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GERAGHTY-CIRIC TYPE MAPPINGS WITH AN APPLICATION

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Abstract. In this paper, we present some fixed point results for Geraghty - Ciric contraction type mappings in orbitally complete metric spaces. As an application, we give an existence and uniqueness for the solution of a nonlinear integral equation.

1. Introduction and Preliminaries

Banach contraction principle is one of the most important results in fixed point theory. Many generalizations and improvements of the contraction mapping principle have been obtained [[1]- [24]].
One of the most interesting improvement is the generalization of the contractive condition using comparison functions instead of constants which was first introduced by Geraghty [9]. By using such functions, Geraghty [9] proved the following theorem.

Let F be the family of all functions $\beta : [0, \infty) \to [0, 1)$ which satisfy the condition

$$\lim_{n\to\infty} \beta(t_n) = 1 \implies \lim_{n\to\infty} (t_n) = 0.$$

Theorem 1.1 [9]. Let (X, d) be a complete metric space and let T be a self mapping on X. Suppose that there exists $\beta \in F$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y), \tag{1}$$

then T has a unique fixed point $x^* \in X$ and $\{T^n x\}$ converges to x^* for all $x \in X$.

Since then, many authors have generalized and extended this result in diverse ways see ([5], [13], [12], [17] and [23]). Meanwhile, Ciric [7] defined the following concepts and proved the following fixed point result.

Definition 1.2 [7]. Let $T : X \to X$ be an mapping on a metric space. For each $x \in X$ and for any positive integer n, $O_T(x, n) = \{x, Tx, ..., T^n x\}$ and $O_T(x, \infty) = \{x, Tx, ..., T^n x, ...\}$. The set $O_T(x, \infty)$ is called the orbit of T at x and the metric space X is called T-orbitally complete if every Cauchy sequence in $O_T(x, \infty)$ is convergent in X.
Note that every complete metric space is T-orbitally complete. But the converse does not hold

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in general. An example in [14] shows that there exist a T-orbitally complete metric space that is not complete.

Although, quasi-contraction mapping is known in literature as a contractive mappings that generalizes several mappings, there has been some improvements of the mapping in recent times, see [6], [8], [16]).

Definition 1.3 [7]. An mapping $T : X \to X$ of a metric space X into itself is said to be a quasi-contraction if and only if there exists a number k, $0 \leq k < 1$, such that

$$d(Tx, Ty) \leq k \max\{d(x, y); d(x, Tx); d(y, Ty); d(x, Ty); d(y, Tx)\} \tag{2}$$

holds for every $x, y \in X$.

Popescu [17], on the other hand introduced the following mappings, as an improvement of the admissible mappings defined in [11] and [20], to obtain some fixed point results for α-Geraghty contraction type mappings in metric spaces.

Definition 1.4 [17]. Let $T : X \to X$ be a self mapping and $\gamma : X \times X \to R^+$ be a function. Then T is said to be

(i) γ-orbital admissible if $\gamma(x, Tx) \geq 1$ implies $\gamma(Tx, T^2x) \geq 1$.
(ii) triangular γ-orbital admissible if T is γ-orbital admissible, $\gamma(x, y) \geq 1$ and $\gamma(y, Ty) \geq 1$ imply $\gamma(x, Ty) \geq 1$.

Lemma 1.5 [17]. Let $T : X \to X$ be a triangular γ-orbital admissible mapping. Assume that there exists $x_1 \in X$ such that $\gamma(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then, we have $\gamma(x_n, x_m) \geq 1$ for all $m, n \in N$ with $n < m$.

Let Φ denote the class of the functions $\phi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

(i) $\phi(t) = 0 \iff t = 0$;
(ii) ϕ is nondecreasing;
(iii) ϕ is continuous.

The purpose of this paper is to prove the existence and uniqueness of solution for a nonlinear integral equation using fixed point results of γ-ϕ-Geraghty quasi-contraction type mappings which are also generalizations of ´Ciric and Geraghty contraction type mappings.

