DOI: 10.28919/ejma.2023.3.3

Eur. J. Math. Appl. (2023)3:3
URL: http://ejma.euap.org

MATHEMATICAL ANALYSIS OF A DETERMINISTIC AND A STOCHASTIC EPIDEMIC MODELS OF DENGUE

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ABSTRACT. In this paper, a comparative study of a deterministic model with its associated stochastic model was carried out. The thresholds of the model considered, denoted $\mathcal{R}_0$, $\mathcal{R}_0^H$ and $\mathcal{R}_0^m$, which can determine the extinction and persistence in mean of dengue, were calculated. Specifically, if $\mathcal{R}_0 < 1$, the deterministic model analysis shows that dengue disappears, while if $\mathcal{R}_0 < 1$ and $\mathcal{R}_0^H > 1$ or $\mathcal{R}_0^m > 1$, the disease persists in the population.

1. INTRODUCTION

Dengue fever, formerly known as « tropical flu », « red fever » or « small malaria » is a viral infection, endemic in tropical countries. Dengue is an arbovirosis, transmitted to humans through the bite of a diurnal mosquito of the genus Aedes, itself infected by a virus of the flavivirus family. This viral infection typically causes fever, headache, muscle and joint pain, fatigue, nausea, vomiting and a skin rash. According to the WHO, there are more than one hundred and ninety million cases of dengue fever per year, of which ninety-six million have clinical manifestations [2]. An estimated 3.9 billion people in 108 countries are at risk of infection [1, 3] with this in mind, we aim to study the dynamics of dengue transmission through a stochastic model obtained by adding two white noises to the contact rates of the deterministic model relatively related to that of Lourdes Esteva al, Cristobal Vargas [8]. Some authors have taken an interest in this topic. We can mention Lourdes Esteva al, Cristobal Vargas in 1997 who studied a deterministic model. Recently in 2021 the authors Anwarud Din , Tahir Khan, Yongjin Li , Hassan Tahir, Asaf Khan, Wajahat Ali Khan constructed a stochastic model by adding white noise to the mortality rates of the deterministic model. They established the existence and uniqueness of positive solution, studied the extinction and stationary ergodic distribution of the model under certain conditions and performed numerical simulations on the proposed model [6]. In this work, the main contributions we make are at three levels. First we propose the deterministic version of the model, we show the stability of the disease-free equilibrium point using a Lyapunov function [9] and other techniques of analysis [4]. Secondly we propose a stochastic model by adding two white noises to the contact rates of the deterministic model, we then show the existence and uniqueness of the positive solution, followed by the study of extinction by establishing the almost sure exponential stability of the disease-free equilibrium equilibrium point and then the persistence in the mean of the stochastic system

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Key words and phrases. dengue fever; Itô's formula; extinction; persistence in the mean.

Received 22/11/2022.

under some assumptions. Finally, we perform numerical simulations to evaluate our results and then compare the two models.

2. Deterministic model

2.1. **Model formulation and preliminary results.** In this section we propose a deterministic model of dengue transmission. According to the epidemiological status of dengue, we distinguish two hosts: the definitive hosts, which are humans, and the intermediate hosts, which are mosquitoes. The interaction between an infectious final host and an intermediate host activates the disease transmission process. Consider N_H et N_m and population size of humans and mosquitoes respectively. We divide the human population into three compartments: $\overline{S}_H$, $\overline{I}_H$, and $\overline{R}_H$ which are the total number of susceptible, infectious and recovered humans respectively. Let a be the mosquito bite rate (the average number of bites) per mosquito per day and let b be the probability that a mosquito will choose a person's blood as a meal. So it is estimated that humans take $ab\dfrac{N_m}{N_H}$ bites per unit time and mosquitoes take ab meals of human blood per unit time. Thus the actual contact rate leading to infection of a susceptible human by mosquitoes is $\lambda_H = abp_m\dfrac{N_m}{N_H}$ and the actual contact rate that causes mosquito infection is $\lambda_m = abp_H$ where, p_H et p_m are respectively the probability of transmission of dengue from a mosquito to a susceptible human and the probability of transmission of dengue from a human to a mosquito. Thus the infection rates per susceptible human and susceptible mosquito are: $\lambda_H\dfrac{\overline{I}_m}{N_m}$ and $\lambda_m\dfrac{\overline{I}_H}{N_H}$ respectively. We thus obtain the following system of ordinary differential equations which describes the above model:

$$
(1) \qquad \begin{cases}
\dfrac{d\overline{S}_H(t)}{dt} = \mu_H N_H - \lambda_H \overline{S}_H(t)\dfrac{\overline{I}_m(t)}{N_m} - \mu_H \overline{S}_H(t), \\
\dfrac{d\overline{I}_H(t)}{dt} = \lambda_H \overline{S}_H(t)\dfrac{\overline{I}_m(t)}{N_m} - (\mu_H + \gamma_H + \alpha_H)\overline{I}_H(t), \\
\dfrac{d\overline{R}_H(t)}{dt} = \gamma_H \overline{I}_H(t) - \mu_H \overline{R}_H(t), \\
\dfrac{d\overline{S}_m(t)}{dt} = \mu_m N_m - \lambda_m \overline{S}_m(t)\dfrac{\overline{I}_H(t)}{N_H} - \mu_m \overline{S}_m(t), \\
\dfrac{d\overline{I}_m(t)}{dt} = \lambda_m \overline{S}_m(t)\dfrac{\overline{I}_H(t)}{N_H} - \mu_m \overline{I}_m(t),
\end{cases}
$$

In this system:
λ_H: is the actual contact rate between susceptible humans and mosquitoes.
ν_H: is the recruitment rate of humans.
γ_H: the recovery rate of humans from dengue.
α_H: represents the death rate of humans induced by dengue.
μ_H: is the natural mortality rate of humans.
λ_m: is the actual contact rate between susceptible mosquitoes and humans.
μ_m: is the natural mortality rate of mosquitoes.
where, the initial conditions $\left(\overline{S}_H(0), \overline{S}_H(0), \overline{I}_H(0); \overline{R}_H(0); \overline{S}_m(0), \overline{I}_m(0)\right) \in \mathbb{R}^5_+$.

We make the following assumptions:
(H_1): the human population and the mosquito population are constant.
(H_2): the recovery rate in humans is higher than the specific dengue mortality rate and the human birth rate ($\gamma_H > \alpha_H$, $\nu_H > \alpha_H$ et $\nu_H = \mu_H$).

