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INTERIOR CO-IDEALS IN Γ-SEMIRINGS WITH APARTNESS

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Abstract. In this paper, as a continuation of research on Γ-semirings with apartness in Bishop's constructive framework, we determine the concepts of interior co-ideals (i.e. a constructive dual of interior ideals) in Γ-semirings with apartness as a generalization of the concept of co-ideals of Γ-semiring and analyze some of their fundamental properties. The class of interior co-ideals coincides with the class of co-ideals in regular and in intra-regular Γ-semirings.

1. Introduction

The concept of Γ-semirings were first introduced and studied by M. M. Krishna Rao [20,21] as a generalization of notion of Γ-rings. Many authors have studies on these algebraic structures. For example: H. Hedayati and K. P. Shum [7] (2011), R. Jagatap and Y. Pawar [8–10, 12, 13] (2009-17) and M. M. Krishna Rao [22] (2018). There is an interest in the academic community to study and publish the results of these research on these algebraic structures, their internal organization as well as their substructures in general, as well as in many specific cases.

The settings of this article is the Bishop's constructive mathematics **Bish** in the sense of books [2, 3, 18, 19] and articles [1, 4, 5, 25–28, 31] including the Intuitionistic logic **IL** ([35, 38]).

Let $(S, =, \neq)$ be a relational system, where the relation $' \neq '$ is an apertness relation - a relation on a set S which is consistent, symmetric and co-transitive:

$(\forall x \in S)\neg(x \neq x)$,
$(\forall x, y \in X)(x \neq y \implies y \neq x)$ and
$(\forall x, y, z \in S)(x \neq z \implies (x \neq y \vee y \neq z))$.

For example, in the field of real numbers $\mathbb{R}$, an apartness relation is introduced ([19]) as follows

$$(\forall a, b \in \mathbb{R})(a \neq b \iff (\exists k \in \mathbb{N})(|a - b| > \frac{1}{k})).$$

In the field of rational numbers $\mathbb{Q}$, apartness relation is introduced as follows

$$(\forall a, b \in \mathbb{Q})(a \neq b \iff \neg(a = b)).$$

The apartness relation on a set S is an extensive relation with respect to the equality relation in S (see, for example [25, 27, 28, 31]) in the following sense:

$$(\forall x, y, z \in S)(x = y \wedge y \neq z) \implies x \neq z)).$$

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This relational system is called '*set with apartness*' or, shortly, a *set* in the Bish framework. Since in this system the logical principle of the TND (LAT: tertium non datur; the principle of excluding the third) is not an axiom, all formulas that (directly or indirectly) contain the equality have their own non-equivalent doubles. These specifics generate greater complexity in many algebraic structures than is the case in classical algebra. For example, article [24] discusses Abel's groups with apartness; papers [25, 26] are dedicated to commutative rings with apartness and modules above them; In paper [29], analyzing semi-valuations in the Heyting field, the concept of co-order relation on Abelian group with apartness is constructed as a constructive dual of order relation.

In articles [32, 33], our intention is to recognize, understand and describe as precisely as possible these specificity on the example of one complex algebraic structure, Γ-semirings structure. In these papers we have dealt with this algebraic structure within the specific principled-logical environment offered by **Bish** orientation. So, we have observed the behavior of these algebraic structures, assuming that all carriers of algebraic structures are sets with apartness, that all relations, operations, and functions that appear in they are strongly extensional with respect to apartneses (see, for example [31]). For example, to illustrate:

- For $f : X \longrightarrow Y$ is said to be a strongly extensional function (*se-function* / *se-map*, briefly) if valid

$$(\forall u, v \in X)(f(u) \neq f(v) \implies u \neq v).$$

We also in [32, 33] analyzed the doubles of the congruence relations, the order relations, the ideals, and the filters in such introduced Γ-semirings with apartness.

In this paper, as a continuation of the previously mentioned reports [32, 33], we offer our reflections on establishing a constructive dual of classical concepts of interior ideals (for example, in sense of [14]) in Γ-semirings with apartness (Definition 3.1 and Theorem 3.1). In addition to the above, the paper also analyzes the interrelationships of co-ideals with this newly introduced class of interior co-ideals in Γ-semirings with apartness (Theorem 3.2). The paper shows (Theorem 3.5) that the family of all interior co-ideals in a Γ-semiring with apartness forms a complete lattice. In addition to the above, it has also been shown that the class of interior co-ideals coincides with the class of co-ideals in regular and intra-regular Γ-semirings.

The notions and notations used in this article but not determined in it, we are take over from previously published articles [4, 5, 30–33].

