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KEY RENEWAL THEOREM AND ASYMPTOTICS OF THE RENEWAL MEASURE ON BOREL SETS

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Abstract. We prove a uniform key renewal theorem with submultiplicative estimate of the remainder. The asymptotic behavior of the renewal measure on Borel sets of arbitrary form is also considered.

1. Notation and assumptions

Let F be a probability distribution on $\mathbb{R}$ with finite positive mean μ and let $U = \sum_{n=0}^{\infty} F^{n*}$ be the corresponding renewal measure; here $F^{1*} := F$, $F^{(n+1)*} := F * F^{n*}$, $n \geq 1$, and $F^{0*} := \delta$, the atomic measure of unit mass at the origin, the symbol $*$ means convolution of measures.

A function $\varphi(x)$, $x \in \mathbb{R}$, is called *submultiplicative* if $\varphi(x)$ is a finite, positive, Borel-measurable function with the following properties: $\varphi(0) = 1$, $\varphi(x + y) \leq \varphi(x)\,\varphi(y)$, x, $y \in \mathbb{R}$. Here are some examples of such functions on $\mathbb{R}_+ := [0, \infty)$: $\varphi(x) = (1 + x)^r$, $r > 0$; $\varphi(x) = \exp(cx^\gamma)$ with $c > 0$ and $\gamma \in (0, 1)$; $\varphi(x) = \exp(rx)$ with $r \in \mathbb{R}$. Moreover, if $R(x)$, $x \in \mathbb{R}_+$, is a positive, ultimately nondecreasing regularly varying function at infinity with a nonnegative exponent γ (i.e., $R(tx)/R(x) \to t^\gamma$ for $t > 0$ as $x \to \infty$ [4, Section VIII.8]), then there exist a nondecreasing submultiplicative function $\varphi(x)$ and a point $x_0 \in (0, \infty)$ such that $c_1 R(x) \leq \varphi(x) \leq c_2 R(x)$ for all $x \geq x_0$, where c_1 and c_2 are some positive constants [6, Proposition]. The product of a finite number of submultiplicative functions is again a submultiplicative function.

It is well known [5, Section 7.6] that

$$-\infty < r_1 := \sup_{x<0} \frac{\log \varphi(x)}{x} \leq \inf_{x>0} \frac{\log \varphi(x)}{x} =: r_2 < \infty.$$

Consider the collection $S(\varphi)$ of all complex-valued measures $\varkappa$ such that

$$\|\varkappa\|_\varphi := \int_{\mathbb{R}} \varphi(x)\,|\varkappa|(dx) < \infty;$$

here $|\varkappa|$ stands for the total variation of $\varkappa$. The collection $S(\varphi)$ is a Banach algebra with norm $\|\cdot\|_\varphi$ by the usual operations of addition and scalar multiplication of measures, the product of two elements ν and $\varkappa$ of $S(\varphi)$ is defined as their convolution $\nu * \varkappa$ [5, Section 4.16]. The unit element of $S(\varphi)$ is the measure δ. Define the Laplace transform of a measure $\varkappa$ as

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$\widehat{\varkappa}(s) := \int_{\mathbb{R}} \exp(sx)\,\varkappa(dx)$. The Laplace transform of any $\varkappa \in S(\varphi)$ converges absolutely with respect to $|\varkappa|$ for all s in the strip $\Pi(r_1, r_2) := \{s \in \mathbb{C} : r_1 \le \Re s \le r_2\}$.

Let ν be a finite complex-valued measure. Denote by $T\nu$ the σ-finite measure with the density $v(x;\nu) := \nu((x,\infty))$ for $x \ge 0$ and $v(x;\nu) := -\nu((-\infty, x])$ for $x < 0$. In case $\int_{\mathbb{R}} |x|\,|\nu|(dx) < \infty$, $T\nu$ is a finite measure whose Laplace transform is given by $\widehat{T\nu}(s) = [\widehat{\nu}(s) - \widehat{\nu}(0)]/s$, $\Re s = 0$, the value $\widehat{T\nu}(0)$ being defined by continuity as $\int_{\mathbb{R}} x\,\nu(dx) < \infty$. Let ν be a measure and $f(x)$ be a function. Denote by $\nu * f(x)$ their convolution $\int_{\mathbb{R}} f(x-y)\,\nu(dy)$.

