interior ideal of a semigroup S if valid

(16) $J \neq \emptyset$,

(17) $(\forall x, y \in S)((x \in J \wedge y \in J) \implies xy \in J)$,

(22) $(\forall x, u, v \in S)((u \in J \wedge v \in J) \implies uvx \in J)$.

- A non-empty subset J of a semigroup S is said to be a weak-interior ideal of S if J is a subsemigroup of S and J is left and right weak-interior ideal of S.

The constructive counterparts of these types of ideals in semigroups with apartbess are introduced by the following definition:

Definition 5.1. Let $S =: ((S, =, \neq), \cdot)$ be a semigroup with apartness and let K be a subset of S.

(i) The strongly extensional subset K is a left weak-interior co-ideal of S if the condition (11) is valid and the following holds

(23) $(\forall x, u, v \in S)(xuv \in K \implies (u \in K \vee v \in K))$.

(ii) The strongly extensional subset K is a right weak-interior co-ideal of S if the condition (11) us valid and the following holds

(24) $(\forall x, u, v \in S)(uvx \in K \implies (u \in K \vee v \in K))$.

(iii) The strongly extensional subset K is a weak-interior co-ideal of S and it is a left and right weak-interior co-ideal of S.

Example 5.1. Let $\mathbb{Q}$ be a field of rational numbers, $S := \left\{ \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \mid b, d \in \mathbb{Q} \right\}$ be a semigroup of matrices over the filed $\mathbb{Q}$. The operation in S is the standard multiplication of matrices. Then S is a semigroup with apartness. Then $K =: \left\{ \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \mid d \in \mathbb{Q} \wedge d \neq 0 \right\}$ is a right weak-interior co-ideal of the semigroup S and K is neither a left co-ideal nor a right co-ideal, not a weak interior co-ideal and not an interior co-ideal of the semigroup S.

Our first proposition shows that the concept of the right weak-interior co-ideal is well defined:

Proposition 5.1. *Let $K\ (\neq S)$ be a right weak-interior co-ideal of a semigroup with apartness S. Then the set $K^{\lhd}$ is a weak-interior ideal of S.*

Proof. It needs to be proven:
$K^{\lhd} \neq \emptyset$,
$(\forall x, y \in S)((x \in K^{\lhd} \wedge y \in K^{\lhd}) \Longrightarrow xy \in K^{\lhd}$, and
$(\forall x, u, v \in S)((u \in K^{\lhd} \wedge v \in K^{\lhd}) \Longrightarrow uvx \in K^{\lhd})$.

The condition $K \neq S$ ensures that the set $K^{\lhd}$ is not empty.

Let $x, y, t \in S$ be such that $x \lhd K$, $y \lhd K$ and $t \in K$. Then $t \neq xy \vee xy \in K$ by strongly extensionality of K in S. The second option would give $x \in K \vee y \in K$ by (11) which is in contradiction with the hypotheses $x \lhd K$ and $y \lhd K$. Therefore, it must be $xy \neq t \in K$. This means $xy \in K^{\lhd}$.

Let $x, u, v, t \in S$ be arbitrary elements such that $u \lhd K$, $v \lhd K$ and $t \in K$. Then $t \neq uvx \vee uvx \in K$ by strongly extensionality of K in S. The second option would give $u \in K \vee v \in K$ according to (24) which is contrary to the hypotheses $u \lhd K$ and $v \lhd K$. So it must be $uvx \neq t \in K$. This means $uvx \in K^{\lhd}$. □

Moreover, this proposition demonstrates the importance of the requirement of strictly extensibility of the subset K in the set S.

Theorem 5.1. *Any right co-ideal of a semigroup with apartness S is a right weak-interior co-ideal of S.*

Proof. Let K be a right co-ideal of S. This means that K satisfies the condition (11). Let $x, u, v \in S$ be such that $uvx \in K$ Then $uv \in K$ because K is a right co-ideal of S. Thus $u \in K \vee v \in K$ by (11). This proves that K is a right weak -interior co-ideal of S. □

Theorem 5.2. *Any interior co-ideal of a semigroup with apartness S is a right weak-interior co-ideal of S.*

Proof. Let K be an interior co-ideal of a semigroup S. Then (11) and (20) are valid formulas. Let $x, y, v \in S$ be such that $uvx \in K$. Then $v \in K$ by (20). Thus $u \in K \vee v \in K$. So, K is a right weak-interior co-ideal of S. □

Theorem 5.3. *The family $\mathfrak{W}_{\mathfrak{r}}\mathfrak{intc}(S)$ of all right weak-interior co-ideals of a semigroup with apartness S forms a complete lattice.*

Proof. Let $\{K_i\}_{i\in I}$ be family of right weak-interior co-ideals of a semigroup with apartness S.

