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ON THE SOLUTIONS AND THE PERIODICITY OF SOME RATIONAL DIFFERENCE EQUATIONS SYSTEMS

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Abstract. In this paper, we get the form of the solutions of the following difference equation systems of order three

$$z_{n+1} = \frac{w_n w_{n-2}}{z_{n-1}(1 + w_n w_{n-2})}, \quad w_{n+1} = \frac{z_n z_{n-2}}{w_{n-1}(\pm 1 \pm z_n z_{n-2})},$$

where the initial conditions $z_{-2},\ z_{-1},\ z_0, w_{-2},\ w_{-1},\ w_0$ are arbitrary non-zero real numbers.

1. Introduction

Many academics' attention has recently been attracted to rational difference equations for a variety of reasons. On the one hand, they give examples of nonlinear equations that can sometimes be solved, but their dynamics have certain new characteristics in comparison to the linear case. On the other hand, some biological models typically use rational equations. As a result, their study is interesting because of the applicability. Ricatti difference equations are a good illustration of both of these facts because of how rich their dynamics are (see [6]). Obviously, higher-order rational difference equations and systems of rational equations have also received a lot of attention, but there are still a lot of aspects to be investigated. There are many papers related to the difference equations systems for example in [5] Clark and Kulenovic investigated the global asymptotic stability

$$z_{n+1} = \frac{z_n}{a + cw_n}, \quad w_{n+1} = \frac{w_n}{b + dz_n}.$$

Elsayed [14] has found the solutions of the following systems of the difference equations

$$z_{n+1} = \frac{z_{n-1}}{\pm 1 \pm z_{n-1} w_n}, \quad w_{n+1} = \frac{w_{n-1}}{\pm 1 \pm w_{n-1} z_n}.$$

In [44] Yalcinkaya looked into the necessary condition for the global asymptotic stability of the following system of difference equations

$$z_{n+1} = \frac{z_n + w_{n-1}}{z_n w_{n-1} - 1}, \quad w_{n+1} = \frac{w_n + z_{n-1}}{w_n z_{n-1} - 1}.$$

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Kurbanli et al. [35] investigated the solutions the following dynamical system of recursive equations

$$z_{n+1}=\frac{z_{n-1}}{w_n z_{n-1}-1},\quad w_{n+1}=\frac{w_{n-1}}{z_n w_{n-1}-1},\quad u_{n+1}=\frac{z_n}{w_n u_{n-1}}.$$

Touafek and Elsayed [40] examined the periodic nature and found the form of the solutions of the following systems of rational difference equations

$$z_{n+1}=\frac{z_{n-3}}{\pm 1\pm z_{n-3}w_{n-1}},\quad w_{n+1}=\frac{w_{n-3}}{\pm 1\pm w_{n-3}z_{n-1}}.$$

Ozban [39] has researched the system of rational difference equations' positive solution

$$z_{n+1}=\frac{a}{w_{n-3}},\quad w_{n+1}=\frac{w_{n-3}}{z_{n-q}w_{n-q}}.$$

El-Dessoky et al. [12] examined the periodicity and got the form of the solutions of the following systems

$$z_{n+1}=\frac{z_n w_{n-3}}{w_{n-2}(\pm 1\pm z_n w_{n-3})},\quad w_{n+1}=\frac{w_n z_{n-3}}{z_{n-2}(\pm 1\pm w_n z_{n-3})}.$$

For more studies for nonlinear difference equations and systems of rational difference equations see [1]- [48].

In this paper, we examine the periodicity and the form of some order three nonlinear difference equation systems' solutions

$$z_{n+1}=\frac{w_n w_{n-2}}{z_{n-1}(1+w_n w_{n-2})},\quad w_{n+1}=\frac{z_n z_{n-2}}{w_{n-1}(\pm 1\pm z_n z_{n-2})},$$

where the initial conditions $z_{-2},\ z_{-1},\ z_0,\ w_{-2},\ w_{-1},\ w_0$ are arbitrary non-zero real numbers.

2. On the system $z_{n+1}=\frac{w_n w_{n-2}}{z_{n-1}(1+w_n w_{n-2})}, w_{n+1}=\frac{z_n z_{n-2}}{w_{n-1}(1+z_n z_{n-2})}$

In this section, we study the solution of two difference equations system

$$z_{n+1}=\frac{w_n w_{n-2}}{z_{n-1}(1+w_n w_{n-2})},\quad w_{n+1}=\frac{z_n z_{n-2}}{w_{n-1}(1+z_n z_{n-2})},\tag{1}$$

where the initial conditions $z_{-2},\ z_{-1},\ z_0,\ w_{-2},\ w_{-1},\ w_0$ are arbitrary non-zero real numbers.