The absolutely continuous part of any distribution F will be denoted by F_c, and its singular component by F_σ, i.e., $F_\sigma = F - F_c$. It is known (see [7, Section 4]) that $T|\nu| \in S(\varphi) \Rightarrow \nu \in S(\varphi)$. Suppose that $r_1 \le 0 \le r_2$, $\varphi(x)/\exp(r_1 x)$ is nonincreasing on $(-\infty, 0)$ and $\varphi(x)/\exp(r_2 x)$ is nondecreasing on $[0,\infty)$. Let F be a distribution with finite positive mean μ such that $TF \in S(\varphi)$. Assume that $\widehat{(F^{m*})_\sigma}(r_i) < 1$, $i = 1, 2$, for some $m \ge 1$. In particular, this means that the distribution F^{m*} has an absolutely continuous component. Let $\widehat{F}(s) \ne 1$ for $s \in \Pi(r_1, r_2) \setminus \{0\}$. Let $\alpha \in S(\varphi)$. Denote by $\mathrm{mes}(A)$ the Lebesgue measure of a Borel set A. Let us call a subset $A \subset \mathbb{R}$ *bounded from the left* if there exists $a \in \mathbb{R}$ such that $A \subseteq [a, \infty)$. Denote by $\mathbf{1}_A(x)$ the indicator of a set A. The relation $a(x) \sim cb(x)$ as $x \to \infty$ means that $a(x)/b(x) \to c$ as $x \to \infty$.

2. Key renewal theorem

We shall need the following lemma.

Lemma 1. *Let ν and $\varkappa$ be finite measures. Then*

$$T(\nu * \varkappa) = (T\nu) * \varkappa + \nu(\mathbb{R}) T\varkappa. \tag{1}$$

Proof. It suffices to show that the densities of both sides of (1) coincide. Let $x \in \mathbb{R}_+$. The density of the left-hand side is equal to

$$\nu * \varkappa((x,\infty)) = \int_{\mathbb{R}} \nu((x-y,\infty))\,\varkappa(dy).$$

Note that if α is a measure with density $a(x)$ and β is a finite measure, then the function $\beta * a(x) = \int_{\mathbb{R}} a(x-y)\,\beta(dy)$ is the density of $\alpha * \beta$. Indeed, let A be a Borel set. Then

$$\begin{aligned}\int_A \int_{\mathbb{R}} a(x-y)\,\beta(dy)\,dx &= \int_{\mathbb{R}} \int_A a(x-y)\,dx\,\beta(dy) \\ &= \int_{\mathbb{R}} \int_{A-y} a(z)\,dz\,\beta(dy) = \int_{\mathbb{R}} \alpha(A-y)\,\beta(dy),\end{aligned}$$

which proves the assertion. Put $\mathbb{R}_- := \mathbb{R} \setminus \mathbb{R}_+$. The density of the right-hand side of (1) is equal to

$$\begin{aligned}&\int_{\mathbb{R}} \big[\nu((x-y,\infty))\mathbf{1}_{\mathbb{R}_+}(x-y) - \nu((-\infty, x-y])\mathbf{1}_{\mathbb{R}_-}(x-y)\big]\,\varkappa(dy) \\ &\qquad + \nu(\mathbb{R})\big[\varkappa((x,\infty))\mathbf{1}_{\mathbb{R}_+}(x) - \varkappa((-\infty,x))\mathbf{1}_{\mathbb{R}_-}(x)\big], \qquad x \in \mathbb{R}.\end{aligned}$$