(a) Let $x, y \in S$ be arbitrary element s such that $x \in \bigcup_{i\in I} K_i$. Then there exists an index $k \in I$ such that $x \in K_k$. Thus $x \neq y \vee y \in K_k \subseteq \bigcup_{i\in I} K_i$. So, the set $\bigcup_{i\in I} K_i$ is strongly extensional.

Let $x, y \in S$ be such that $xy \in \bigcup_{i\in I} K_i$. Then there exists an index $k \in I$ such that $xy \in K_k$. Thus $x \in K_k \subseteq \bigcup_{i\in I} K_i$ or $y \in K_k \subseteq \bigcup_{i\in I} K_i$. this means that the set $\bigcup_{i\in I} K_i$ is a co-subsemigroup of S.

Let $u, v, x \in S$ be such that $uvx \in \bigcup_{i\in I} K_i$. Then there exists an index $k \in I$ such that $uvx \in K_k$. Thus $u \in K_k \subseteq \bigcup_{i\in I} K_i$ or $v \in K_k \subseteq \bigcup_{i\in I} K_i$. Thus, the set $\bigcup_{i\in I} K_i$ satisfies the condition (24).

Therefore, the set $\bigcup_{i\in I} K_i$ is a right weak-interior co-ideal of S.

(b) Let X be the family of all right weak-interior co-ideals of S contained in $\bigcap_{i\in I} K_i$. Then $\cup X$ is a maximal right weak-interior co-ideal of S contained in $\bigcap_{i\in I} K_i$ according to part (a) of this proof.

(c) If we put $\sqcup_{i\in I} K_i = \bigcup_{i\in I} K_i$ and $\sqcap_{i\in I} K_i = \cup X$, then $(\mathfrak{W}_{\mathfrak{r}}\mathfrak{intc}(S), \sqcup, \sqcap)$ is a complete lattice. □

Corollary 5.1. *For any subset X of a semigroup S there is a maximal right weak-interior co-ideal of S contained in X .*

Corollary 5.2. *For any element $a \in S$ there is a maximal right weak-interior co-ideal K_a of S such that $a \lhd K_a$.*

Remark 5.1. Claims for (left) weak-interior co-ideals of a semigroup with apartness can be designed without major difficulties analogously to previous claims.

5.2. **Weak-interior co-ideals in co-quasiordered semigroups.** Let $S =: ((S, =, \neq), \cdot, \not\preceq)$ be a semigroup with apartness ordered under a co-quasiorder relation $\not\preceq$. The constructive counterparts of (left, right) weak-interior ideals in a co-quasiordered semigroups with apartbess are introduced by the following definition:

Definition 5.2. Let S be a semigroup with apartness ordered under a co-quasiorder $\not\preceq$ and let K be a subset of S.

(iv) The subset K is a left weak-interior co-ideal of S if the conditions (11), (13) are valid and the following holds

(23) $(\forall x, u, v \in S)(xuv \in K \implies (u \in K \vee v \in K))$.

(v) The subset K is a right weak-interior co-ideal of S if the conditions (11), (13) are valid and the following holds

(24) $(\forall x, u, v \in S)(uvx \in K \implies (u \in K \vee v \in K))$.

(vi) The subset K is a weak-interior co-ideal of S and it is a left and right weak-interior co-ideal of S.

As it can be seen, the requirement that the subset K be strongly extensioanl is omitted. However, the (left, right) weak-interior co-ideal is certainly a strongly extensional subset in S. The extensiveness with to the apartness of these co-ideals follows from the presence of the co-quasiorder relation in the semigroup.