2. Preliminaries: Γ-semirings with apartness

Looking at the definition of Γ-semigring in the classical sense ([7, 20, 21]), we first introduce the concept of Γ-semirings with apartness which will be used throughout this paper. Let $(R, +, \cdot)$ and $(\Gamma, +, \cdot)$ be commutative semigroups with apartness. About the 'apartness' reader can consult the following books [2, 3, 19]. By this we mean that the sets $R \equiv (R, =_R, \neq_R)$ and $\Gamma \equiv (\Gamma, =_\Gamma, \neq_\Gamma)$ are supplied by apartness relations and that the internal operations in them are strongly extensional total functions. In the following, we do not use indices in the equation relations and apartness relations, except in cases where it is necessary to distinguish them so as not to cause confusion. About the relations, functions and operations in the system **Bish** a reader can consult some of our previously published articles such as [27, 28], or any of our bibliographic units listed in the literature of this article [4, 5, 24–26, 29–31].

Definition 2.1. ([32], definition 2.1) We call R a Γ-*semiring with apartness* if there exists a map $R \times \Gamma \times R \longrightarrow R$, written image of (x, a, y) by xay, such that it satisfies the following axioms:

(1) $(\forall x, y, z \in R)(\forall a \in \Gamma)(xa(y + z) = xay + xaz$ and $(x + y)az = xaz + yaz)$,
(2) $(\forall x, y \in R)(\forall a, b \in \Gamma)(x(a + b)y = xay + xby)$,
(3) $(\forall x, y, z \in R)(\forall a, b \in \Gamma)((xay)bz = xa(ybz))$.

Remark 2.1. As can be seen, the definition of Γ-semirings with apartness is completely identical to the definition of Γ-semiring in the classical case. However, they do not determine the same algebraic structure. The reader should always keep in mind that the logical setting are different and that the manipulation with them takes place with the previously acceptance of the various principles-philosophical orientations. In this environment, the following implication is valid

$$(\forall x, y, u, v \in R)(x + u \neq y + v \implies (x \neq y \vee u \neq v)) \text{ and}$$

$$(\forall x, y, u, v \in R)(\forall a, b \in \Gamma)(xay \neq ubv \implies (x \neq u \vee a \neq b\, y \neq v)).$$

A Γ-semiring with apartness R is said to have a *zero element* if there exists an element $0 \in R$ such that the following

$$(\forall x \in R)(\forall a \in \Gamma)(0 + x = x = x + 0 \text{ and } 0ax = 0 = xa0)$$

is valid. Of course, we also have

$$(\forall x, y \in R)(x + y \neq 0 \implies (x \neq 0 \vee y \neq 0))$$

and

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \neq 0 \implies (x \neq 0 \wedge y \neq 0)).$$

Also, a Γ-semiring with apartness R is said to be *commutative* if the following holds

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay = yax).$$

About the slogan '*a function f is an embedding*', which we will use in the following definition, the reader can consult with some of our previously published texts [25–28, 31]:

$$(\forall u, v \in S)(u \neq v \implies f(u) \neq f(v)).$$

In what follows, the following specific constructive notion will play an important role. It can be seen as the extensiveness of the predicate which determined the subset B of the set R in relation to the apartness relation.

- For the subset B of the set R it is said to be strongly extensional in R (shorter, *se-subset*) if valid

$$(\forall u, v \in R)(u \in B \implies (u \neq v \vee v \in B)).$$

Also, the following notions, taken from [32, 33], will play an important role. Let R be a Γ-semiring with apartness:

- A se-subset B of R is a *cosub*-Γ-*semiring* of R if B is an additive cosub-semigroup of R and the following holds

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \in B \implies (x \in B \vee y \in B));$$

- A se-subset B of R is a *right* Γ-*coideal* of R if B is a additive cosub-semigroup of R and the following holds

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \in B \implies y \in B).$$

- A se-subset B of R is a *left* Γ-*coideal* of R if B is a additive cosub-semigroup of R and the following holds

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \in B \implies x \in B).$$

- A se-subset B of R is a Γ-*coideal* of R if B is a additive cosub-semigroup of R and the following holds

$$(\forall x, y \in R)(\forall a \in \Gamma)(xay \in B \implies (x \in B \wedge y \in B)).$$

- An element $1 \in S$ is said to be unity if for each $x \in S$ there exists $a \in \Gamma$ such that $xa1 = 1ax = x$.