Considering the system (1) on the sides of $\mathbb{R}^5_+$, we get:

$$(2)\qquad \begin{cases} \dfrac{d\overline{S}_H(t)}{dt}|_{\overline{S}_H=0} = \mu_H N_H \geq 0, \\ \dfrac{d\overline{I}_H(t)}{dt}|_{\overline{I}_H=0} = \lambda_H \overline{S}_H(t)\dfrac{\overline{I}_m(t)}{N_m} \geq 0, \\ \dfrac{d\overline{R}_H(t)}{dt}|_{\overline{R}_H=0} = \gamma_H \overline{I}_H(t) \geq 0, \\ \dfrac{d\overline{S}_m(t)}{dt}|_{\overline{S}_m=0} = \mu_m N_m \geq 0, \\ \dfrac{d\overline{I}_m(t)}{dt}|_{\overline{I}_m=0} = \lambda_m \overline{S}_m(t)\dfrac{\overline{I}_H(t)}{N_H} \geq 0. \end{cases}$$

Therefore, Proposition 2.1 of [10] implies that every solution of system (1) remains in $\mathbb{R}^5_+$. Let us now introduce the proportions $S_H(t) = \dfrac{\overline{S}_H(t)}{N_H}$, $I_H(t) = \dfrac{\overline{I}_H(t)}{N_H}$, $R_H(t) = \dfrac{\overline{R}_H(t)}{N_H}$, $S_m(t) = \dfrac{\overline{S}_m(t)}{N_m}$ and $I_m(t) = \dfrac{\overline{I}_m(t)}{N_m}$ and also, taking into account the equalities $S_H(t) + I_H(t) + R_H(t) = 1$ and $S_m(t) + I_m(t) = 1$, we obtain $R_H(t) = 1 - S_H(t) - I_H(t)$ and $S_m(t) = 1 - I_m(t)$. Thus the system (1) reduces to

$$(3)\qquad \begin{cases} \dfrac{dS_H(t)}{dt} = \mu_H\left(1 - S_H(t)\right) - \lambda_H S_H(t) I_m(t), \\ \dfrac{dI_H(t)}{dt} = \lambda_H S_H(t) I_m(t) - M_H I_H(t), \\ \dfrac{dI_m(t)}{dt} = \lambda_m\left(1 - I_m(t)\right) I_H(t) - \mu_m I_m(t), \end{cases}$$

where, $M_H = \mu_H + \gamma_H + \alpha_H$. So, the system (3) describes the model.

Proposition 2.1. *Any solution $(S_H(t), I_H(t), I_m(t))$ for all $t \geq 0$ of the system (3) with initial condition $(S_H(0), I_H(0), I_m(0))$ is positive.*

Proof. For the proof of this proposition, we are inspired by the proof of Lemma 3.2 see [14]. Let $x(t) = (S_H(t), I_H(t), I_m(t))$ be a solution of the system (3) with initial values $x(0) = (S_H(0), I_H(0), I_m(0))$ in $\mathbb{R}^3_+$. Thanks to the continuous dependence of the solution on the initial conditions, we will simply show that if $S_H(0) > 0, I_H(0) > 0$ et $I_m(0) > 0$, then $S_H(t) > 0, I_H(t) > 0$ and $I_m(t) > 0$ for all $t > 0$.
Let $m(t) = \min_{t\geq 0}\{S_H(t), I_H(t), I_m(t)\}$. Let $S_H(0), I_H(0), I_m(0) > 0$. Then $m(0) > 0$. Suppose there exists a $t_1 > 0$ such that $m(t_1) \leq 0$ and $m(t) > 0$ for all $t \in [0, t_1)$.
Using the first equation of the system 3 we arrive at:

$$\frac{dS_H(t)}{dt} = \mu_H - (\mu_H + \lambda_H I_m) S_H.$$

If $m(t_1) = S_H(t_1)$, since $I_m(t) > 0$ for all $t \in [0, t_1)$, it follows that:

$$(4)\qquad S'_H(t) \geq -(\mu_H + \lambda_H I_m(t)) S_H(t), \forall t \in [0, t_1).$$

By multiplying each member of the inequality (4) by $\exp(\int_0^t (\mu_H + \lambda_H I_m(s))ds$ and after a few arrangements, we get

$$S'_H(t)\exp(\int_0^t (\mu_H + \lambda_H I_m(s))ds) + (\mu_H + \lambda_H I_m(t)) S_H(t)\exp(\int_0^t (\mu_H + \lambda_H I_m(s))ds) \geq 0, \forall t \in [0, t_1)$$

and therefore

$$\Big[S_H(t)\exp(\int_0^t(\mu_H+\lambda_H I_m(s))ds)\Big]'\geq 0.$$

By integrating on $[0, t_1)$ we get

$$\int_0^{t_1}\Big[S_H(t)\exp(\int_0^t(\mu_H+\lambda_H I_m(s))ds\Big]' dt\geq 0.$$

After calculation we find

$$S_H(t_1)\exp(\int_0^{t_1}(\mu_H+\lambda_H I_m(t))dt)-S_H(0)\geq 0.$$

Thus,

$$S_H(t_1)\geq S_H(0)\exp(-\int_0^{t_1}(\mu_H+\lambda_H I_m(t))dt)>0.$$

That is to say

$$0\geq S_H(t_1)\geq S_H(0)\exp(-\int_0^{t_1}(\mu_H+\lambda_H I_m(t))dt)>0.$$

This is absurd, therefore $m(t_1)\neq S_H(t_1)$. Finally $S_H(t)>0$ for all $t>0$. Similar contradictions can be inferred in the following cases $m(t_1)=I_H(t_1)$ et $m(t_1)=I_m(t_1)$. So we conclude that $\forall t>0,\ S_H(t), I_H(t), I_m>0$. □

Proposition 2.2. *The set* $\Omega=\Big\{(S_H, I_H, I_m)\in\mathbb{R}^3_+ : 0\leq S_H+I_H\leq 1, 0\leq I_m\leq 1\Big\}$ *is positively invariant for the system* (3).

Proof. Since $S_H(t)+I_H(t)+R_H(t)=1$ et $S_m(t)+I_m(t)=1$ pour tout $t\geq 0$, then it is easy to deduce that:

$$\limsup_{t\to\infty}(S_H(t)+I_H(t))\leq 1, \tag{5}$$

$$\limsup_{t\to\infty}(I_m(t))\leq 1. \tag{6}$$

From the relations (5) and (6) any solution of system (3) is bounded. Moreover according to the proposition 2.1, any solution of system (3) is positive. Therefore Ω is positively invariant. □

2.2. **Stability analysis of the disease-free equilibrium point.** The free equilibrium point of system (3) is given by $E_0=(1,0,0)$. The new infection matrix F and the transmission matrix V are given by:

$$F=\begin{pmatrix}0 & \lambda_H\\ \lambda_m & 0\end{pmatrix}\ \text{et}\ V=\begin{pmatrix}M_H & 0\\ 0 & \mu_m\end{pmatrix}.$$

The number of basic reproductions $\mathcal{R}_0$ is defined as the spectral radius of the "next generation" matrix FV^{-1} see [7] that is $\mathcal{R}_0=\rho(FV^{-1})=\sqrt{\dfrac{\lambda_H\lambda_m}{M_H\mu_m}}$.