Lemma 5.1. *Any (left, right) weak-interior co-ideal of a co-quasiordered semigroup with apartness S is a strongly extensional.*

Proof. Let $x, y \in S$ be such that $x \in K$. Then $x \not\leq y \vee y \in K$ by (13). Thus $x \neq y \vee y \in K$ by (7). So, the set K is a strongly extensional subset in S. □

We will first show that the right weak-interior co-ideal of a co-quasiordered semigroup with apartness is correctly determined. As it is common in constructive mathematics, we will show that the complement $K^{\lhd}$ of a right weak-interior co-ideal K of a co-quasiordered semigroup with apartness S is a right weak-interior ideal of S and, moreover, it satisfies the condition

(25) $(\forall x, y \in S)((x \not\leq^{\lhd} y \wedge y \in K^{\lhd}) \implies x \in K^{\lhd})$.

This author has analyzed the concept of weak-interior ideals in ordered semigroups ([24]).

Proposition 5.2. *Let K ($\neq S$) be a right weak-interior co-ideal of a co-quasi-ordered semigroup with apartness S. Then the set $K^{\lhd}$ is a right weak-interior ideal of S and (25) holds.*

Proof. In the Proposition 5.1 it is shown that $K^{\lhd}$ is a right weak-interior ideal of S. It remains to be shown that $K^{\lhd}$ satisfies the condition (25).

Let $x, y, t \in S$ be arbitrary elements such that $x \not\leq^{\lhd} y$, $y \lhd K$ and $t \in K$. Then $t \neq x \vee x \in K$ by strongly extensionality of K in S. The second option would be to give $x \not\leq y \vee y \in K$ by (13). Both of these possibilities contradict to the hypotheses. Therefore, it must be $x \neq t \in K$. This means $x \lhd K$. Thus proving the validity of the condition (25). □

Naturally, the validity of Theorem 5.1 and Theorem 5.2 is preserved in the case of right weak-interior co-ideals in a co-quasiordered semigroup with apartness.

Theorem 5.4. *Any right co-ideal of a co-quasiordered semigroup with apartness S is a right weak-interior co-ideal of S.*

Proof. Let K be a right co-ideal of S. This means that K satisfies the conditions (11), (14) and (13). Let $x, u, v \in S$ be such that $uvx \in K$ Then $uv \in K$ because K is a right co-ideal of S. Thus $u \in K \vee v \in K$ by (11). This proves that K is a right weak-interior co-ideal of S. □

The reversal of the previous theorem can be demonstrated in one special case:

Theorem 5.5. *Let S be a co-quasiordered semigroup with apartness which satisfies the condition*

(OBS) $(\forall x \in S)(x \not\leq^{\lhd} x^2)$.

Then the right co-ideals and the right weak-interior co-ideals in S coincide.

Proof. Suppose that S is a co-quasiordered semigroup with apartness which satisfies the condition (OBS) and K is a right weak-interior co-ideal of S. Let $x, y \in S$ be arbitrary elements such that $xy \in K$. Then $xy \not\leq x^2y \vee x^2y \in K$ by (13). The first option would give $x \not\leq x^2$ by (9) which is contrary to the condition (OBS). So, it has to be $x^2y \in K$. Thus $x \in K$ by (24). This means that K is a right co-ideal od S. □

Theorem 5.6. *Any interior co-ideal of a co-quasiordered semigroup with apartness S is a right weak-interior co-ideal of S.*

We conclude this subsection with the following theorem:

Theorem 5.7. *The family $\mathfrak{W}_{\mathfrak{or}}\mathfrak{intc}(S)$ of all right weak-interior co-ideals of a co-quasiordered semigroup with apartness S forms a complete lattice.*

Proof. Let $\{K_i\}_{i \in I}$ be family of right weak-interior co-ideals of a co-quasi- ordered semigroup with apartness S. It is only necessary to prove that $\bigcup_{i \in I} K_i$ is a right weak-interior co-ideal of S because in Theorem 5.3 it is proved that $\bigcup_{i \in I} K_i$ is a right weak-interior co-ideal of S.

Let $x, y \in S$ be such that $x \in \bigcup_{i \in I} K_i$. Then there exists an index $k \in I$ such that $x \in K_k$. Thus $x \npreceq y \vee y \in K_k \subseteq \bigcup_{i \in I} K_i$ by (13). Thus, the right weak-interior co-ideal $\bigcup_{i \in I} K_i$ satisfies condition (13). □

Remark 5.2. Appropriate claims for (left) weak-interior co-ideals of a co-quasiordered semigroup with apartness can be designed without major difficulties analogously to previous claims.