2. Main Results

Firstly, we restate the class of mappings introduced in Umudu et al. [23] and fixed point results as follows:

Definition 2.1. Let (X, d) be a metric space and $\gamma : X \times X \to \mathbb{R}^+$. A self mapping $T : X \to X$ is called a γ-Geraghty quasi-contraction type mapping if there exists $\beta \in F$ such that for all $x, y \in X$,

$$\gamma(x, y)(d(Tx, Ty)) \leq \beta(M_T(x, y))(M_T(x, y)) \tag{3}$$

where $M_T(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$.

Definition 2.2. Let (X, d) be a metric space and $\gamma : X \times X \to \mathbb{R}^+$. A self mapping $T : X \to X$ is called a γ-ϕ-Geraghty quasi-contraction type mapping if there exists $\beta \in F$ such that for all

$x, y \in X$,

$$\gamma(x,y)\phi(d(Tx,Ty)) \leq \beta(\phi(M_T(x,y)))\phi(M_T(x,y)), \tag{4}$$

where $M_T(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$, and $\phi \in \Phi$.

Note that since $\beta : [0,\infty) \to [0,1)$, inequality 3 reduces to 2 if $\beta(M_T(x,y))$ is replaced with a constant k, $k \in [0,1)$ for any $x, y \in X$ and $\gamma(x,y) = 1$.

Theorem 2.3. Let (X,d) be a T-orbitally complete metric space, $\gamma : X \times X \to \mathbb{R}^+$ be a function, and $T : X \to X$ be a self mapping. Suppose that the following conditions are satisfied:

(i) T is a γ-ϕ-Geraghty quasi-contraction type mapping.
(ii) T is triangular γ-orbital admissible mapping.
(iii) There exists $x_1 \in X$ such that $\gamma(x_1, Tx_1) \geq 1$.
(iv) For all $x \neq y \in Fix(T)$, there exists $w \in X$ such that $\alpha(x,w) \geq 1, \alpha(y,w) \geq 1$ and $\alpha(w,Tw) \geq 1$.

Then T has a unique fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Taking $\phi(t) = t$ in inequality 4, T becomes a γ-Geraghty quasi-contraction type mapping and Theorem 2.3 reduces to the following.

Corollary 2.4. Let (X,d) be a T-orbitally complete metric space, $\gamma : X \times X \to \mathbb{R}^+$ be a function, and let $T : X \to X$ be a self mapping. Suppose that the following conditions are satisfied:

(i) T is a γ-Geraghty quasi-contraction type mapping.
(ii) T is triangular γ-orbital admissible mapping.
(iii) There exists $x_1 \in X$ such that $\gamma(x_1, Tx_1) \geq 1$.
(iv) For all $x \neq y \in Fix(T)$, there exists $w \in X$ such that $\alpha(x,w) \geq 1, \alpha(y,w) \geq 1$ and $\alpha(w,Tw) \geq 1$.

Then T has a unique fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

3. An application

Consider the following nonlinear integral equation:

$$x(t) = \int_a^b f(t,s)g(t,s,x(s))ds, \tag{5}$$

where $f : [a,b] \times [a,b] \to \mathbb{R}$ and $g : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ are well defined, continuous functions.

Theorem 3.1. With the assumption of Theorem 2.3 and considering the nonlinear integral equation 5, suppose the following conditions hold:

(i) $T^2 x(s) \geq 0$, for all $x \in X$, $s \in [a,b]$ and $x(s) \geq 0$.

(ii) For all $t, s \in [a, b]$ and $x, y \in X$ such that $x(s), y(s) \in [0, \infty)$, we have

$$\begin{aligned} |g(t,s,x(s)) - g(t,s,y(s))| &\leq f(t,s)\max\{|x(s)-y(s)|, |x(s)-Tx(s)|, |y(s)-Ty(s)|, \\ &\quad |x(s)-Ty(s)|, |y(s)-Tx(s)|\} \\ &\leq \frac{\ln(1+(|x(s)-y(s)|)}{2}. \end{aligned}$$

(iii) $\max_{a\leq t\leq b}\int_a^b f(t,s)ds \leq \frac{1}{(b-a)}$, for all $t, s \in [a, b]$:

(iv) There exists $x_1 \in X$ such that $x_1(t) \geq 0$ and $Tx_1(t) \geq 0$ for all $t \in [a, b]$.