Proposition 2.3. *We consider the system* (3). *The disease-free equilibrium point* $E_0=(1,0,0)$ *is locally asymptotically stable if and only if* $\mathcal{R}_0<1$.

Proof. The linearized system associated with system (3) at the equilibrium point $E_0 = (1,0,0)$ is
$x'(t) = D_f(E_0)x(t)$ where,

$$D_f(E_0) = \begin{pmatrix} -\mu_H & 0 & -\lambda_H \\ 0 & -M_H & \lambda_H \\ 0 & \lambda_m & -\mu_m \end{pmatrix}$$

is the Jacobian matrix associated with the system 3 in E_0. The second additive component of the matrix $D_f(E_0)$ (see [20] and [21]) is given by

$$D_f^{[2]}(E_0) = \begin{pmatrix} -\mu_H - M_H & \lambda_H & \lambda_H \\ \lambda_m & -\mu_H - \mu_m & 0 \\ 0 & 0 & -M_H - \mu_m \end{pmatrix}.$$

Since $\mathcal{R}_0 < 1$, then

$$\begin{aligned} trace(D_f(E_0)) &= -(\mu_H + M_H + \mu_m) < 0, \\ det(D_f(E_0)) &= -\mu_H\mu_m M_H\left(1 - {\mathcal{R}_0}^2\right), \\ det(D_f^{[2]}(E_0)) &= -(M_H + \mu_m)\left[\mu_H(\mu_H + M_H + \mu_m) + \mu_m M_H(1 - {\mathcal{R}_0}^2)\right] < 0. \end{aligned}$$

Given that $trace(D_f(E_0)) < 0$, $det(D_f(E_0) < 0$ and $det(D_f^{[2]}(E_0)) < 0$, the proposition of [7] guarantees that E_0 is locally asymptotically stable. □

Proposition 2.4. *The disease-free equilibrium E_0 of the system (3) is globally asymptotically stable on Ω when $\mathcal{R}_0 \leq 1$.*

Proof. Consider the Lyapunov candidate function V defined by

$$V(S_H(t), I_H(t), I_m(t)) = \frac{\lambda_H}{\mu_m} I_m(t) + I_H(t).$$

It is easy to see that

$$\begin{aligned} \text{i. } V(S_H(t), I_H(t), I_m(t)) &\geq 0 \text{ for all } (S_H(t), I_H(t), I_m(t)) \in \Omega, \\ V(S_H(t), I_H(t), I_m(t)) &= 0 \text{ if and only if } I_m(t) = I_H(t) = 0 \text{ and } S_H(t) = 1 \text{ thus we have} \\ \text{ii. } V(S_H(t), I_H(t), I_m(t)) &= 0 \text{ if and only if } (S_H(t), I_H(t), I_m(t)) = (1,0,0) = E_0. \end{aligned}$$

The orbital derivative of V along the solution of the system (3) is given by

$$\nabla V(S_H(t), I_H(t), I_m(t)) = \begin{pmatrix} 0 \\ 1 \\ \frac{\lambda_H}{\mu_m} \end{pmatrix}.$$

Let

$$X(S_H(t), I_H(t), I_m(t)) = \begin{pmatrix} \mu_H(1 - S_H(t)) - \lambda_H S_H(t) I_m(t) \\ \lambda_H S_H(t) I_m(t) - M_H I_H(t) \\ \lambda_m(1 - I_m(t)) - \mu_m I_m(t) \end{pmatrix}.$$

$$\begin{aligned} \text{So, } \dot{V}(S_H(t), I_H(t), I_m(t)) &= \langle \nabla V(S_H(t), I_H(t), I_m(t)), X(S_H(t), I_H(t), I_m(t)) \rangle \\ &= \lambda_H S_H(t) I_m(t) - \lambda_H I_m(t) - M_H I_H(t) + \frac{\lambda_H \lambda_m}{\mu_m}(1 - I_m(t)) I_H(t) \\ &= \lambda_H I_m(t)(S_H(t) - 1) - M_H \left[1 - \frac{\lambda_H \lambda_m}{M_H \mu_m}(1 - I_m(t))\right] I_H(t) \\ &= \lambda_H I_m(t)(S_H(t) - 1) - M_H \left[1 - {\mathcal{R}_0}^2(1 - I_m(t))\right] I_H(t). \end{aligned}$$

Since, $(S_H(t), I_m(t)) \in [0, 1]^2$, then $\lambda_H I_m(t)(S_H(t) - 1) \leq 0$
and $-M_H \left[1 - {\mathcal{R}_0}^2(1 - I_m(t))\right] I_H(t) \leq 0$.

Thus, iii. $\dot{V}(S_H(t), I_H(t), I_m(t)) \leq 0$, for all $(S_H(t), I_H(t), I_m(t)) \in \Omega$.

In addition for all $(S_H(t), I_H(t), I_m(t)) \in \Omega \setminus \{E_0\}$, we get $\dot{V}(S_H(t), I_H(t), I_m(t)) < 0$.

Therefore V is a Lyapunov function in the strict sense of E_0 on Ω when $\mathcal{R}_0 \leq 1$.
By Lyapunov's asymptotic stability theorem [17] E_0 is globally asymptotically stable on Ω when $\mathcal{R}_0 \leq 1$. □

Proposition 2.5. *The disease-free equilibrium E_0 of the system (3) is unstable when $\mathcal{R}_0 > 1$.*

Proof. Here the tool used for the proof is Chetaev's theorem. Consider the function V defined by

$$V(x(t)) = I_m(t) + \frac{1 + {\mathcal{R}_0}^2}{2} \times \frac{\mu_m}{\lambda_H} \times I_H(t),$$

where, $x(t) = (S_H(t), I_H(t), I_m(t))$. Let U_{E_0} a neighbourhood of E_0 in Ω. Then it is clear that $V(x(t)) \geq 0$ for all $(S_H(t), I_H(t), I_m(t))$ element of U_{E_0}. We have