Theorem 1. *Assume that the solutions to system (1) are $\{z_n, w_n\}_{n=-2}^{\infty}$. Then, for $n=0,1,2,...$, we observe that the following formulas provide all of the solutions to the system (1):*

$$z_{4n-2}=\frac{c\prod_{i=0}^{n-1}(1+(4i)ac)}{\prod_{i=0}^{n-1}(1+(4i+2)ac)},\qquad z_{4n-1}=\frac{b\prod_{i=0}^{n-1}(1+(4i+1)fd)}{\prod_{i=0}^{n-1}(1+(4i+3)fd)},$$

$$z_{4n}=\frac{a\prod_{i=0}^{n-1}(1+(4i+2)ac)}{\prod_{i=0}^{n-1}(1+(4i+4)ac)},\qquad z_{4n+1}=\frac{fd\prod_{i=0}^{n-1}(1+(4i+3)fd)}{b\ (1+fd)\prod_{i=0}^{n-1}(1+(4i+5)fd)},$$

$$w_{4n-2}=\frac{d\prod_{i=0}^{n-1}(1+(4i)fd)}{\prod_{i=0}^{n-1}(1+(4i+2)fd)},\qquad z_{4n-1}=\frac{e\prod_{i=0}^{n-1}(1+(4i+1)ac)}{\prod_{i=0}^{n-1}(1+(4i+3)ac)},$$

$$w_{4n} = \frac{f \prod_{i=0}^{n-1}(1+(4i+2)fd)}{\prod_{i=0}^{n-1}(1+(4i+4)fd)}, \qquad w_{4n+1} = \frac{ac \prod_{i=0}^{n-1}(1+(4i+3)ac)}{e\ (1+ac) \prod_{i=0}^{n-1}(1+(4i+5)ac)},$$

where $z_{-2} = c,\ z_{-1} = b,\ z_0 = a,\ w_{-2} = d,\ w_{-1} = e,\ w_0 = f.$

Proof. For n = 0 the result holds. Now suppose that n > 0 and that our assumption holds for n − 1, that is

$$z_{4n-6} = \frac{c \prod_{i=0}^{n-2}(1+(4i)ac)}{\prod_{i=0}^{n-2}(1+(4i+2)ac)}, \qquad z_{4n-5} = \frac{b \prod_{i=0}^{n-2}(1+(4i+1)fd)}{\prod_{i=0}^{n-2}(1+(4i+3)fd)},$$

$$z_{4n-4} = \frac{a \prod_{i=0}^{n-2}(1+(4i+2)ac)}{\prod_{i=0}^{n-2}(1+(4i+4)ac)}, \qquad z_{4n-3} = \frac{fd \prod_{i=0}^{n-2}(1+(4i+3)fd)}{b\ (1+fd) \prod_{i=0}^{n-2}(1+(4i+5)fd)},$$

$$w_{4n-6} = \frac{d \prod_{i=0}^{n-2}(1+(4i)fd)}{\prod_{i=0}^{n-2}(1+(4i+2)fd)}, \qquad z_{4n-5} = \frac{e \prod_{i=0}^{n-2}(1+(4i+1)ac)}{\prod_{i=0}^{n-2}(1+(4i+3)ac)},$$

$$w_{4n-4} = \frac{f \prod_{i=0}^{n-2}(1+(4i+2)fd)}{\prod_{i=0}^{n-2}(1+(4i+4)fd)}, \qquad w_{4n-3} = \frac{ac \prod_{i=0}^{n-2}(1+(4i+3)ac)}{e\ (1+ac) \prod_{i=0}^{n-2}(1+(4i+5)ac)},$$

From system (1) we have

$$\begin{aligned} z_{4n-2} &= \frac{w_{4n-3}w_{4n-5}}{z_{4n-4}(1+w_{4n-3}w_{4n-5})} \\ &= \frac{1}{z_{4n-4}(\frac{1}{w_{4n-3}w_{4n-5}}+1)}. \end{aligned}$$

$$\begin{aligned} w_{4n-3}w_{4n-5} &= \frac{ac}{e(1+ac)} \times \prod_{i=0}^{n-2} \frac{(1+(4i+3)ac)}{(1+(4i+5)ac)} \times e \prod_{i=0}^{n-2} \frac{(1+(4i+1)ac)}{(1+(4i+3)ac)} \\ &= \frac{ac}{(1+ac)} \prod_{i=0}^{n-2} \frac{(1+(4i+1)ac)}{(1+(4i+5)ac)} \\ &= \frac{ac}{(1+ac)} \times \frac{(1+ac)(1+5ac)...(1+(4n-7)ac)}{(1+5ac)...(1+(4n-7)ac)(1+(4n-3)ac)} \\ &= \frac{ac}{(1+(4n-3)ac)}. \end{aligned}$$