Then the nonlinear integral equation 5 has a unique solution in X.

Proof. Let X be the set of all real continuous functions on [a,b] and let $T : X \to X$ be a self-mapping defined by:

$$Tx(t) = \int_a^b f(t,s)g(t,s,x(s))ds.$$

Let $d : X \times X \to [0, \infty)$ be defined by:

$$d(x,y) = \max_{a\leq t\leq b}|x(t)-y(t)|, \quad x, y \in X$$

and define a mapping $\gamma : [a, b] \times [a, b] \to \mathbb{R}^+$ by

$$\gamma(x,y) = \begin{cases} 1, & x(s), y(s) \in [0,\infty) \quad for \quad all \quad x \in [a,b]; \\ 0, & otherwise. \end{cases}$$

(1) We claim that T is a γ-ϕ-Geraghty quasi-contraction type mapping. From inequality (5), we get

$$\begin{aligned} \gamma(x,y)\phi(d(Tx,Ty)) &= \phi(|Tx(s)-Ty(s)|) \\ &= \phi\left(\max_{a\leq t\leq b}|Tx(t)-Ty(t)|\right) \\ &\leq \phi\left(\max_{a\leq t\leq b}\left\{\left|\int_a^b f(t,s)g(t,s,x(s))ds - \int_a^b f(t,s)g(t,s,y(s))ds\right|\right\}\right) \\ &\leq \phi\left(\max_{a\leq t\leq b}\left\{\int_a^b |f(t,s)||g(t,s,x(s))-g(t,s,y(s))|ds\right\}\right) \\ &\leq \phi\max_{a\leq t\leq b}\int_a^b |f(t,s)|ds \max\{|x(s)-y(s)|, |x(s)-Tx(s)|, |y(s)-Ty(s)|, \\ &\quad |x(s)-Ty(s)|, |y(s)-Tx(s)|\} \\ &\leq \phi\left(\frac{1}{b-a}\left(\frac{\ln(1+|x(s)-y(s)|)}{2}\right)\right) \\ &\leq \frac{1}{b-a}\ln(1+2M_T(x,y)) \\ &\leq \ln(1+2M_T(x,y)). \end{aligned}$$

Therefore,

$$\begin{aligned}\gamma(x,y)\phi(d(Tx,Ty)) &\leq \frac{\ln(1+2M_T(x,y))}{M_T(x,y)}M_T(x,y)\\ &\leq \beta(\phi(M_T(x,y)))\phi(M_T(x,y)).\end{aligned}$$

Thus, T is an γ-ϕ-Geraghty quasi-contraction type mapping with $\beta = \frac{\ln(1+t)}{t}, t > 0$ and $\phi(t) = 2t$.

(2) Next, we show that T is triangular γ-orbital admissible mapping. For $x \in [a,b]$ such that $\gamma(x,Tx) \geq 1$, we have $x(s) \geq 0$ for all $x \in [a,b]$. It follows from condition (i) that $T^2x(s) \geq 0$. Therefore, $\gamma(x,T^2x) \geq 1$ and hence T is γ-orbital admissible mapping. Furthermore, for $x,y \in [a,b]$ such that $\gamma(x,y) \geq 1$ and $\gamma(y,Ty) \geq 1$, we have $x(s), y(s), Ty(s) \geq 0$ for all $s \in [a,b]$. It implies that $\gamma(x,Ty) \geq 1$. Thus, T is a triangular γ-orbital mapping.

(3) By condition (iv), we have that $\gamma(x,Tx_1) \geq 1$.

Thus, all the conditions of Theorem 2.3 are satisfied and T has a fixed point in X and hence equation 5 has a unique solution $x \in [a,b]$.

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