$$V(x(t)) = 0 \text{ if and only if } (S_H(t), I_H(t), I_m(t)) = E_0,$$

Hence V is positive on U_{E_0}.
Then the orbital derivative of V is given by

$$\begin{aligned} \dot{V}(x(t)) &= \langle \nabla V(x(t)), X(S_H(t), I_H(t), I_m(t)) \rangle \text{ where,} \\ \nabla V(x(t)) &= \begin{pmatrix} 0 \\ \dfrac{1 + {\mathcal{R}_0}^2}{2} \times \dfrac{\mu_m}{\lambda_H} \\ 1 \end{pmatrix} \text{ and} \\ X(S_H(t), I_H(t), I_m(t)) &= \begin{pmatrix} \mu_H(1 - S_H(t)) - \lambda_H S_H(t) I_m(t) \\ \lambda_H S_H(t) I_m(t) - M_H I_H(t) \\ \lambda_m(1 - I_m(t)) - \mu_m I_m(t) \end{pmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned}
\dot{V}(x(t)) &= \frac{1+{\mathcal{R}_0}^2}{2} \times \frac{\mu_m}{\lambda_H}\left[\lambda_H I_m(t) S_H(t) - M_H I_H(t)\right] - \mu_m I_m(t) + \lambda_m (1-I_m(t)) I_H(t) \\
&= -\frac{1+{\mathcal{R}_0}^2}{2} \times \frac{\mu_m M_H}{\lambda_m \lambda_H} \times \lambda_m I_H(t) + (1-I_m(t))\lambda_m I_H(t) + \frac{1+{\mathcal{R}_0}^2}{2} \times \mu_m I_m(t) S_H(t) \\
&\quad -\mu_m I_m(t) \\
&= \left[(1-I_m(t)) - \frac{1+{\mathcal{R}_0}^2}{2} \times \frac{\mu_m M_H}{\lambda_H \lambda_m}\right] \lambda_m I_H(t) + \left[\frac{1+{\mathcal{R}_0}^2}{2} \times S_H(t) - 1\right] \mu_m I_m(t) \\
&= \left[(1-I_m(t)) - \frac{1+{\mathcal{R}_0}^2}{2} \times \frac{1}{{\mathcal{R}_0}^2}\right] \lambda_m I_H(t) + \left[S_H(t) - \frac{2}{1+{\mathcal{R}_0}^2}\right] \times \frac{1+{\mathcal{R}_0}^2}{2} \mu_m I_m(t) \\
&= \left[(1-I_m(t)) - \frac{1}{2}\left(1+\frac{1}{{\mathcal{R}_0}^2}\right)\right] \lambda_m I_H(t) + \left[S_H(t) - \frac{2}{1+{\mathcal{R}_0}^2}\right] \times \frac{1+{\mathcal{R}_0}^2}{2} \mu_m I_m(t).
\end{aligned}$$

Since $\mathcal{R}_0 > 1$, then we get

$$\frac{1}{{\mathcal{R}_0}^2} < 1 \text{ which leads to } 1 + \frac{1}{{\mathcal{R}_0}^2} < 2 \text{ and therefore } \frac{1}{2}\left(1+\frac{1}{{\mathcal{R}_0}^2}\right) < 1.$$

In addition

$$\begin{aligned}
\mathcal{R}_0 &> 1 \quad \text{means that } {\mathcal{R}_0}^2 + 1 > 2 \text{ by going the other way around we get:} \\
\frac{1}{{\mathcal{R}_0}^2+1} &< \frac{1}{2} \text{ and therefore } \frac{2}{{\mathcal{R}_0}^2+1} < 1.
\end{aligned}$$

Given that $S_H, I_m \in [0,1]$, then we get the following framing:

$$-\frac{2}{{\mathcal{R}_0}^2+1} \leq S_H - \frac{2}{{\mathcal{R}_0}^2+1} \leq 1 - \frac{2}{{\mathcal{R}_0}^2+1} \tag{7}$$

$$-\frac{1}{2}\left(1+\frac{1}{{\mathcal{R}_0}^2}\right) \leq 1 - I_m - \frac{1}{2}\left(1+\frac{1}{{\mathcal{R}_0}^2}\right) \leq 1 - \frac{1}{2}\left(1+\frac{1}{{\mathcal{R}_0}^2}\right). \tag{8}$$

From the relations (7) and (8) we deduce that there is a neighbourhood U_{E_0} of E_0 such that for $(S_H(t), I_H(t), I_m(t))$ element of $U_{E_0} \setminus \{E_0\}$ the expressions in square brackets are strictly positive. As a result $\dot{V}(x(t)) > 0$ for all $x(t) = (S_H(t), I_H(t), I_m(t))$ belonging to $U_{E_0} \setminus \{E_0\}$. So, Chetaev's instability theorem applies. Hence E_0 is unstable when $\mathcal{R}_0 > 1$. □

Outside of the disease-free equilibrium point, the system 3 has an endemic equilibrium point. By direct calculation, we show that the system 3 has an endemic equilibrium point given by $E_1 = \left(\frac{a+b}{a{\mathcal{R}_0}^2+b}, \frac{{\mathcal{R}_0}^2-1}{a{\mathcal{R}_0}^2+b}, \frac{b({\mathcal{R}_0}^2-1)}{(a+b){\mathcal{R}_0}^2}\right)$ in $\overset{\circ}{\Omega}$, where, $a = \frac{M_H}{\mu_H}$ and $b = \frac{\lambda_m}{\mu_m}$.

3. Stochastic model

In this section we propose a stochastic model of dengue by adding two white noises to the contacts λ_H and λ_m. In the rest of this paper, we consider $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space with $(\mathcal{F}_t)_{t\geq 0}$ filtration that satisfying the usual conditions. We note
$\mathbb{R}^5_+ = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 > 0, x_2 > 0, x_3 > 0, x_4 > 0, x_5 > 0\}$ and

$\Gamma_0 = \left\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5_+ : x_1 + x_2 + x_3 < 1, x_4 + x_5 < 1\right\}$. The following stochastic system is considered:

$$dX(t) = f(t, X(t))dt + g(t, X(t))dB(t), \tag{9}$$

for $t \geq t_0$ with$X(t_0) = X_0 \in \mathbb{R}^n$, $B(t)$ denotes n dimensional standard Brownian motion defined on the above probability space. Define the differential operator $\mathcal{L}$ associated to (9) by:

$$(\mathcal{L}V)(t, X) = \frac{\partial V(t, X)}{dt} + f^T \frac{\partial V(t, X)}{dX} + \frac{1}{2} Tr\left[g^T \frac{\partial^2 V(t, X)}{dX^2} g\right] \text{ où } V(t, X) \in \mathcal{C}^{1,2}\left(\mathbb{R} \times \mathbb{R}^m\right). \tag{10}$$

The stochastic version of the deterministic system (1) is given by

$$\begin{cases} dS_H(t) = \left[\mu_H - \lambda_H S_H(t) I_m(t) - \mu_H S_H(t)\right] dt - \sigma_1 S_H(t) I_m(t) dB_1(t), \\ dI_H(t) = \left[-(\mu_H + \gamma_H + \alpha_H) I_H(t) + \lambda_H S_H(t) I_m(t)\right] dt + \sigma_1 S_H(t) I_m(t) dB_1(t), \\ dR_H(t) = -\left[\mu_H R_H(t) - \gamma_H I_H(t)\right] dt, \\ dS_m(t) = \left[\mu_m - \lambda_m S_m(t) I_H(t) - \mu_m S_m(t)\right] dt - \sigma_2 S_m(t) I_H(t) dB_2(t), \\ dI_m(t) = \left[\lambda_m S_m(t) I_H(t) - \mu_m I_m(t)\right] dt + \sigma_2 S_m(t) I_H(t) dB_2(t), \end{cases} \tag{11}$$

where B_1 and B_2 are mutually independent Brownians and σ_1 and σ_2 are their respective intensities.