It is known ([20]: pp. 51, [7]: Theorem 4.5) that if ρ is a congruence relation on a Γ-semiring R, then $R/\rho := \{[x]_\rho : x \in R\}$ is also a Γ-semiring where are

$$(\forall x, y \in R)([x]_\rho + [y]_\rho = [x + y]_\rho),$$

$$(\forall x, y \in R)(\forall a \in \Gamma)([x]_\rho a[y]_\rho = [xay]_\rho).$$

In our case, in the case of Γ-semiring with apartness and a co-congruence relation on it, the situation is more complex (see, for example [32, 33]):

- A relation q on a set with apartness $(R, =, \neq)$ is a co-equivalence on R if the following holds:

$(\forall x, y \in R)((x, y) \in q \implies x \neq y)$ (consistency),
$(\forall x, y \in R)((x, y) \in q \implies (y, x) \in q)$ (symmetry), and
$(\forall x, y, z \in R)((x, z) \in q \implies ((x, y) \in q \vee (y, z) \in q))$ (co-transitivity).

About this class of relations in sets with apartness, a reader can find in the papers [27, 28, 31].

- A co-equality relation q on Γ-semiring with apartness R is said to be a co-congruence if the following conditions

$(\forall x, y, z \in R)((x + z; y + z) \in q \vee (z + x, z + y) \in q) \implies (x, y) \in q)$ and
$(\forall x, y, z \in R)(\forall a \in \Gamma)(((xaz, yaz) \in q \vee (zax, zay) \in q) \implies (x, y) \in q)$

are satisfied. Co-congruence relations in some of the algebraic structures designed on sets with apartness are discussed in the papers [25, 26, 30, 31].

- If q is a co-congruence on a Γ-semiring with apartness R, then the relation $q^\lhd := \{(x, y) \in R \times R : (\forall (u.v) \in R \times R)((x, y) \neq (u, v))\}$ is a congruence on R.

- Let q be a co-congruence on a Γ-semigrong with apartness R. Then the set $R/(q^\lhd, q) := \{xq^\lhd : x \in R\}$ is a Γ-semigring with apaerness where

$$(\forall x, y \in R)((xq^\lhd =_1 yq^\lhd \iff (x, y) \lhd q) \wedge (xq^\lhd \neq_1 yq^\lhd \iff (x, y) \in q)),$$

$$(\forall x, y \in R)(\forall a \in \Gamma)((xq^\lhd +_1 yq^\lhd =_1 (x + y)q^\lhd) \wedge (xq^\lhd\, a\, yq^\lhd =_1 (xay)q^\lhd).$$

- Let q be a co-congruence on a Γ-semigring with apartness R. Then the set $[R : q] := \{xq : x \in R\}$ is a Γ-semiring with apaerness where

$$(\forall x, y \in R)((xq =_2 yq \iff (x, y) \lhd q) \wedge (xq \neq_2 yq \iff (x, y) \in q)),$$

$$(\forall x, y \in R)(\forall a \in \Gamma)((xq +_2 yq =_2 (x + y)q) \wedge (xq\, a\, yq =_2 (xay)q).$$

It should be noted that Γ-semiring with apartness $[R : q]$ has no counterpart in the classical theory.

Historical notes. The concept of co-ideals in rings was first introduced by W. Ruitenberg in his dissertation [35]. D. A. Romano then analyzed this concept in commutative rings with apartness in his dissertation [23] which enabled him to design the concepts of co-equality and co-congruence relations (for example, [25–27]). These substructures in algebraic structures were also the subject of interest of A. S. Troelstra and D. van Dalen in Chapret 8 of the book [38].

The concepts of co-substructures in various algebraic structures as well as the concepts of co-equivalences and co-congruences have been in the focus of the author's interest for a long time (for example, [4, 5, 28–31]).

3. The main results

3.1. **Interior ideals in the classical case.** M. M. Krishna Rao [20, 21] defined and studied Γ-semiring. Dutta and Sardar [6] studied different types of ideals in a Γ-semiring. Quasi-ideals and bi-ideals in a Γ-semiring were studied by R. D. Jagatap [9, 10, 12]. Lajos [17] defined the concept of an interior ideal in a semigroup. Interior ideal in a semigroup was studied by Szasz in [36, 37] also. Interior ideals in ordered semigroups and the interior ideal elements in poe-semigroups were discussed by Kehayopulu in [15, 16]. This author in [34] also wrote about interior ideals in quasi-ordered semigroup.

The concepts of an interior ideal and minimal interior ideal in a Γ-semiring are introduced in paper [14] (Definition 2.1) by R. D. Jagatap:

- A non-empty subset J of a Γ-semiring S is an *interior ideal* of S if J is an additive subsemigroup of S and $S\Gamma J\Gamma S \subseteq J$.

In other words, the following formulas

$$(\forall u, v \in S)((u \in J \wedge v \in J) \implies u + v \in J),$$

$$(\forall u, v, x \in S)(\forall a, b \in \Gamma)(x \in J \implies uaxbv \in J)$$

are valid formulas. Every ideal J on a Γ-semiring S is an interior ideal of S but not conversely ([14], Remark 2.2).