6. The main result: Quasi-interior co-ideals

Determining the concept of quasi-interior co-ideals of a semigroup with apartness is somewhat different from the description of the concept of quasi-interior co-ideals of a co-quasiordered semigroup with apartness. This is the reason why the definitions and analyses of these two classes of concepts we consider separately. In order to be able to discuss the constructive counterparts of (left, right) quasi-interior ideals in co-quasiordered semigroup with apartness, we will assume that a (left, right) quasi-interior ideal J in a ordered semigroup S meet one additional requirement

$$(\forall x, y \in S)((x \preceq y \wedge y \in J) \implies x \in J).$$

This author has analyzed the concept of quasi-interior ideals in ordered semigroups ([24]).

6.1. **Quasi-interior co-ideals.**

Let S be a semigroup. The concept of (left, right) quasi-interior ideals was analyzed in papers [14–16] by M. M. Krishna Rao:

- A non-empty subset J of a semigroup S is said to be a left quasi-interior ideal of S if J is a subsemigroup of S and holds $SJSJ \subseteq J$. Thus means

(16) $J \neq \emptyset$,

(17) $(\forall x, y \in S)((x \in J \wedge y \in J) \implies xy \in J)$,

(26) $(\forall x, y, u, v \in S)((u \in J \wedge v \in J) \implies xuyv \in J)$.

- A non-empty subset J of S is said to be a right quasi-interior ideal of S if J is a subsemigroup of S and holds $JSJS \subseteq J$. This means

(16) $J \neq \emptyset$,

(17) $(\forall x, y \in S)((x \in J \wedge y \in J) \implies xy \in J)$,

(27) $(\forall x, y, u, v \in S)((u \in J \wedge v \in J) \implies uxvy \in J)$.

- A non-empty subset J of a semigroup S is said to be a quasi-interior ideal of S if it is both a left quasi-interior ideal and a right quasi-interior ideal of S.

The constructive counterparts of these notions are introduced in a semigroup with apartness by the following way:

Definition 6.1. Let $S =: ((S, =, \neq), \cdot)$ be a semigroup with apartness and let K be a strongly extensional subset of S.

(vii) The subset K is a left quasi-interior co-ideal of S if the condition (11) is valid and the following holds

(28) $(\forall x, y, u, v \in S)(xuyv \in K \implies (u \in K \vee v \in K))$.

(viii) The subset K is a right quasi-interior co-ideal of S if the condition (11) is valid and the following holds

(29) $(\forall x, y, u, v \in S)(uxvy \in K \implies (u \in K \vee v \in K))$.

(ix) The subset K is a quasi-interior co-ideal of S and it is a left and right quasi-interior co-ideal of S.

Example 6.1. Let $\mathbb{Q}$ be a field of rational numbers, $S := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,|\, a, b, d \in \mathbb{Q} \right\}$ be the semigroup of matrices over the filed $\mathbb{Q}$. The operation in S is the standard multiplication of matrices. Then S is a semigroup with apartness. Then $K =: \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \,|\, a \in \mathbb{Q} \wedge a \neq 0 \right\}$ is a left quasi-interior co-ideal of the semigroup S.

Let us show, for the sake of illustration, that the concept of left quasi-interior co-ideals of a semigroup with apartness is correctly defined:

Proposition 6.1. *Let $K\,(\neq S)$ be a left quasi-interior co-ideal of a semigroup with apartness S. Then the set $K^{\lhd}$ us a left quasi-interior ideal of S.*

Proof. It needs to be prove:

$K^{\lhd} \neq \emptyset$,

$(\forall x, y \in S)((x \lhd K\, y \lhd K) \implies xy \lhd K)$,

$(\forall x, y, u, v \in S)((u \lhd K \wedge v \lhd K) \implies xuyv \lhd K)$.

The condition $K \neq S$ ensures that the set $K^{\lhd}$ is not empty.

Let $x, y, t \in S$ such that $x \lhd K$, $y \lhd K$ and $t \in K$. Then $xy \in K$ or $xy \neq t \in K$ by strongly extensionality of K. The first option would give $x \in K \vee y \in K$ by (11). We got a contradiction according to the hypotheses. So, must be $xy \neq t \in K$. This means $xy \lhd K$.