3.1. Existence of a positive global solution.

Theorem 3.1. *For all initial values $x(0) = (S_H(0), I_H(0), R_H(0), S_m(0), I_m(0)) \in \Gamma_0$, there is a unique solution $x(t) = (S_H(t), I_H(t), R_H(t), S_m(t), I_m(t))$ for the system (11) such that $\mathbb{P}(x(t) \in \Gamma_0) = 1$ for all $t \geq 0$.*

Proof. Let's call it $N_H(t) = S_H(t) + I_H(t) + R_H(t)$ the sum of the respective proportions of susceptible, infected and recovered humans at time t and $N_m(t) = S_m(t) + I_m(t)$ that of the proportions of susceptible and infected mosquitoes at time t. For all $x(s) = (S_H(s), I_H(s), R_H(s), S_m(s), I_m(s))$ belongs to $\mathbb{R}^5_+$ a.s we have

$$\begin{aligned} dN_H(s) &= \mu_H - \alpha_H I_H - \mu_H N_H(s) ds \\ &\leq \mu_H - \mu_H N_H(s) ds \ a.s., \qquad (12) \\ dN_m(s) &= \mu_m - \mu_m N_m(s) ds \ a.s. \qquad (13) \end{aligned}$$

Using Gronwall's lemma , we get:

$$\begin{aligned} N_H(s) &\leq 1 + (N_H(0) - 1)\exp(-\mu_H s) \ a.s., \\ N_m(s) &= 1 + (N_m(0) - 1)\exp(-\mu_m s) \ a.s. \end{aligned}$$

Since $(S_H(0), I_H(0), R_H(0), S_m(0), I_m(0)) \in \Gamma_0$, then $N_H(s) < 1$ a.s. and $N_m(s) < 1$ a.s. So, $x(s) \in (0, 1)^5$ for all $s \in [0, t]$. Moreover, since the coefficients of the system 11 are locally Lipschitzian, there is a unique solution $(S_H(t), I_H(t), R_H(t), S_m(t), I_m(t))$ on any fixed interval $[0, t]$.

Let $x(t) = (S_H(t), I_H(t), R_H(t), S_m(t), I_m(t))$ a solution of system 11 where, $t \in [0, \tau_e)$ and τ_e is the explosion time. To show that $x(t)$ is global, we need only show that $\tau_e = \infty$. Let's define the stopping time τ^* see [18]:

$$\tau^* = \inf\left\{t \in [0, \tau_e) : \ S_H(t) \leq 0 \ or \ I_H(t) \leq 0 \ or \ R_H(t) \leq 0 \ or \ S_m(t) \leq 0 \ or \ I_m(t) \leq 0\right\} \tag{14}$$

where in this paper we assume that $\inf(\emptyset) = \infty$. Thus it is clear that $\tau^* \leq \tau_e$. If we can verify that $\tau^* = \infty$ a.s, then $\tau_e = \infty$ and $x(t) = (S_H(t), I_H(t), R_H(t), S_m(t), I_m(t)) \in \Gamma_0$, $\forall t \geq 0$. If this assertion is not true, then there is a constant $T > 0$ such that $\mathbb{P}(\{\tau^* \leq T\}) > 0$.
We define the function V of class $\mathcal{C}^2$ of Γ_0 in $\mathbb{R}^+$ by

$$V(x(t)) = V_{S_H} + V_{I_H} + V_{R_H} + V_{S_m} + V_{I_m},$$

where,

$$\begin{aligned} V_{S_H} &= -\ln(S_H(t)), \\ V_{I_H} &= -\ln(I_H(t)), \\ V_{R_H} &= -\ln(R_H(t)), \\ V_{S_m} &= -\ln(S_m(t)), \\ V_{I_m} &= -\ln(I_m(t)), \ \forall\ t \geq 0. \end{aligned}$$

Using Itô's formula and for all $t \geq 0$ fixed and $s \in [0, t]$, we get

$$\begin{aligned} dV(x(s)) &= \mathcal{L}V(x(s))ds + \left[\frac{\sigma_1 I_H(s) I_m(s) - \sigma_1 S_H(s) I_m(s)}{I_H(s)}\right] dB_1(s) \\ &\quad + \left[\frac{\sigma_2 I_H(s) I_m(s) - \sigma_2 S_m(s) I_H(s)}{I_m(s)}\right] dB_2(s), \end{aligned}$$

where,

$$\begin{aligned} \mathcal{L}V(x(s))ds &= \left[-\frac{\mu_H}{S_H(s)} + \mu_H + \lambda_H I_m(s) + \frac{1}{2}\sigma_1^2 I_m^2(s)\right] + \\ &\quad \left[(\mu_H + \gamma_H + \alpha_H) - \frac{\lambda_H S_H(s)}{I_H(s)} + \frac{1}{2}\left(\frac{\sigma_1 S_H(s)}{I_H(s)}\right)^2 I_m^2(s)\right] + \\ &\quad \left[\mu_H - \frac{\gamma_H I_H(s)}{R_H(s)}\right] + \left[-\frac{\mu_m}{S_m(s)} + \mu_m + \lambda_m I_H(s) + \frac{1}{2}\sigma_2^2 I_m^2(s)\right] + \\ &\quad \left[\mu_m - \frac{\lambda_m S_m(s)}{I_m(s)} + \frac{1}{2}\left(\frac{\sigma_2 S_m(s)}{I_m(s)}\right)^2 I_H^2(s)\right] \end{aligned}$$

$$\begin{aligned} \mathcal{L}V(x(s))ds &\leq \left[\mu_H + \lambda_H I_m(s) + \frac{1}{2}\sigma_1^2 I_m^2(s) + (\mu_H + \gamma_H + \alpha_H)\right] + \\ &\quad \left[\frac{1}{2}\left(\frac{\sigma_1 S_H(s)}{I_H(s)}\right)^2 I_m^2(s) + \mu_H\right] + \\ &\quad \left[\mu_m + \lambda_m I_H(s) + \frac{1}{2}\sigma_2^2 I_m^2(s)\right] + \left[\mu_m + \frac{1}{2}\left(\frac{\sigma_2 S_m(s)}{I_m(s)}\right)^2 I_H^2(s)\right] \end{aligned}$$