3.2. **Interior co-ideals of Γ-semirings with apartness.** Here we define the concept of interior co-ideals of a Γ-semiring with apartness.

Definition 3.1. Let $R := (R, =, \neq, +, \cdot)$ be a Γ-semiring with apartness over the set $\Gamma := (\Gamma, =, \neq)$ with apartness. A se-subset K of a Γ-semiring R is an interior co-ideal of R if

(8) K is an additive co-subsemigroup of R and

(9) $(\forall u, v, x \in R)(\forall a, b \in \Gamma)(uaxbv \in K \implies x \in K)$

holds.

Our first theorem connects the terms 'interior ideal' and 'interior co-ideal' in a Γ-semiring with apartness.

Theorem 3.1. *If $K\,(\neq R)$ is an interior co-ideal of a Γ-semiring with apartness R, then the set*

$$K^{\lhd} = \{x \in R : x \lhd K\} = \{x \in R : (\forall u \in K)(x \neq u)\}$$

is an interior ideal of R.

Proof. It should be shown that the set $K^{\lhd}$ satisfies the following conditions:

$K^{\lhd} \neq \emptyset$,

$(\forall x, y \in R)(x \in K^{\lhd} \wedge y \in K^{\lhd} \implies x + y \in K^{\lhd})$ and

$(\forall x, u, v \in R)(\forall a, b \in \Gamma)(x \in K^{\lhd} \implies uaxbv \in K^{\lhd})$.

The condition $K \neq X$ ensures that the set $K^{\lhd}$ is inhabited.

Let $x, y, u \in R$ be arbitrary elements such that $u \in K$, $x \lhd K$ and $y \lhd K$. Then $u \neq x + y$ or $x + y \in K$ by strongly extensionality of K in R. The second option $x + y \in K$ would give $x \in K \vee y \in K$ by (8), which contradicts the assumptions. Therefore, it must be $x+y \neq u \in K$. This means that $x + y \lhd K$ holds.

Let $x, u, v, t \in R$ and $a, b \in \Gamma$ be arbitrary elements such that $t \in K$, $x \lhd K$. Then $t \neq uaxbv$ or $uaxbv \in K$ by strongly extensionality of K in R. The second option $uaxbv \in K$ would give $x \in K$ by (9), which contradicts the assumptions. Therefore, it must be $uaxbv \neq u \in K$. This means that $uaxbv \lhd K$ holds. □

The reverse implication of the implication proved in the previous theorem is not valid in the general case.

Example 3.1. Let $R = \{0, 1, 2, 3, 4\}$ and operations '+' and '$\cdot$' defined on R as follows:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	2
2	2	3	4	2	3
3	3	4	2	3	4
4	4	3	2	4	2

and

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	3	2
3	0	3	3	3	3
4	0	4	2	3	4

For $\Gamma = R$, both R and Γ are additive commutative semigroups. A mapping $R \times \Gamma \times R \longrightarrow R$ is defined as xay = usual product of $x, y, a \in R$. Then R forms a R-semiring ([14], First example in Section 2). By direct verification one can establish that the sets $\{1, 2, 3, 4\}$, $\{1, 2, 4\}$ and $\{1\}$ are interior co-ideals in Γ-semiring R.

The following theorem connects the concepts of co-ideals and interior co-ideals in a Γ-semiring with apartness.

Theorem 3.2. *Every co-ideal is an interior co-ideal in a Γ-semiring.*

Proof. Let K be a co-ideal in a Γ-semiring R. It suffices to prove that for K we hold (9) because (8) is satisfied by assumption. Let $u, v, x \in R$ and $a, b \in \Gamma$ be such that $uaxbv \in K$. Then $uax \in K$ since K is a right co-ideal in R. Also, from $uax \in K$ it follows $x \in K$ since K is a left co-ideal in R. So, K is an interior co-odeal in R. □

The inverse of the previous theorem is not valid as the following example shows:

Example 3.2. Let $R = \{a, b, c, d\}$ and operations '+' and '$\cdot$' defined on R as follows:

+	a	b	c	d
a	a	c	b	d
b	c	d	d	d
c	b	d	d	d
d	d	d	d	d

and

·	a	b	c	d
a	a	c	d	d
b	c	d	d	d
c	b	d	d	d
d	d	d	d	d

For $\Gamma = R$, both R and Γ are additive commutative semigroups. A mapping $R \times \Gamma \times R \longrightarrow R$ is defined as xay = usual product of $x, y, a \in R$. Then R forms a R-semiring ([14], Second example in Section 2). By direct verification one can establish that the sets $\{a, c\}$ and $\{a, v\}$ are interior co-ideals in R-semiring R but they are neither left co-ideals nor right co-ideals in R.