Let $x, y, u, v, t \in S$ be arbitrary elements such that $u \lhd K$, $v \lhd K$ and $t \in K$. Then $xuyv \in K \vee xuyv \in K$ by strongly extensionality of K in S. The second option would be gibe $u \in K \vee v \in K$ by (28). This is impossible. So, must be $xuyv \neq t \in K$. This means $xuyv \lhd K$. □

Theorem 6.1. *Every left co-ideal of a semigroup with apartness S is a left quasi-interior co-ideal of S.*

Proof. let K be a left co-ideal of a semigroup with apartness S and let $x, y, t, v \in S$ be such that $xuyv \in K$. Then $v \in K$ by (12). Thus $u \in K \vee v \in K$. So, K is a left quasi-interior co-ideal of S. □

Theorem 6.2. *Every interior co-ideal of a semigroup with apartness S is a left quasi-interior co-ideal of S.*

Proof. Let K be an interior co-ideal of a semigroup with apartness S and let $x, y, u, v \in S$ such that $xu(yv) = xuyv \in K$. Then $u \in K$ by (20). Thus $u \in K \vee v \in K$. So, K is a left quasi-interior co-ideal of S. □

Theorem 6.3. *The family $\mathfrak{Q}_{\mathfrak{l}}\mathfrak{intc}(S)$ of all left quasi-interior co-ideals of a semigroup with apartness S forms a complete lattice.*

Proof. Let $\{K_i\}_{i\in I}$ be a family of left quasi-interior co-ideals of a semigroup with apartness S.

(a) Let $x, y \in S$ be such that $x \in \bigcup_{i\in I} K_i$. Then there exists an index $k \in I$ such that $x \in K_k$. Thus $x \neq y\, y \in K_k \subseteq \bigcup_{i\in I} K_i$ by Strongly extensionality of K_k in S. This means that the set $\bigcup_{i\in I} K_i$ is a strongly extensional subset in S.

Let $x, y \in S$ be such that $xy \in \bigcup_{i\in I} K_i$. Then there exists an index $k \in I$ such that $xy \in K_k$. Thus $x \in K_k \subseteq \bigcup_{i\in I} K_i$ or $y \in K_k \subseteq \bigcup_{i\in I} K_i$ by (11). This means that the set $\bigcup_{i\in I} K_i$ is a co-subsemigroup of S.

Let $x, y, t, v \in S$ be arbitrary elements such that $xuyv \in \bigcup_{i\in I} K_i$. then there exists an index $k \in I$ such that $xuyv \in K_k$. Thus $u \in K_k \subseteq \bigcup_{i\in I} K_i$ or $v \in K_k \subseteq \bigcup_{i\in I} K_i$ by (28).

This proves that the set $\bigcup_{i\in I} K_i$ is a left quasi-interior co-ideal of S.

(b) Let X be the family of all lest quasi-interior co-ideals of S included in $\bigcap_{i\in I} K_i$. Then the set $\cup X$ is a maximal lest quasi-interior co-ideal of S included in $\bigcap_{i\in I} K_i$.

(c) If we put $\sqcup_{i\in I} K_i = \bigcup_{i\in I} K_i$ and $\sqcap_{i\in I} K_i = \cup X$, then $(\mathfrak{Q}_{\mathfrak{l}}\mathfrak{intc}(S), \sqcup, \sqcap)$ is a complete lattice. □

Corollary 6.1. *For any subset X of a semigroup S there is a maximal lett quasi-interior co-ideal of S contained in X .*

Corollary 6.2. *For any element $a \in S$ there is a maximal left quasi-interior co-ideal K_a of S such that $a \lhd K_a$.*

Remark 6.1. Claims for (right) quasi-interior co-ideals of a semigroup with apartness can be designed without major difficulties analogously to previous claims.

6.2. **Quasi-interior co-ideals of co-quasiordered semigroups.** This subsection is dedicated to creating the concept of (left, right) quasi-interior co-ideals of a semigroup with apartness ordered under a co-quasiorder and analyzing its basic features.

The constructive counterpart of notion of quasi-interior ideals of an ordered semigroup is the concept of (left, right) quasi-interior co-ideals of a semigroup with apartness ordered under a co-quasiorder relation:

Definition 6.2. Let $S =: ((S, =, \neq), \cdot, \not\leq)$ be a semigroup with apartness ordered under a co-quasiorder relation and let K be a subset of S.

(x) The subset K is a left quasi-interior co-ideal of S if the conditions (11), (13) and (28) are valid.

(xi) The subset K is a right quasi-interior co-ideal of S if the conditions (11), (13) and (29) are valid.