Set $c_1 = \inf\limits_{s\in[0,t]} \{I_H(s)\}$ et $c_2 = \inf\limits_{s\in[0,t]} \{I_m(s)\}$. We obtain

$$\begin{aligned} \mathcal{L}V(X(s))ds &\leq (3\mu_H + \gamma_H + \alpha_H) + \lambda_H I_m(s) + \frac{1}{2}\sigma_1^2 I_m^2(s) + \frac{1}{2}\left(\frac{\sigma_1}{c_1}\right)^2 S_H^2(s) I_m^2(s) \\ &\quad + 2\mu_m + \lambda_m I_H(s) + \frac{1}{2}\sigma_2^2 I_m^2(s) + \frac{1}{2}\left(\frac{\sigma_2}{c_2}\right)^2 S_m^2(s) I_H^2(s) \end{aligned}$$

Using the fact that $(S_H(s), I_H(s), S_m(s), I_m(s)) \in (0,1)^4$, we get:

$$\begin{aligned}\mathcal{L}V(x(s))ds &\leq 3\mu_H + \gamma_H + \alpha_H + \lambda_H + \frac{1}{2}\sigma_1^2 + 2\mu_m + \lambda_m + \frac{1}{2}\sigma_2^2 \\ &\quad + \frac{1}{2}\left(\frac{\sigma_1}{c_1}\right)^2 + \frac{1}{2}\left(\frac{\sigma_2}{c_2}\right)^2 := \ell.\end{aligned}$$

Hence,

$$dV(x(s)) \leq \ell ds + \left[\frac{\sigma_1 I_H(s)I_m(s) - \sigma_1 S_H(s)I_m(s)}{I_H(s)}\right] dB_1(s) + \left[\frac{\sigma_2 I_H(s)I_m(s) - \sigma_2 S_m(s)I_H(s)}{I_m(s)}\right] dB_2(s).$$

By integrating both sides of this inequality from 0 to t we get:

$$\begin{aligned}V(x(t)) \leq V(x(0)) + \ell t &+ \int_0^t \left[\frac{\sigma_1 I_m(s)(I_H(s) - S_H(s))}{I_H(s)}\right] dB_1(s) + \\ &+ \int_0^t \left[\frac{\sigma_2 I_H(s)(I_m(s) - S_m(s))}{I_m(s)}\right] dB_2(s) \ a.s.\end{aligned}$$

Let

$$\begin{aligned}\mathcal{A}_1(S_H(s), I_H(s), I_m(s)) &= \frac{\sigma_1 I_m(s)(I_H(s) - S_H(s))}{I_H(s)}, \ \forall s \in [0,t] \text{ and} \\ \mathcal{A}_2(S_m(s), I_H(s), I_m(s)) &= \frac{\sigma_2 I_H(s)(I_m(s) - S_m(s))}{I_m(s)}, \ \forall s \in [0,t].\end{aligned}$$

Then

$$\sup_{s\in[0,t]} \{\mathcal{A}_1(S_H(s), I_H(s), I_m(s))\} < \infty, \text{ and } \sup_{s\in[0,t]} \{\mathcal{A}_2(S_m(s), I_H(s), I_m(s))\} < \infty.$$

Let

$$\sup_{s\in[0,t]} \{\mathcal{A}_1(S_H(s), I_H(s), I_m(s))\} = K_1, \text{ and } \sup_{s\in[0,t]} \{\mathcal{A}_2(S_m(s), I_H(s), I_m(s))\} = K_2.$$

So,

$$V(x(t)) \leq V(x(0)) + \ell t + K_1 B_1(t) + K_2 B_2(t), \ a.s. \tag{15}$$

Noticing that some components of $x(\tau^*)$ equal 0. Thus, $\lim_{t\to\tau^*} V(x(t)) = \infty$. Letting $t \to \tau^*$ in (15) leads to

$$\infty \leq V(x(0)) + \ell t + K_1 B_1(\tau^*) + K_2 B_2(\tau^*) < \infty; \tag{16}$$

which yields the contradiction. Hence we derive $\tau^* = \infty$, a.s. This completes the proof. □

3.2. Almost sure exponential stability of the disease-free equilibrium.

Theorem 3.2. *Let* $(S_H(0), I_H(0), R_H(0), S_m(0), I_m(0)) \in \Gamma$. *Then* $(I_H(t), R_H(t), I_m(t))$ *exponentially converges almost surely to* $(0,0,0)$ *when* $\mathcal{R}_0 < 1$.

Proof. Let $\theta_2 > 0$. Set $I(t) = I_H(t) + \theta_1 I_m(t) + \theta_2 R_H(t)$, $\forall t \geq 0$ where, $\theta_1 = \dfrac{\lambda_H}{\mu_m}$. Using Itô's formula and, $\forall t \geq 0$ fixed and $u \in [0,t]$, we get:

$$\begin{aligned}
d\ln(I(u)) &= \frac{1}{I(u)}\left[-(\mu_H+\gamma_H+\alpha_H)I_H(u)+\lambda_H S_H(u)I_m(u)\right]du \\
&\quad +\frac{\theta_2}{I(u)}\left[-\mu_H R_H(u)+\gamma_H I_H(u)\right]dt+\frac{\theta_1}{I(u)}\left[\lambda_m S_m(u)I_H(u)-\mu_m I_m(u)\right]du \\
&\quad +\frac{1}{I(u)}\left(\sigma_1 S_H(u)I_m(u)\right)dB_1(u)+\frac{\theta_1}{I(u)}\left(\sigma_2 S_m(u)I_H(u)\right)dB_2(u) \\
&\quad -\frac{1}{2}\frac{1}{I(u)^2}\left(\sigma_1 S_H(u)I_m(u)\right)^2du-\frac{1}{2}\frac{\theta_1}{I(u)^2}\left(\sigma_2 S_m(u)I_H(u)\right)^2du \\
&\leq \frac{1}{I(u)}\left[-(\mu_H+\gamma_H+\alpha_H)I_H(u)+\lambda_H S_H(u)I_m(u)\right]du \\
&\quad +\frac{\theta_2}{I(u)}\left[-\mu_H R_H(u)+\gamma_H I_H(u)\right]du+\frac{\theta_1}{I(u)}\left[\lambda_m S_m(u)I_H(u)-\mu_m I_m(u)\right]du \\
&\quad +\frac{1}{I(u)}\left(\sigma_1 S_H(u)I_m(u)\right)dB_1(u)+\frac{\theta_1}{I(u)}\left(\sigma_2 S_m(t)I_H(u)\right)dB_2(u).
\end{aligned}$$