However, in regular Γ-semirings the reverse inclusion is also valid. In what follows we need the notion of regular Γ-semiring with apartness ([13]): An element x of a Γ-semiring R is said to be regular if $x \in x\Gamma R\Gamma x$. If all elements of Γ-semiring R are regular, then R is known as a regular Γ-semiring.

Theorem 3.3. *If R is a regular Γ-semiring with apartness, then any interior co-ideal in R is a co-ideal in R.*

Proof. Let K be an interior co-ideal of a regular Γ-semiring with apartness. This means that (8) and (9) are valid. Let us prove that K is a co-ideal in R.

Let $x, y \in R$ and $a \in \Gamma$ such that $xay \in K$. Since R is regular, there are $b, c, d, e \in \Gamma$ and $u, v \in R$ such that $x = xbucx$ and $y = ydvey$. Then $xay = (xbucx)ay = (xbu)cxay \in K$ and $xay = xa(ydvey) = xayd(vey) \in K$. It follows from here $x \in K$ and $y \in K$ according to (9). □

The assertion stated in the previous theorem can also be proved by assuming that R is an intra-regilar Γ-semiring with apartness. A Γ-semiring with apartness R is said to be an intra-regular Γ-semiring with apartness if for any $x \in R$ holds $x \in R\Gamma x\Gamma x\Gamma R$ ([11, 13]).

Theorem 3.4. *If R is an intra-regular Γ-semiring with apartness, then any interior co-ideal in R is a co-ideal in R.*

That the family $\mathfrak{Intc}(R)$ of all interior co-ideals of a Γ-semiring with apartness R is not empty, because $R \in \mathfrak{Intc}(R)$. Also, we accept that $\emptyset \in \mathfrak{Intc}(R)$ is valid. Further on, we have the following propositions:

Proposition 3.1. *Let X be any subset of a Γ-semiring with apartness R such that $R\Gamma X\Gamma R$ is a strongly extensional Γ-semigroup of R. Then, $R\Gamma X\Gamma R$ is an interior co-ideal of R.*

Proof. Let $u, v, x \in R$ and $a, b \in \Gamma$ be arbitrary elements such that $uaxbv \in R\Gamma(R\Gamma X\Gamma R)\Gamma R$. Then there exist elements $u', v' \in R$, $x' \in X$ and $a', b' \in \Gamma$ such that $x = u'a'x'b'v' \in R\Gamma X\Gamma R$. Thus

$$R\Gamma(R\Gamma X\Gamma R)\Gamma R \ni uaxbv = ua(u'a'x'b'v')bv = (uau')a'x'b'(v'bv) \in R\Gamma X\Gamma R$$

because $uau' \in R\Gamma R \subseteq R$ and $v'bv \in R\Gamma R \subseteq R$ by definition. □

Proposition 3.2. *If K is an interior co-ideal and T is a sub-semiring of Γ-semiring with apartness R, then $K \cap T$ is an interior co-ideal of T.*

Proof. The first, it should be proved that $K \cap T$ is an additive co-subsemigroup of R. Let $x, y \in T$ be arbitrary elements such that $x + y \in K \cap T\,(\subseteq K)$. Then $x \in K$ or $y \in K$. Thus $x \in K \cap T$ or $y \in K \cap T$. This shows that $K \cap T$ is an additive co-subsemigroup of R.

Let $u, v, x \in T$ and $a, b \in \Gamma$ be such that $uaxbv \in T\Gamma(K \cap T)\Gamma T$. Then $T\Gamma(K \cap T)\Gamma T \subseteq T\Gamma K\Gamma T \subseteq R\Gamma K\Gamma R$. Thus $x \in K$ because K is an interior co-ideal of R. Therefore, $x \in K \cap T$. This shows that $K \cap T$ is an interior co-ideal of T. □

Theorem 3.5. *The family $\mathfrak{Intc}(R)$ of all interior co-ideals of a Γ-semiring with apartness R forms a complete lattice.*

Proof. Let $\{K_i\}_{i\in I}$ be a family of interior co-ideals of a Γ-semiring with apartness R.

(a) (i) Let $u, v \in R$ be such that $u \in \bigcup_{i\in I}$. Then there exists an index $k \in I$ such that $u \in K_k$. Thus $u \neq v$ or $v \in K_k \subseteq \bigcup_{i\in I} K_i$ by strongly extensionality of the co-ideal K_k in R. This means that the set $\bigcup_{i\in I} K_i$ is a strongly extensional subset in R.