For $x \in \mathbb{R}_+$, it is equal to

$$\int_{-\infty}^{x} \nu((x-y,\infty))\,\varkappa(dy) - \int_{x}^{\infty} \nu((-\infty,x-y])\,\varkappa(dy) + \nu(\mathbb{R})\varkappa((x,\infty))$$
$$= \int_{-\infty}^{x} \nu((x-y,\infty))\,\varkappa(dy) - \int_{x}^{\infty} [\nu(\mathbb{R}) - \nu((x-y,\infty))]\,\varkappa(dy)$$
$$+ \nu(\mathbb{R})\varkappa((x,\infty)) = \int_{\mathbb{R}} \nu((x-y,\infty))\,\varkappa(dy),$$

which establishes the equality of both densities on $\mathbb{R}_+$. A similar argument applies when $x \in \mathbb{R}_-$. Let $x \in \mathbb{R}_-$. The density of the left-hand side of (1) is equal to $-\nu * \varkappa((-\infty,x])$, whereas the right-hand side has the density

$$\int_{\mathbb{R}} \big[\nu((x-y,\infty))\mathbf{1}_{\mathbb{R}_+}(x-y) - \nu((-\infty,x-y])\mathbf{1}_{\mathbb{R}_-}(x-y)\big]\,\varkappa(dy)$$
$$+ \nu(\mathbb{R})\big[\varkappa((x,\infty))\mathbf{1}_{\mathbb{R}_+}(x) - \varkappa((-\infty,x])\mathbf{1}_{\mathbb{R}_-}(x)\big]$$
$$= \int_{-\infty}^{x} \nu((x-y,\infty))\,\varkappa(dy) - \int_{x}^{\infty} \nu((-\infty,x-y])\,\varkappa(dy) - \nu(\mathbb{R})\varkappa((-\infty,x])$$
$$= \int_{-\infty}^{x} [\nu(\mathbb{R}) - \nu((x-y,\infty))]\,\varkappa(dy) - \int_{x}^{\infty} \nu((x-y,\infty))]\,\varkappa(dy)$$
$$- \nu(\mathbb{R})\varkappa((-\infty,x]) = -\int_{\mathbb{R}} \nu((\infty,x-y])\,\varkappa(dy) = -\nu * \varkappa((-\infty,x]).$$

Both densities also coincide on $\mathbb{R}_-$ and hence they coincide on the whole of $\mathbb{R}$. □

We now state the main theorem. In comparison with Theorem 3.1 in [7], it involves a less restrictive condition on the underlying distribution F.

Theorem 1. *Let the assumptions of Section 1 be satisfied. Suppose that $\alpha \in S(\varphi)$ and that $g(x) \geq 0$, $x \in \mathbb{R}$, is a Borel-measurable function with the properties $g \cdot \varphi \in L_1(\mathbb{R})$ and $g(x)\varphi(x) \leq C < \infty$, $x \in \mathbb{R}$.*

I. *If $g(x)\varphi(x) \to 0$ as $x \to \infty$, then*

$$\sup_{f:|f|\leq g} \left| U * \alpha * f(x) - \frac{\alpha(\mathbb{R})}{\mu} \int_{\mathbb{R}} f(y)\,dy \right| = o\left(\frac{1}{\varphi(x)}\right) \qquad as \quad x \to \infty, \tag{2}$$

the f's being Borel measurable.

II. *If $g(x)\varphi(x) \to 0$ as $x \to -\infty$, then*

$$U * \alpha * g(x) = o\left(\frac{1}{\varphi(x)}\right) \qquad as \quad x \to -\infty. \tag{3}$$

Proof. Let L be the restriction of Lebesgue measure to $\mathbb{R}_+$. Put $\mathcal{A} = S(\varphi)$ in Theorem 3.1 [7]. We have $U = U_1 + U_2$, where $U_2 \in S(\varphi)$ and $U_1 = L/\mu + rTU_2$ for some $r > r_2$. By Lemma 1,

$$rTU_2 * \alpha = T(rU_2 * \alpha) - r\widehat{U}_2(0)T\alpha = rT[U_2 * \alpha - \widehat{U}_2(0)\alpha],$$

whence

$$U * \alpha = \frac{L * \alpha}{\mu} + rT[U_2 * \alpha - \widehat{U}_2(0)\alpha] + U_2 * \alpha =: \frac{L * \alpha}{\mu} + TU_3 + U_4, \tag{4}$$

where both U_3 and U_4 belong to $S(\varphi)$. It follows from (4) that

$$U * \alpha * f(x) = \frac{L * \alpha * f(x)}{\mu} + TU_3 * f(x) + U_4 * f(x),$$