Set $M_H = (\mu_H + \gamma_H + \alpha_H)$ and use θ_1 value, it follows that:

$$\begin{aligned}
d\ln(I(u)) &\leq \frac{1}{I(u)}\left[-M_H I_H(u)+\lambda_H S_H(u)I_m(u)\right]du \\
&\quad +\frac{1}{I(u)}\left[-\mu_H\theta_2 R_H(u)+\gamma_H\theta_2 I_H(u)\right]dt+\frac{1}{I(u)}\left[\frac{\lambda_H\lambda_m}{\mu_m}S_m(u)I_H(u)-\lambda_H I_m(u)\right]du \\
&\quad +\frac{1}{I(u)}\left(\sigma_1 S_H(u)I_m(t)\right)dB_1(u)+\frac{\theta_1}{I(u)}\left(\sigma_2 S_m(u)I_H(u)\right)dB_2(u) \\
&\leq \frac{1}{I(u)}\left[-M_H I_H(u)+\frac{\lambda_H\lambda_m}{\mu_m}S_m(u)I_H(u)+\gamma_H\theta_2 I_H(u)\right]du \\
&\quad +\frac{1}{I(u)}\left[\lambda_H S_H(u)I_m(u)-\lambda_H I_m(u)-\mu_H\theta_2 R_H(u)\right]du \\
&\quad +\frac{1}{I(u)}\left(\sigma_1 S_H(u)I_m(u)\right)dB_1(u)+\frac{\theta_1}{I(u)}\left(\sigma_2 S_m(u)I_H(u)\right)dB_2(u) \\
&\leq \frac{1}{I(u)}\left[-M_H\left(1-\frac{\lambda_H\lambda_m}{M_H\mu_m}S_m(u)\right)+\gamma_H\theta_2\right]I_H(u)du \\
&\quad +\frac{1}{I(u)}\left[-\lambda_H I_m(u)\left(1-S_H(u)\right)-\mu_H\theta_2 R_H(u)\right]du \\
&\quad +\frac{1}{I(u)}\left(\sigma_1 S_H(u)I_m(u)\right)dB_1(u)+\frac{\theta_1}{I(u)}\left(\sigma_2 S_m(u)I_H(u)\right)dB_2(u).
\end{aligned}$$

Since $\mathcal{R}_0^2 = \dfrac{\lambda_H\lambda_m}{M_H\mu_m} < 1$ and $S_H(u), S_m(u) \in (0,1)$, then $(1 - S_H(u)), (1 - \mathcal{R}_0^2 S_m(u)) > 0$. Thus it follows

$$\begin{aligned}
&d\ln(I(u)) \\
&\leq \frac{1}{I(u)}\left[-\left(M_H\left(1-S_H(u)\right)-\gamma_H\theta_2\right)I_H(u)-\mu_m\left(1-\mathcal{R}_0^2 S_m(u)\right)\frac{\lambda_H}{\mu_m}I_m(u)-\mu_H\theta_2 R_H(u)\right]du \\
&\quad +\frac{1}{I(u)}\left(\sigma_1 S_H(u)I_m(u)\right)dB_1(u)+\frac{\theta_1}{I(u)}\left(\sigma_2 S_m(u)I_H(u)\right)dB_2(u).
\end{aligned}$$

Since $M_H(1 - S_H(u)) > 0$, $\forall u \in [0,t]$ then you can choose $\theta_{2,u} < \theta_2$ very small such as $(M_H(1 - S_H(u)) - \gamma_H \theta_{2,u}) > 0$. Letting $\psi_1 = \inf_{u\in[0,t]}\{(M_H(1 - S_H(u)) - \gamma_H \theta_{2,u})\}$ and $\psi_2 = \inf_{u\in[0,t]}\{\mu_m\left(1 - \mathcal{R}_0^2 S_m(u)\right)\}$ then

$$\begin{aligned} d\ln(I(u)) &\leq \frac{1}{I(u)}\left[-\psi_1 I_H(u) - \psi_2\frac{\lambda_H}{\mu_m}I_m(u) - \mu_H\theta_2 R_H(u)\right]du \\ &\quad + \frac{1}{I(u)}(\sigma_1 S_H(u)I_m(u))\,dB_1(u) + \frac{\theta_1}{I(u)}(\sigma_2 S_m(u)I_H(u))\,dB_2(u). \end{aligned}$$

By taking into account that $\theta_1 = \dfrac{\lambda_H}{\mu_m}$, we get

$$\begin{aligned} d\ln(I(u)) &\leq \frac{1}{I(u)}\left[-\psi_1 I_H(u) - \psi_2\theta_1 I_m(u) - \mu_H\theta_2 R_H(u)\right]du \\ &\quad + \frac{1}{I(u)}(\sigma_1 S_H(u)I_m(u))\,dB_1(u) + \frac{\theta_1}{I(u)}(\sigma_2 S_m(u)I_H(u))\,dB_2(u). \end{aligned}$$

By posing $\psi^* = \min\{\psi_1, \psi_2, \mu_H\}$, we get

$$\begin{aligned} d\ln(I(u)) &\leq \frac{1}{I(u)}\left[-\psi^* I_H(u) - \psi^*\theta_1 I_m(u) - \psi^*\theta_2 R_H(u)\right]du \\ &\quad + \frac{1}{I(u)}(\sigma_1 S_H(u)I_m(u))\,dB_1(u) + \frac{\theta_1}{I(u)}(\sigma_2 S_m(u)I_H(u))\,dB_2(u). \end{aligned}$$

By replacing $I(u)$ by $I_H(u) + \theta_1 I_m(u) + \theta_2 R_H(u)$ and integrating the above inequality from 0 to t on both sides yields

$$\begin{aligned} \ln(I_H(t) + \theta_1 I_m(t) + \theta_2 R_H(t)) &\leq -\psi^* t + \ln(I_H(0) + \theta_1 I_m(0) + \theta_2 R_H(0)) \\ &\quad + M_1(t) + M_2(t), \end{aligned} \tag{17}$$

where,

$$M_1(t) = \int_0^t \frac{\sigma_1 S_H(s)I_m(s)}{I_H(s) + \theta_1 I_m(s) + \theta_2 R_H(s)}dB_1(s) \text{ and } M_2(t) = \int_0^t \frac{\sigma_2\theta_1 S_m(s)I_H(s)}{I_H(s) + \theta_1 I_m(s) + \theta_2 R_H(s)}dB_2(s).$$

The stochastic process $(M_1(t))_{t\geq 0}$ and $(M_2(t))_{t\geq 0}$ are local martingales (see [13]). The quadratic variation of the stochastic integral $M_1(t)$ is

$$\langle M_1(t), M_1(t)\rangle = \int_0^t \frac{\sigma_1^2 S_H^2(s) I_m^2(s)}{(I_H(s) + \theta_1 I_m(s) + \theta_2 R_H(s))^2}ds \tag{18}$$

$$\leq \int_0^t \frac{\sigma_1^2}{(I_H(s) + \theta_1 I_m(s) + \theta_2 R_H(s))^2}ds, \tag{19}$$

because $S_H(s), I_m(s) \in (0,1)$.
As the maps I_H, I_m and R_H are continous then by using the Weierstrass theorem we obtain

$$\begin{aligned} \inf_{s\in[0,t]}\{I_H(s) + \theta_1 I_m(s) + \theta_2 R_H(s)\} &= C \\ &< \infty. \end{aligned}$$