(ii) Let $x, y \in R$ be such that $x + y \in \bigcup_{i\in I} K_i$. Then there exists an index $k \in I$ such that $x + y \in K_k$. Thus $x \in K_k \subseteq \bigcup_{i\in I} K_i$ or $y \in K_k \subseteq \bigcup_{i\in I} K_i$ by (8). This means that the set $\bigcup_{i\in I} K_i$ is an additive co-subsemigroup of R.

(iii) Let $u, v, x \in R$ and $a \in \Gamma$ be such that $uaxbv \in \bigcup_{i\in I} K_i$. Then there exists an index $k \in I$ such that $uaxbv \in K_k$. Thus $x \in K_k \subseteq \bigcup_{i\in I} K_i$ by (9). This shows that the set $\bigcup_{i\in I} K_i$ satisfies the condition (9).

Based on (i), (ii) and (iii), we conclude that the set $\bigcup_{i\in I} K_i$ is an interior co-ideal of R.

(b) Let X be the family of all interior co-ideals of Γ-semiring with apartness R contained in $\bigcap_{i\in I} K_i$. Then $\cup X$ is the maximal interior co-ideal of R contained in $\bigcap_{i\in I} K_i$, according to (a).

(c) If we put $\sqcup_{i\in I} K_i = \bigcup_{i\in I} K_i$ and $\sqcap_{i\in I} K_i = \cup X$, then $(\mathfrak{Intc}(R), \sqcup, \sqcap)$ a is a complete lattice. □

Corollary 3.1. *For any subset X of Γ-semiring with apartness R there is the maximal co-ideal contained in X.*

Proof. The proof of this Corollary is obtained directly from part (b) of the evidence in the previous theorem. □

Corollary 3.2. *For any element $x \in R$ there is the maximal interior co-ideal K_x in Γ-semiring with apartness R such that $x \lhd K_x$.*

Proof. One should take $X = \{u \in R : u \neq x\}$ and apply the previous corollary. □

Let R be a Γ-semiring and T a Λ-semiring. Then $(f, \varphi) : (R, \Gamma) \longrightarrow (T, \Lambda)$ is called a *se-homomorphism* if $f : R \longrightarrow T$ and $\varphi : \Gamma \longrightarrow \Lambda$ are strongly extensional homomorphisms of semigroups such that

$$(\forall x, y \in R)(\forall a \in \Gamma)((f, \varphi)(xay) = f(x)\varphi(a)f(y))$$

holds ([32], Definition 2.3). The mapping (f, φ) is called an *epimorphism* if (f, φ) is a homomorphism and f and φ are epimorphisms of semigroups. Similarly, we can define a monomorphism. A homomorphism (f, φ) is an isomorphism if (f, φ) is an epimorphism and a monomorphism and f and φ are embeddings.

Theorem 3.6. *Let (f, φ) be a se-homomorphism from Γ-semiring with apartness R into Λ-semiring with apartness T. If K is an interior co-ideal of T, then $f^{-1}(K)$ is an interior co-ideal of R.*

Proof. The proof of this theorem is derived by direct verification:

Let $x, y \in R$ be such that $x \in f^{-1}(K)$. Then $f(x) \in K$ and $f(y) \in T$. Thus $f(x) \neq f(y) \vee f(y) \in K$ because K a strongly extensional subset in T. The first option gives $x \neq y$ since f is a strongly extensional homomorphism. Therefore, we have $x \neq y \vee y \in f^{-1}(K)$ which shows that $f^{-1}(K)$ is a strongly extensional subset in R.

Let $x, y \in R$ be such that $f(x) + f(y) = f(x + y) \in K$. Then $f(x) \in K$ or $f(y) \in K$. Thus $x \in f^{-1}(K) \vee y \in f^{-1}(K)$.

Let $x, y \in R$ and $a \in \Gamma$ be such that $xay \in f^{-1}(K)$. Then $f(x)\varphi(a)f(y) \in K$. Thus $f(x) \in K \vee f(y) \in K$. So, $x \in f^{-1}(K) \vee y \in f^{-1}(K)$.

Let $u, v, x \in R$ and $a, b \in \Gamma$ be such that $uaxbv \in f^{-1}(K)$. Then

$$f(u)\varphi(a)f(x)\varphi(b)f(v) \in K \text{ and } \varphi(a) \wedge \varphi(b) \in \Lambda.$$

Thus $f(x) \in K$ since K is an interior co-ideal of T. This means $x \in f^{-1}(K)$.