(xii) The subset K is a quasi-interior co-ideal of S and it is ad left and right quasi-interior co-ideal of S.

Lemma 6.1. *Any left (right) quasi-interior co-ideal of S is a strongly extensional subset in S.*

Proof. Indeed, strongly extensionality of a left quasi-interior co-ideal follows directly from the validity of formula (13). Indeed, let K be a left quasi-interior co-ideal of a semigroup with apartness ordered under a co-quasiordeer relation $\not\leq$. Let $x, y \in S$ be such that $x \in K$. Then $x \not\leq y \vee y \in K$ by (13). Thus $x \neq y \vee y \in K$ by (7). This means that the set K is a strongly extensional subset in S. □

The following proposition shows that the concept of left quasi-interior co-ideals of a semigroup with apaerness ordered under a co-quasiorder is well defined:

Proposition 6.2. *Let K be a left quasi-interior co-ideal of a semigroup with apartness S ordered under a co-quasiorder $\not\leq$. Then the set $K^{\lhd}$ is a left quasi-interior ideal of S which satisfies the condition (25).*

Proof. To prove this Proposition, it suffices to prove that the set $K^{\lhd}$ satisfies the condition (25) since $K^{\lhd}$ is a left quasi-interior ideal of S according to Proposition 6.1.

Let $x, y, t \in S$ be arbitrary elements such that $x \not\leq^{\lhd} y$, $y \lhd K$ and $t \in K$. Then $t \not\leq y \vee y \in K$ by (13). The second option is impossible. Thus $t \not\leq x \vee x \not\leq y$ by co-transitivity of $\not\leq$. Since the second option is not possible, we have $t \not\leq x$, hence $x \neq t \in K$. This means that $x \lhd K$. □

The following theorems can be proved based on the corresponding theorems in the subsection 6.1, so we will omit their proofs.

Theorem 6.4. *Every left co-ideal of a co-quasiordered semigroup with apartness S is a left quasi-interior co-ideal of S.*

Theorem 6.5. *Every interior co-ideal of a co-quasiordered semigroup with apartness S is a left quasi-interior co-ideal of S.*

The reverse of the previous theorem can be proved if the co-quasiordered semigroup with apartness S satisfies one additional condition.

Theorem 6.6. *Suppose that a co-quasiordered semigroup with apartness S satisfies one additional condition:*

(C) *For every elements $a, b \in S$ the following holds $a \not\leq^{\lhd} ab$.*

Then the interior co-ideals and the left quasi-interior co-ideals in S coincide.

Proof. Suppose that a co-quasiordered semigroup with apartness S satisfies the (C) and let K be a left quasi-interior co-ideal of S. Let $u, v, x \in S$ be arbitrary elements such that $uxv \in K$. Then $uxv \not\leq uxvx \vee uxvx \in K$ by (13). The first option would give $v \not\leq vx$ which contradicts to the hypothesis (C). So, must be $uxvx \in K$. It follows $x \in K$ from here because K is a left quasi-interior co-ideal of S. This means that K is an interior co-ideal of S. □

Theorem 6.7. *The family $\mathfrak{Q}_{\mathfrak{ol}}\mathfrak{intc}(S)$ of all left quasi-interior co-ideals of a co-quasiordered semigroup with apartness S forms a complete lattice.*

Remark 6.2. Claims for (right) quasi-interior co-ideals of a co-quasiordered semigroup with apartness can be designed without major difficulties analogously to previous claims.

Thus, for example, we transform Theorem 6.6 into the following theorem:

Theorem 6.8. *Suppose that a co-quasiordered semigroup with apartness S satisfies one additional condition:*

(D) *For every elements $a, b \in S$ the following holds $a \not\leq^{\lhd} ba$.*

Then the interior co-ideals and the right quasi-interior co-ideals in S coincide.

7. Concluding comments

As the concepts of ideals are important in the classical theory of (ordered) semigroups, so the concept of co-ideals is an important substructure in the theory of semigroups with apartness in Bishop's constructive framework. The family of substructures in semigroups with apartness ordered under a co-quasiorder relation is enriched by a special class of co-ideals in such semigroups.

The contribution of this report to the theory of semigroups with apartness within Bishop's constructive orientation is a discussion of interior co-ideals, (left, right) weak-interior co-ideals and (left, right) quasi-interior co-ideals in co-quasi- ordered semigroups with apartness.

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