Now

$$L*\alpha*f(x)=\int_0^\infty \alpha*f(x-y)\,dy=\alpha(\mathbb{R})\int_{\mathbb{R}}f(y)\,dy-\int_{-\infty}^0\alpha*f(x-y)\,dy. \tag{5}$$

Equalities (4) and (5) imply

$$\begin{aligned}U*\alpha*f(x)-\frac{\alpha(\mathbb{R})}{\mu}\int_{\mathbb{R}}f(y)\,dy&=TU_3*f(x)+U_4*f(x)\\&-\frac{1}{\mu}\int_{-\infty}^0\alpha*f(x-y)\,dy=:I_1(x)+I_2(x)-\frac{1}{\mu}I_3(x).\end{aligned}$$

Further,

$$I_1(x)=\int_0^\infty f(x-y)U_3((y,\infty))\,dy-\int_{-\infty}^0 f(x-y)U_3((-\infty,y])\,dy=:I_4(x)-I_5(x).$$

We have

$$\begin{aligned}|I_4(x)|&\le\frac{1}{\varphi(x)}\int_0^\infty\varphi(x-y)g(x-y)\varphi(y)|U_3|((y,\infty))\,dy\\&\le\frac{1}{\varphi(x)}\int_0^\infty\varphi(x-y)g(x-y)\int_y^\infty\varphi(u)\,|U_3|(du)\,dy\\&=\frac{1}{\varphi(x)}\int_{-\infty}^x\varphi(v)g(v)\int_{x-v}^\infty\varphi(u)\,|U_3|(du)\,dv\\&=\frac{1}{\varphi(x)}\int_{\mathbb{R}}\mathbf{1}_{(-\infty,x]}(v)\varphi(v)g(v)\int_{x-v}^\infty\varphi(u)\,|U_3|(du)\,dv.\end{aligned}$$

The integrand tends to zero as $x\to\infty$ and is majorized by $\varphi(v)g(v)\|U_3\|_\varphi\in L_1(\mathbb{R})$. By Lebesgue's bounded convergence theorem, the integral tends to zero as $x\to\infty$ and we have

$$\sup_{f:|f|\le g}|I_4(x)|=o\Big(\frac{1}{\varphi(x)}\Big)\qquad\text{as}\quad x\to\infty. \tag{6}$$

Similarly,

$$\begin{aligned}|I_5(x)|&\le\frac{1}{\varphi(x)}\int_{-\infty}^0\varphi(x-y)g(x-y)\int_{-\infty}^y\varphi(u)\,|U_3|(du)\,dy\\&\le\frac{\|U_3\|_\varphi}{\varphi(x)}\int_{-\infty}^0\varphi(x-y)g(x-y)\,dy\\&=\frac{\|U_3\|_\varphi}{\varphi(x)}\int_x^\infty\varphi(v)g(v)\,dv=o\Big(\frac{1}{\varphi(x)}\Big)\qquad\text{as}\quad x\to\infty.\end{aligned} \tag{7}$$

It follows from (6) and (7) that

$$\sup_{f:|f|\le g}|I_1(x)|=o\Big(\frac{1}{\varphi(x)}\Big)\qquad\text{as}\quad x\to\infty. \tag{8}$$

Consider $I_2(x)$: $|I_2(x)|\le\dfrac{1}{\varphi(x)}\displaystyle\int_{\mathbb{R}}\varphi(x-y)g(x-y)\varphi(y)\,|U_4|(dy)$. By hypotheses, the integrand tends to zero as $x\to\infty$ and is majorized by the $|U_4|$-integrable function $C\varphi(x)$. By Lebesgue's bounded convergence theorem, the integral tends to zero as $x\to\infty$ and hence

$$\sup_{f:|f|\le g}|I_2(x)|=o\Big(\frac{1}{\varphi(x)}\Big)\qquad\text{as}\quad x\to\infty. \tag{9}$$