This proves that $f^{-1}(K)$ is an interior co-ideal of Γ-semiring with apartness R. □

In what follows, we need the following lemmas:

Lemma 3.1 ([32]). *Let $(f, \varphi) : (R, \Gamma) \longrightarrow (T, \Lambda)$ be a strongly extensional homomorphism. Then there exists the se-epimorphism*

$$(\pi, \iota) : R \longrightarrow R/(Ker(f), Coker(f))$$

and the embedding embedding se-monomorphism

$$(g, \iota) : R/(Ker(f), Coker(f)) \longrightarrow T$$

such that

$$(f, \varphi) = (g, \varphi) \circ (\pi, \iota).$$

Lemma 3.2 ([32]). *Let $(f, \varphi) : (R, \Gamma) \longrightarrow (T, \Lambda)$ be a strongly extensional homomorphism. Then there exists the se-epimorphism*

$$(\vartheta, \iota) : R \longrightarrow [R : Coker(f)]$$

and the embedding se-monomorphism

$$(h, \iota) : [R : Coker(f)] \longrightarrow T$$

such that

$$(f, \varphi) = (h, \varphi) \circ (\vartheta, \iota).$$

Combining Theorems 3.5, Lemma 3.1 and lemma 3.2 gives the following results:

Theorem 3.7. *Let $(f, \varphi) : (R, \Gamma) \longrightarrow (T, \Lambda)$ be a strongly extensional homomorphism. There are correspondences*

$$\mathfrak{Intc}(T) \longrightarrow \mathfrak{Intc}(R/(Ker(f), Coker(f))) \text{ and}$$

$$\mathfrak{Intc}(T) \longrightarrow \mathfrak{Intc}([R : Coker(f)]).$$

4. Final comments and conclusions

The concept of Γ-semiring with apartness was introduced in 2019 by this author. This algebraic structure is designed on a relational system $(R, =, \neq)$ as carriers of that algebraic structure with acceptance of the Bishop's logical-principles work environment which implies Intuitionistic logic instead of Classical logic. Here, the relation $\neq$ is an apartness relation on R compatible with the equality relation $=$. Besides, since in Intuitionist logic, the principle of excluding the third is not a valid axiom of that logic, in designing the algebraic structure of Γ-semiring with apartness, two groups of substructures and the relations between them appear naturally. In this report, which is a continuation of our papers [32,33], we deal with the design of a constructive dual of interior ideals.

It is quite justified to ask the question:

Are Γ*-semirings and* Γ*-semirings with apartness one and the same class of algebraic structures?*

If we look at these algebraic structures through the eyes of a traditional mathematician, then these classes differ since the carriers of these structures differ. If we analyze the class of the Γ-semirings through the glasses of Intuitionist logic, then it is one and the same class of algebraic structures. So, it can be said that the latter is just a different way of looking at it.

The next paper could contain our reflections on designing a constructive dual of weak interior ideals in Γ-semirings with apartness (weak interior co-ideals) and its connection to constructive duals of ideals and interior ideals (co-ideals and interior co-ideals) in them.

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References

[1] [1] M. Mitrovic, M.N. Hounkonnou, M.A. Baroni, Theory of constructive semigroups with apartness – foundations, development and practice, (2022). http://arxiv.org/abs/2008.11008.
[2] E. Bishop, Foundations of Constructive Analysis. New York: McGraw-Hill, 1967.
[3] D.S. Bridges, F. Richman, Varieties of Constructive Mathematics, Cambridge: London Mathematical Society Lecture Notes, No. 97, Cambridge University Press, 1987.
[4] S. Crvenković, M. Mitrović and D. A. Romano, Semigroups with apartness, Math. Log. Quart. 59 (2013) 407-414. https://doi.org/10.1002/malq.201200107.
[5] S. Crvenković, M. Mitrović, D. A. Romano, Basic notions of (constructive) semigroups with apartness, Semigroup Forum. 92 (2016) 659–674. https://doi.org/10.1007/s00233-016-9776-y.
[6] T. K. Dutta and S. K. Sardar, Semi-prime ideals and irreducible ideals of Γ-semiring, Novi Sad J. Math. 30 (2000) 97–108.
[7] H. Hedayati and K. P. Shum, An introduction to Γ-semirings, Int. J. Algebra, 5 (2011) 709–726.
[8] R.D. Jagatap, Y.S. Pawar, Quasi-ideals and minimal quasi-ideals in Γ-semirings, Novi Sad J. Math. 39 (2009) 79-–87.
[9] R.D. Jagatap, Y.S. Pawar, Quasi-ideals in regular Γ-semirings, Bull. Kerala Math. Assoc. 6 (2010) 51-–61.
[10] R. Jagatap, Y.S. Pawar, Right ideals of Γ-semirings, Novi Sad J. Math. 43 (2013) 11–19.
[11] R.D. Jagatap, Intra-regilar Γ-semirings, Int. J. Latest Res. Sci. Technol. 4 (2015) 79–83.
[12] R.D. Jagatap, Y.S. Pawar, Bi-ideals in Γ-semirings, Bull. Int. Math. Virtual Inst. 6 (2016) 169-–179.