$$z_{4n-2} = \frac{1}{a \prod_{i=0}^{n-2} \frac{(1+(4i+2)ac)}{(1+(4i+4)ac)}(\frac{(1+(4n-3)ac)}{ac}+1)}$$

$$= \frac{c\prod_{i=0}^{n-2}(1+(4i+4)ac)}{\prod_{i=0}^{n-2}(1+(4i+2)ac)(1+(4n-2)ac)}$$
$$= \frac{c\prod_{i=0}^{n-1}(1+(4i)ac)}{\prod_{i=0}^{n-1}(1+(4i+2)ac)}.$$

$$\begin{aligned} w_{4n-2} &= \frac{z_{4n-3}z_{4n-5}}{w_{4n-4}(1+z_{4n-3}z_{4n-5})} \\ &= \frac{1}{w_{4n-4}(\frac{1}{z_{4n-3}z_{4n-5}}+1)}. \end{aligned}$$

$$\begin{aligned} z_{4n-3}z_{4n-5} &= \frac{fd}{b(1+fd)} \times \prod_{i=0}^{n-2}\frac{(1+(4i+3)fd)}{(1+(4i+5)fd)} \times b\prod_{i=0}^{n-2}\frac{(1+(4i+1)fd)}{(1+(4i+3)fd)} \\ &= \frac{fd}{(1+fd)}\prod_{i=0}^{n-2}\frac{(1+(4i+1)fd)}{(1+(4i+5)fd)} \\ &= \frac{fd}{(1+fd)} \times \frac{(1+fd)(1+5fd)...(1+(4n-7)fd)}{(1+5fd)...(1+(4n-7)fd)(1+(4n-3)fd)} \\ &= \frac{fd}{(1+(4n-3)fd)}. \end{aligned}$$

$$\begin{aligned} w_{4n-2} &= \frac{1}{f\prod_{i=0}^{n-2}\frac{(1+(4i+2)fd)}{(1+(4i+4)fd)}(\frac{(1+(4n-3)fd)}{fd}+1)} \\ &= \frac{d\prod_{i=0}^{n-2}(1+(4i+4)fd)}{\prod_{i=0}^{n-2}(1+(4i+2)fd)(1+(4n-2)fd)} \\ &= \frac{d\prod_{i=0}^{n-1}(1+(4i)fd)}{\prod_{i=0}^{n-1}(1+(4i+2)fd)}. \end{aligned}$$

So, we can prove the other relations and the proof is completed. □

3. On the system $z_{n+1} = \frac{w_n w_{n-2}}{z_{n-1}(1+w_n w_{n-2})}$, $w_{n+1} = \frac{z_n z_{n-2}}{w_{n-1}(1-z_n z_{n-2})}$

Here, we get the solutions of the following system of the difference equations

$$z_{n+1} = \frac{w_n w_{n-2}}{z_{n-1}(1+w_n w_{n-2})}, \qquad w_{n+1} = \frac{z_n z_{n-2}}{w_{n-1}(1-z_n z_{n-2})}, \tag{2}$$

where the initial conditions z_{-2}, z_{-1}, z_0, w_{-2}, w_{-1}, w_0 are arbitrary non-zero real numbers.

Theorem 2. *System (2) has a periodic solution of period four. Moreover* $\{z_n, w_n\}_{n=-2}^{\infty}$ *takes the form*

$$z_n = \{c, b, a, \frac{fd}{b(1+fd)}, c, b, ...\},$$

$$w_n = \{d, e, f, \frac{ac}{e(1-ac)}, d, e, ...\},$$

or $z_{4n-2} = c, z_{4n-1} = b, z_{4n} = a, z_{4n+1} = \frac{fd}{b(1+fd)}$ and $w_{4n-2} = d, w_{4n-1} = e, w_{4n} = f, w_{4n+1} = \frac{ac}{e(1-ac)}$, where $z_{-2} = c$, $z_{-1} = b$, $z_0 = a$, $w_{-2} = d$, $w_{-1} = e$, $w_0 = f$.

Proof. For n = 0 the result holds. Now suppose that n > 0 and that our assumption holds for n − 1, that is

$$z_{4n-6} = c, z_{4n-5} = b, z_{4n-4} = a, z_{4n-3} = \frac{fd}{b(1+fd)},$$

$$w_{4n-6} = d, w_{4n-5} = e, w_{4n-4} = f, w_{4n-3} = \frac{ac}{e(1-ac)}.$$

Now, it follows from system (2) that

$$\begin{aligned} z_{4n-2} &= \frac{w_{4n-3}w_{4n-5}}{z_{4n-4}(1+w_{4n-3}w_{4n-5})} \\ &= \frac{\frac{ac}{e(1-ac)}.e}{a(1+\frac{ac}{(1-ac)})} \\ &= \frac{\frac{c}{1-ac}}{\frac{1}{1-ac}} = c, \end{aligned}$$