The integral $I_3(x)$ is equal to $\int_x^\infty \alpha * f(y)\,dy$. The condition $g\cdot\varphi \in L_1(\mathbb{R})$ implies that the measure, G, with density g belongs to $S(\varphi)$. Therefore, the measure $|\alpha| * G$ with density $|\alpha| * g$ also belongs to $S(\varphi)$ and we have

$$\begin{aligned}(10)\qquad \sup_{f:|f|\le g} |I_3(x)| &\le \int_x^\infty |\alpha| * g(y)\,dy\\ &\le \frac{1}{\varphi(x)}\int_x^\infty \varphi(y)|\alpha| * g(y)\,dy = o\Big(\frac{1}{\varphi(x)}\Big) \qquad \text{as} \quad x\to\infty.\end{aligned}$$

Summing up relations (8)–(10), we arrive at the desired conclusion (2). The remaining relation (3) is proved similarly. □

3. Asymptotics of the renewal measure on Borel sets

Blackwell's renewal theorem states that if G is a nonarithmetic distribution with positive mean μ_G and U_G is the renewal measure generated by G, then $U_G((x, x+h]) \to h/\mu_G$ as $x\to\infty$, for fixed $h > 0$ (see [3, Theorem 1] and [4, Chapter XI, Section 1, Theorem 1]).

Theorem 2. *Suppose that the hypotheses of Theorem 1 with $\varphi(x) \equiv 1$ for $x \le 0$ are satisfied and let A be a Borel set bounded from the left which has finite Lebesgue measure. Then*

$$(11)\qquad \sup_{B\subseteq A}\Big|U * \alpha(B + x) - \frac{\alpha(\mathbb{R})\mathrm{mes}(B)}{\mu}\Big| = o\Big(\frac{1}{\varphi(x)}\Big) \qquad \textit{as} \quad x\to\infty,$$

where $B + x := \big\{y \in \mathbb{R} \colon y - x \in B\big\}$.

Proof. We have $TF \in S(\varphi) \Rightarrow F \in S(\varphi)$. Put $g(x) = \mathbf{1}_{-A}(x)$, where $-A := \{x \in \mathbb{R} : -x \in A\}$, and put $f(x) = \mathbf{1}_{-B}(x)$. Obviously,

$$U * \alpha(B + x) = U * \alpha * f(x), \qquad \int_{\mathbb{R}} f(x)\,dx = \mathrm{mes}(-B) = \mathrm{mes}(B).$$

Theorem 1 implies (11). □

Putting $\alpha = \delta$ in Theorem 2 we get the following analog of Blackwell's theorem with submultiplicative estimate of the remainder, even for possibly unbounded Borel sets.

Corollary 1. *Suppose that the hypotheses of Theorem 2 are satisfied. Then*

$$U(A + x) - \frac{\mathrm{mes}(A)}{\mu} = o\Big(\frac{1}{\varphi(x)}\Big) \qquad \textit{as} \quad x\to\infty.$$

Remark 1. The requirement that the distribution F^{m*} have an absolutely continuous component for some $m \ge 1$ is also *necessary* for the validity of relation (11). Indeed, if the requirement is not fulfilled, then the measure U is concentrated on a set B of Lebesgue measure zero. Take as A the set $B \cap [0, 1]$ and put $\alpha = \delta$. By Blackwell's renewal theorem for nonarithmetic distributions, the left-hand side in (11) tends to $1/\mu \ne 0$ as $x\to\infty$ while the right-hand side tends to zero, i.e., relation (11) does not hold.

Remark 2. Let F be a probability distribution on $\mathbb{R}_+$ with finite mean μ such that for some $m \ge 1$ the distribution F^{m*} has an absolutely continuous component and let I be a bounded interval. Then

$$\lim_{t\to\infty}\sup_{B\subset I}\big|G * U(t + B) - \mu^{-1}\mathrm{mes}(B)\big| = 0,$$

where G is an arbitrary initial distribution and B is a Borel set [2, Corollary 2]. If F is a probability distribution on $\mathbb{R}$ with positive mean μ such that for some $m \geq 1$ the distribution F^{m*} has an absolutely continuous component, then $\lim_{t\to\infty} U(t + B) = \mu^{-1}\mathrm{mes}(B)$ for all bounded Borel sets B (see the remark after the proof of Theorem 2.6.4 in [1]). These results also follow from Theorem 2 with $\varphi(x) \equiv 1$.

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