[13] R.D. Jagatar, Y.S. Pawar, Regular, intra-regular and duo Γ-semirings, Asia Pac. J. Math. 4 (2017) 62–74.

[14] R.D. Jagatap, Interior ideals in *Gamma*-semirings, Malaya J. Mat. 06 (2018) 731–735. https://doi.org/10.26637/mjm0604/0005.

[15] N. Kehayopulu, Interior ideals and interior ideal elements in ordered semigroups, Pure Math. Appl. 10 (1999) 323–329.

[16] N. Kehayopulu, Note on interior ideals, ideal elements in ordered semigroups, Sci. Math. 2 (1999) 407–409.

[17] S. Lajos, $(m; k; n)$-ideals in semigroups, In: Notes on Semigroups II, Karl Marx Univ. Econ., Dept. Math. Budapest, 1 (1976) 12-–19.

[18] H. Lombardi, C. Quitté, Commutative algebra: constructive methods: finite projective modules, Springer Netherlands, Dordrecht, 2015. https://doi.org/10.1007/978-94-017-9944-7.

[19] R. Mines, F. Richman, W. Ruitenburg, A course of constructive algebra, New York: Springer-Verlag, 1988.

[20] M.M. Krishna Rao, Γ-semirings–I, Southeast Asian Bull. Math. 19 (1995) 49–54.

[21] M. M. Krishna Rao, Γ-semirings–II, Southeast Asian Bull. Math. 21 (1997) 281–287.

[22] M.M. Krishna Rao, Ideals in ordered Γ-semirings, Discuss. Math. - Gen. Algebra Appl. 38 (2018) 47-–68. https://doi.org/10.7151/dmgaa.1284.

[23] D.A. Romano, Constructive algebra - algebraic structures, Ph.D. Thesis, Faculty of Mathematics, University of Belgrade, Belgrade 1985. (In Serbian)

[24] D.A. Romano, Aspect of constructive Abelian groups, In: Z. Stojaković (Ed.). Proceedings of the 5th conference "Algebra and Logic", Cetinje 1986 (pp. 167-174). Novi Sad: University of Novi Sad, Institute of Mathematics, 1987.

[25] D.A. Romano, Rings and fields, a constructive view, Math. Logic Quart. (Formerly: Z. Math. Logik Grundl. Math.) 34 (1988) 25–40. https://doi.org/10.1002/malq.19880340105.

[26] D.A. Romano, Rings and modules, a constructive view, In: S. Crvenković (Ed.). Proceedings of the 6th conference "Algebra and Logic", Sarajevo 1987 (pp. 113–140). Novi Sad: University of Novi Sad, Institute of Mathematics, 1989.

[27] D.A. Romano, Coequality relations, a survey, Bull. Soc. Math. Banja Luka, 3 (1996) 1–36.

[28] D.A. Romano, A note on two problems in constructive commutative coideal theory, Sci. Rev. Ser. Math. 19-20(1996) 91–96.

[29] D.A. Romano, Semivaluation on Heyting field, Kragujevac J. Mat. 20(1998) 24–40.

[30] D.A. Romano, Semilattice-ordered semiring with apartness, J. Adv. Math. Stud. 11 (2018) 496–502.

[31] D.A. Romano, Some algebraic structures with apartness, A review, J. Int. Math. Virtual Inst. 9 (2019) 361–395.

[32] D.A. Romano, Γ-semirings with apartness, Rom. J. Math. Comput. Sci. 9 (2019) 108–112.

[33] D.A. Romano, Ordered Γ-semirings with apartness under co-order, Rom. J. Math. Comput. Sci. 12 (2022) 10-17.

[34] D. A. Romano, A note on weak-interior and quasi-interior ideals in quasi-ordered semigroups, Discuss. Math., Gen. Algebra Appl. 43(2023) (In press).

[35] W. Ruitenburg, Intuitionistic algebra, Ph.D. Thesis, University of Utrecht, Utrecht 1982.

[36] G. Szasz. Interior ideals in semigroups, In: Notes on semigroups IV, Karl Marx Univ. Econ., Dept. Math. Budapest, 5(1977) 1-–7.

[37] G. Szasz, Remark on interior ideals of semigroups, Stud. Sci. Math. Hung. 16 (1981) 61-–63.

[38] A.S. Troelstra, D. van Dalen, Constructivism in Mathematics: An Introduction, Amsterdam: North-Holland, 1988.