$$\begin{aligned} z_{4n-1} &= \frac{w_{4n-2}w_{4n-4}}{z_{4n-3}(1+w_{4n-2}w_{4n-4})} \\ &= \frac{df}{\frac{fd}{b(1+fd)}(1+fd)} = b, \end{aligned}$$

also

$$\begin{aligned} w_{4n-2} &= \frac{z_{4n-3}z_{4n-5}}{w_{4n-4}(1+z_{4n-3}z_{4n-5})} \\ &= \frac{\frac{fd}{b(1+fd)}.b}{f(1-\frac{fd}{1+fd})} \\ &= \frac{\frac{d}{1+fd}}{\frac{1}{1+fd}} = d, \end{aligned}$$

$$\begin{aligned} w_{4n-1} &= \frac{z_{4n-2}z_{4n-4}}{w_{4n-3}(1+z_{4n-2}z_{4n-4})} \\ &= \frac{ac}{\frac{ac}{e(1-ac)}(1-ac)} = e. \end{aligned}$$

The other relations can be proved by similar way. The proof is completed. □

4. On the system $z_{n+1} = \frac{w_n w_{n-2}}{z_{n-1}(1+w_n w_{n-2})}, w_{n+1} = \frac{z_n z_{n-2}}{w_{n-1}(-1+z_n z_{n-2})}$

In this section, we deal with the solutions of the system of the difference equations

$$z_{n+1} = \frac{w_n w_{n-2}}{z_{n-1}(1 + w_n w_{n-2})}, \qquad w_{n+1} = \frac{z_n z_{n-2}}{w_{n-1}(-1 + z_n z_{n-2})}, \tag{3}$$

where the initial conditions $z_{-2}, z_{-1}, z_0, w_{-2}, w_{-1}, w_0$ are arbitrary non-zero real numbers.

Theorem 3. *Suppose that $\{z_n, w_n\}_{n=-2}^{\infty}$ are solutions of system (3). Then for n = 0, 1, 2, ...,*

$$z_{4n-2} = \frac{c}{(-1+2ac)^n}, \quad z_{4n-1} = \frac{b(1+fd)^n}{(-1+fd)^n},$$
$$z_{4n} = a(-1+2ac)^n, \quad z_{4n+1} = \frac{fd(-1+fd)^n}{b(1+fd)^{n+1}},$$
$$w_{4n-2} = (-1)^n d\ , \ w_{4n-1} = (-1)^n e, \ w_{4n} = (-1)^n f, \quad w_{4n+1} = \frac{(-1)^n ac}{e(-1+ac)},$$

where $z_{-2} = c$, $z_{-1} = b$, $z_0 = a$, $w_{-2} = d$, $w_{-1} = e$, $w_0 = f$ with $w_0 w_{-2} \neq \pm 1$ and $z_0 z_{-2} \neq 1$, 0.5.

Proof. By using mathematical induction. The result holds for $n = 0$. Suppose that the result holds for $n - 1$

$$z_{4n-6} = \frac{c}{(-1+2ac)^{n-1}}, \quad z_{4n-5} = \frac{b(1+fd)^{n-1}}{(-1+fd)^{n-1}},$$
$$z_{4n-4} = a(-1+2ac)^{n-1}, \quad z_{4n-3} = \frac{fd(-1+fd)^{n-1}}{b(1+fd)^n},$$
$$w_{4n-6} = (-1)^{n-1} d\ , \ w_{4n-5} = (-1)^{n-1} e, \ w_{4n-4} = (-1)^{n-1} f, \quad w_{4n-3} = \frac{(-1)^{n-1} ac}{e(-1+ac)}.$$

From system (3) we have

$$\begin{aligned} z_{4n-2} &= \frac{w_{4n-3} w_{4n-5}}{z_{4n-4}(1 + w_{4n-3} w_{4n-5})} \\ &= \frac{\frac{(-1)^{n-1} ac}{e(-1+ac)} \times (-1)^{n-1} e}{a(-1+2ac)^{n-1}(1 + \frac{(-1)^{2n-2} ac}{-1+ac})} \\ &= \frac{\frac{(-1)^{2n-2} c}{-1+ac}}{(-1+2ac)^{n-1} \frac{(-1+2ac)}{-1+ac}} \\ &= \frac{c}{(-1+2ac)^n}. \end{aligned}$$

$$\begin{aligned} z_{4n-1} &= \frac{w_{4n-2} w_{4n-4}}{z_{4n-3}(1 + w_{4n-2} w_{4n-4})} \\ &= \frac{(-1)^n d\ (-1)^{n-1} f}{\frac{fd(-1+fd)^{n-1}}{b(1+fd)^n}(1 + (-1)^{2n-1} fd)} \\ &= \frac{(-1)^{2n-1} fd\ \ b\ (1+fd)^n}{fd(-1+fd)^{n-1}(1-fd)} \\ &= \frac{b(1+fd)^n}{(-1+fd)^n}. \end{aligned}$$

Also

$$\begin{aligned} w_{4n-2} &= \frac{z_{4n-3}z_{4n-5}}{w_{4n-4}(1+z_{4n-3}z_{4n-5})} \\ &= \frac{\frac{fd(-1+fd)^{n-1}}{b(1+fd)^{n}} \times \frac{b(1+fd)^{n-1}}{(-1+fd)^{n-1}}}{(-1)^{n-1}f\ (-1+\frac{fd}{1+fd})} \\ &= \frac{\frac{d}{1+fd}}{(-1)^{n-1}(\frac{-1}{1+fd})} \\ &= (-1)^{n}d. \end{aligned}$$

$$\begin{aligned} w_{4n-1} &= \frac{z_{4n-2}z_{4n-4}}{w_{4n-3}(1+z_{4n-2}z_{4n-4})} \\ &= \frac{\frac{e}{(-1+2ac)^{n}} \times a(-1+2ac)^{n-1}}{\frac{(-1)^{n-1}ac}{e(-1+ac)}(-1+\frac{ac}{-1+2ac})} \\ &= \frac{\frac{ac}{-1+2ac}}{\frac{(-1)^{n-1}ac}{e(-1+ac)}(1-ac)} \\ &= \frac{e\ (-1+ac)}{(-1)^{n}(-1+ac)} \\ &= (-1)^{n}e. \end{aligned}$$

Similarly we can prove other relations and the proof is completed. □

5. On the system $z_{n+1} = \frac{w_n w_{n-2}}{z_{n-1}(1+w_n w_{n-2})}, w_{n+1} = \frac{z_n z_{n-2}}{w_{n-1}(-1-z_n z_{n-2})}$

We study in this section the solutions of the system of two difference equations

$$z_{n+1} = \frac{w_n w_{n-2}}{z_{n-1}(1+w_n w_{n-2})}, \quad w_{n+1} = \frac{z_n z_{n-2}}{w_{n-1}(-1-z_n z_{n-2})}, \tag{4}$$

where the initial conditions $z_{-2}, z_{-1}, z_0, w_{-2}, w_{-1}, w_0$ are arbitrary non-zero real numbers.

Theorem 4. *Assume that $\{z_n, w_n\}_{n=-2}^{\infty}$ is a solution to system (4) and let $z_{-2} = c$, $z_{-1} = b$, $z_0 = a$, $w_{-2} = d$, $w_{-1} = e$, $w_0 = f$ with $w_0 w_{-2} \neq -1,-0.5$ and $z_0 z_{-2} \neq \pm 1$. Then, for n = 0, 1, ..., we have*

$$z_{4n-2} = (-1)^n c, \quad z_{4n-1} = (-1)^n b, \quad z_{4n} = (-1)^n a, \quad z_{4n+1} = \frac{(-1)^n fd}{b(1+fd)},$$

$$w_{4n-2} = \frac{(-1)^n d}{(1+2fd)^n}, \qquad w_{4n-1} = \frac{e\ (1+ac)^n}{(-1+ac)^n},$$

$$w_{4n} = (-1)^n\ f\ (1+2fd)^n, \qquad w_{4n+1} = \frac{-ac\ (-1+ac)^n}{e(1+ac)^{n+1}}.$$

Proof. For n = 0 the result holds. Now suppose that n > 0 and that our assumption holds for $n-1$. That is,

$$z_{4n-6} = (-1)^{n-1}c, \quad z_{4n-5} = (-1)^{n-1}b, \quad z_{4n-4} = (-1)^{n-1}a, \quad z_{4n-3} = \frac{(-1)^{n-1}fd}{b(1+fd)},$$

$$w_{4n-6}=\frac{(-1)^{n-1}d}{(1+2fd)^{n-1}},\qquad w_{4n-5}=\frac{e\ (1+ac)^{n-1}}{(-1+ac)^{n-1}},$$

$$w_{4n-4}=(-1)^{n-1}\ f\ (1+2fd)^{n-1},\qquad w_{4n-3}=\frac{-ac\ (-1+ac)^{n-1}}{e\ (1+ac)^{n}}.$$

From system (4) we can prove as follow

$$\begin{aligned} z_{4n-2} &= \frac{w_{4n-3}w_{4n-5}}{z_{4n-4}(1+w_{4n-3}w_{4n-5})} \\ &= \frac{\frac{-ac\ (-1+ac)^{n-1}}{e(1+ac)^{n}}\times\frac{e\ (1+ac)^{n-1}}{(-1+ac)^{n-1}}}{(-1)^{n-1}a\ (1-\frac{ac}{1+ac})} \\ &= \frac{-c}{(-1)^{n-1}}=(-1)^{n}c. \end{aligned}$$

$$\begin{aligned} z_{4n-1} &= \frac{w_{4n-2}w_{4n-4}}{z_{4n-3}(1+w_{4n-2}w_{4n-4})} \\ &= \frac{\frac{(-1)^{n}d}{(1+2fd)^{n}}\times(-1)^{n-1}\ f\ (1+2fd)^{n-1}}{\frac{(-1)^{n-1}fd}{b(1+fd)}\ (1+\frac{(-1)^{2n-1}fd}{1+2fd})} \\ &= \frac{\frac{(-1)^{n}b(1+fd)}{1+2fd}}{1-\frac{fd}{1+2fd}} \\ &= (-1)^{n}b. \end{aligned}$$

$$\begin{aligned} w_{4n-2} &= \frac{z_{4n-3}z_{4n-5}}{w_{4n-4}(1+z_{4n-3}z_{4n-5})} \\ &= \frac{\frac{(-1)^{n-1}fd}{b(1+fd)}\times(-1)^{n-1}b}{(-1)^{n-1}\ f\ (1+2fd)^{n-1}(-1-\frac{(1)^{2n-2}fd}{1+fd})} \\ &= \frac{\frac{(-1)^{n-1}d}{1+fd}}{(1+2fd)^{n-1}(-1-\frac{fd}{1+fd})} \\ &= \frac{\frac{(-1)^{n-1}d}{1+fd}}{(1+2fd)^{n-1}(\frac{-1-2fd}{1+fd})} \\ &= \frac{(-1)^{n}d}{(1+2fd)^{n}}. \end{aligned}$$

$$\begin{aligned} w_{4n-1} &= \frac{z_{4n-2}z_{4n-4}}{w_{4n-3}(1+z_{4n-2}z_{4n-4})} \\ &= \frac{(-1)^{n}c\ \ (-1)^{n-1}a}{\frac{-ac\ (-1+ac)^{n-1}}{e(1+ac)^{n}}(-1-(-1)^{2n-1}ac)} \\ &= \frac{-e(1+ac)^{n}}{-(-1+ac)^{n-1}(-1+ac)} \\ &= \frac{e(1+ac)^{n}}{(-1+ac)^{n}}. \end{aligned}$$

Similarly we can prove other relations and the proof is completed. □

Remark 1. *System (4) has a periodic solution of period eight.*

6. Numerical Examples

In order to illustrate the results of the previous sections and to support our theoretical discussions, we consider several interesting numerical examples in this section.

Example 1. *We take the initial conditions, for the system (1), as follows* $z_{-2} = -1$,$z_{-1} = 10$, $z_0 = 5$, $w_{-2} = 20$, $w_{-1} = 11$ *and* $w_0 = -2$.*See Fig. 1.*

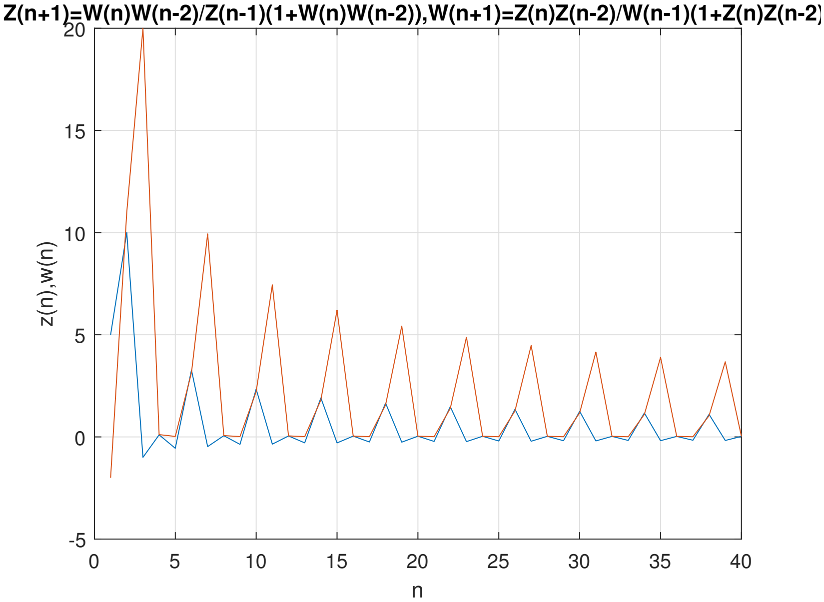

Fig. 1.

Example 2. *We Put the following initial conditions* $z_{-2} = -1.5$,$z_{-1} = 13$, $z_0 = 3$, $w_{-2} = -26$, $w_{-1} = 9$ *and* $w_0 = 2.3$ *on system (1). See Fig. 2.*

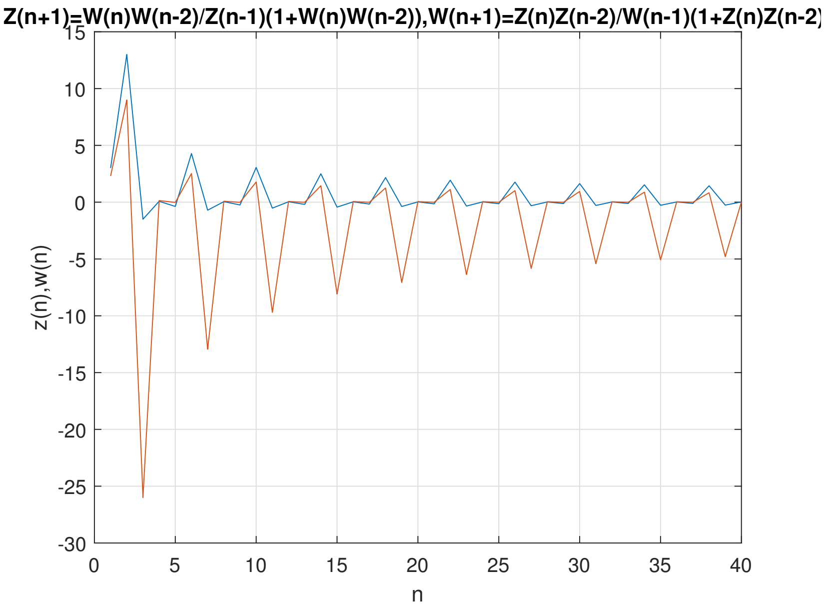

Fig. 2.

Example 3. *See Fig. 3, we put the initial condition* $z_{-2} = -8$,$z_{-1} = 3.5$, $z_0 = 4$, $w_{-2} = -25$, $w_{-1} = 10$ *and* $w_0 = 1.75$ *on the difference system(2).*

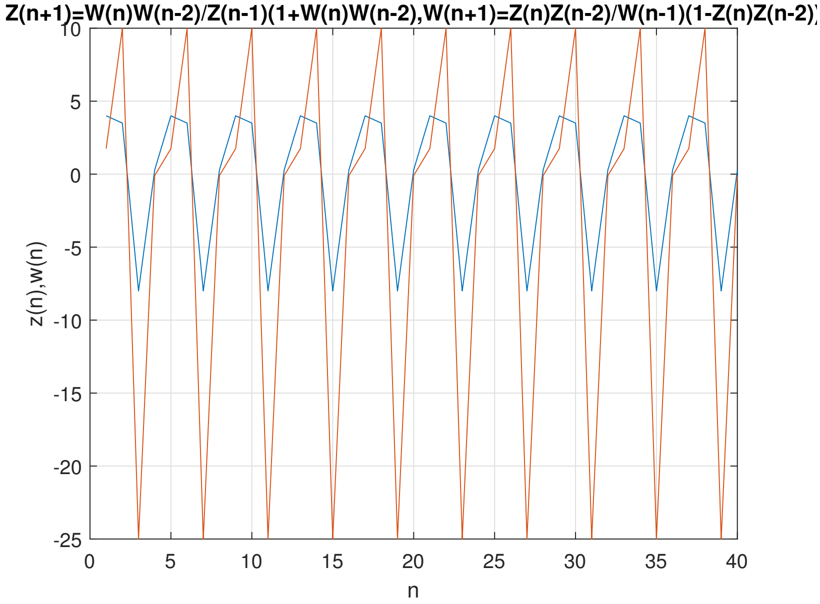

FIG. 3.

Example 4. *Fig. 4 shows the behavior of the solution of the difference system (2) with the initial conditions* $z_{-2} = 7$,$z_{-1} = 22$, $z_0 = 13$, $w_{-2} = 26$, $w_{-1} = 12$ *and* $w_0 = 5$.

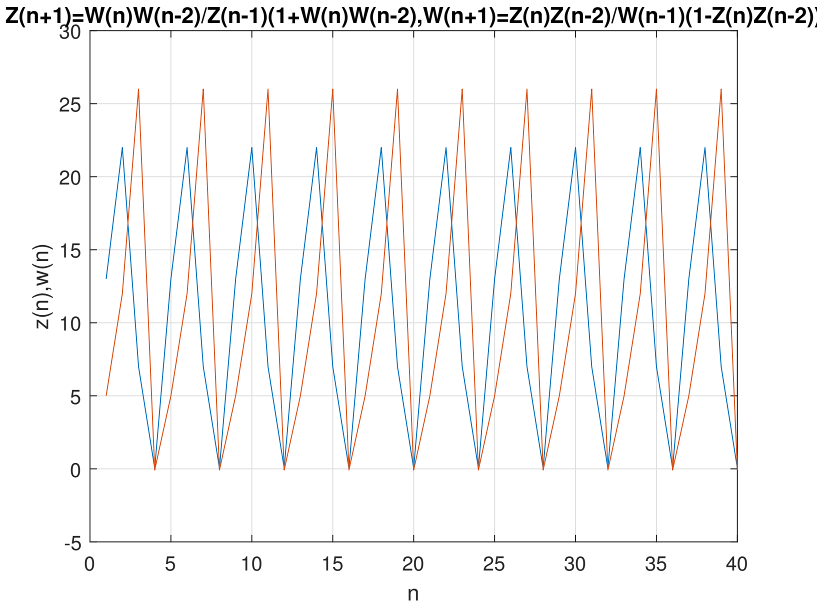

FIG. 4.

Example 5. *We consider numerical example for the difference system (3) with the initial conditions* $z_{-2} = 1.22$,$z_{-1} = 0.22$, $z_0 = 0.1$, $w_{-2} = 0.03$, $w_{-1} = 0.1$ *and* $w_0 = 1.3$. *See Fig. 5.*

Example 6. *We take the initial conditions, for the system (4), as follows* $z_{-2} = 1.22$,$z_{-1} = 2.22$, $z_0 = 0.01$, $w_{-2} = 1.03$, $w_{-1} = 4.01$ *and* $w_0 = 0.03$.*See Fig. 6.*

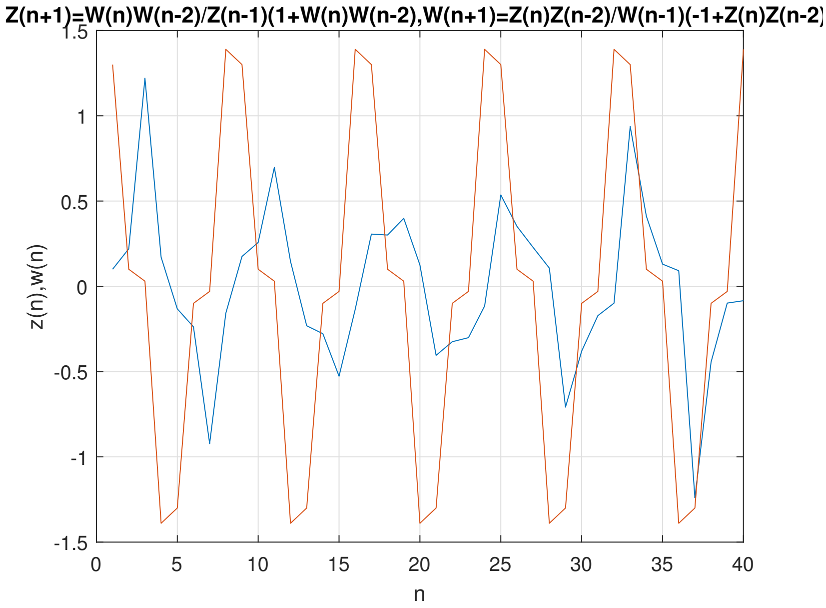

Fig. 5.

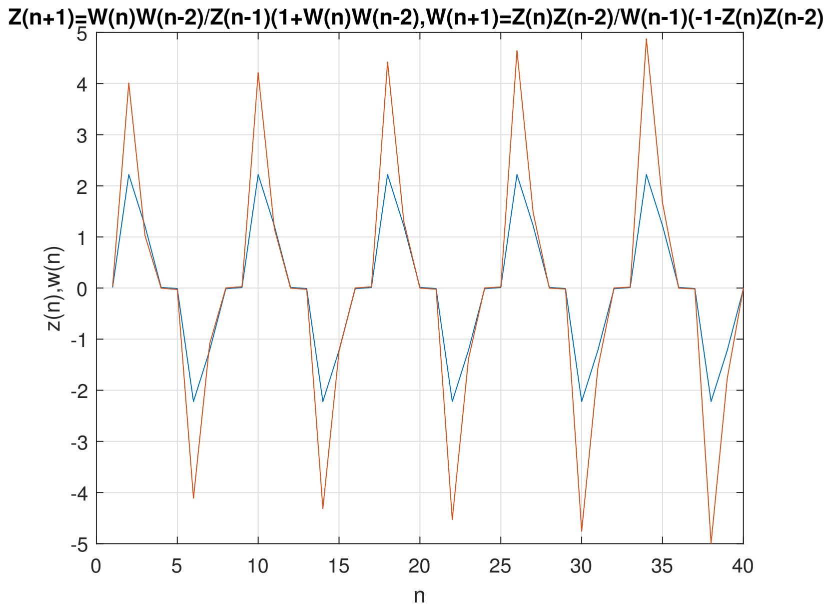

Fig. 6.

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