Thus

$$\langle M_1(t), M_1(t)\rangle < \frac{\sigma_1^2}{C}t.$$

By application of the strong law of large numbers for local martingales [15], we conclude that:

$$\lim_{t\to+\infty}\frac{M_1(t)}{t}=0\ a.s. \tag{20}$$

In the same way

$$\lim_{t\to+\infty}\frac{M_2(t)}{t}=0\ a.s. \tag{21}$$

From the relations (17), (20) and (21) we deduce that

$$\limsup_{t\to+\infty}\frac{1}{t}\ln\left(I_H(t)+\theta_1 I_m(t)+\theta_2 R_H(t)\right)\ \leq\ -\psi^*<0.$$

So,

$$\limsup_{t\to+\infty}\frac{\ln\left(I_H(t)\right)}{t}<0, \limsup_{t\to+\infty}\frac{\ln\left(I_m(t)\right)}{t}<0 \text{ and } \limsup_{t\to+\infty}\frac{\ln\left(R_H(t)\right)}{t}<0.$$

□

To study the convergence of $(S_H(t))_{t\geq 0}$, we use the non-negative semi-martingale convergence theorem established by Liptser and Shiryaev [22].

Theorem 3.3. *If $\mathcal{R}_0<1$, then any solution $x(t)=(S_H(t),I_H(t),R_H(t),S_m(t),I_m(t))$ with initial condition $x(0)=(S_H(0),I_H(0),R_H(0),S_m(0),I_m(0))\in\Gamma_0$ almost surely converges to the equilibrium point $(1,0,0,1,0)$.*

For the proof of this theorem, we need the following lemma (see [19]).

Lemma 3.1. *Let $\{A_t\}_{t\geq 0}$ and $\{U_t\}_{t\geq 0}$ two increasing continuous processes and adapted with $A_0=U_0=0$ a.s. Let $\{M_t\}_{t\geq 0}$ a local continuous real-valued martingale with $M_0=0$ a.s. Let ξ a non-negative variable and $\mathcal{F}_0-$mesurable. Define*

$$X_t=\xi+A_t-U_t+M_t,\ \textit{for } t\geq 0.$$

If X_t is non-negative, then

$$\left\{\lim_{t\to+\infty}A_t<\infty\right\}\subset\left\{\lim_{t\to+\infty}X_t \textit{ exists and finished}\right\}\cap\left\{\lim_{t\to+\infty}U_t<\infty\right\}a.s.,$$

where, $C\subset D$ a.s. means $\mathbb{P}(C\cap D^c)=0$. In particular, if $\lim_{t\to+\infty}A_t<\infty$ a.s., then, $\forall\omega\in\Omega$, $\lim_{t\to+\infty}X_t(\omega)$ exists and finished and $\lim_{t\to+\infty}U_t<\infty$.

Let us now present the proof of the previous theorem.

Proof. Using the results of the theorem 3.2, we just need to show that

$$\lim_{t\to\infty}(1-S_H(t))=\lim_{t\to\infty}(1-S_m(t))=0.$$

By integrating the two sides of the first equation of the system (11), we get:

$$1-S_H(t)=1-S_H(0)+\int_0^t\lambda_H S_H(s)I_m(s)ds-\int_0^t\mu_H(1-S_H(s))ds+\int_0^t\sigma_1 S_H(s)I_m(s)dB_1(s).$$

Since $S_H(t)<1$, then we get

$$\lim_{t\to+\infty}\int_0^t\lambda_H S_H(s)I_m(s)ds<\lim_{t\to+\infty}\int_0^t\lambda_H I_m(s)ds.$$

Moreover, since $I_m(t)$ almost surely converges exponentially to 0, then there exists $c_1,c_2>0$ such that

$$I_m(s)<c_1\exp(-c_2 s)\ \forall s\geq 0.$$

So,

$$\lim_{t\to+\infty}\int_0^t I_m(s)ds < \int_0^{+\infty} c_1\exp(-c_2 s)ds.$$

Thus,

$$\lim_{t\to+\infty}\int_0^t \lambda_H S_H(s)I_m(s)ds < \lambda_H\int_0^{+\infty} c_1\exp(-c_2 s)ds < \infty.$$

Using the results of the lemma 3.1, we arrive at the conclusion

$$\lim_{t\to+\infty}(1-S_H(t)) \;<\; \infty\ a.s. \text{ and } \lim_{t\to+\infty}\int_0^t \mu_H(1-S_H(s))ds < \infty\ a.s.$$

$$\text{i.e} \int_0^\infty (1-S_H(s))ds \;<\; \infty\ a.s. \tag{22}$$

Assume that $(S_H(t))_{t\geq 0}$ does not converge to 1. Then there exists $C\subset\Omega$ with $\mathbb{P}(C)>0$ such as, $\forall\omega\in C$,

$$\liminf_{t\to\infty}(1-S_H(t,\omega)) = \varrho(w) > 0.$$

Thus there exists $T=T_\omega>0$ such that $(1-S_H(t,\omega)) = \frac{1}{2}\varrho(w)>0$, $\forall t\geq T$. So,

$$\begin{aligned}\int_0^\infty (1-S_H(s,\omega))ds &= \int_0^T (1-S_H(s,\omega))ds + \int_T^\infty (1-S_H(s,\omega))ds\\ &> \int_T^\infty (1-S_H(s,\omega))ds = \infty.\end{aligned}$$

This implies that: $C\subset D$ where, $D := \left\{\omega\in\Omega : \int_0^\infty (1-S_H(s)(\omega))ds = \infty\right\}$. Yet inequality (22), $\mathbb{P}(D)=0$, leads to a contradiction. So, $\lim_{t\to\infty}(1-S_H(t)) = 0\ a.s.$ Using similar reasoning, we show that $\lim_{t\to\infty}(1-S_m(t)) = 0\ a.s.$ This completes the proof. □

3.3. **Persistence of dengue fever.** Before establishing the persistence results, we will state a lemma that will be used in the proofs.

Lemma 3.2. *Let $(S_H(.), I_H(.), R_H(.), S_m(.), I_m(.))$ a solution of system (11) with initial conditions*
$(S_H(0), I_H(0), I_H(0), S_m(0), I_m(0)) \in (0;1)^5$. *Then*

$$\lim_{t\to+\infty}\frac{S_H(t)+I_H(t)+R_H(t)+S_m(t)+I_m(t)}{t} = 0,\ \ a.s. \tag{23}$$

So,

$$\lim_{t\to+\infty}\frac{S_H(t)}{t} \;=\; 0, \tag{24}$$

$$\lim_{t\to+\infty}\frac{I_H(t)}{t} \;=\; 0, \tag{25}$$

$$\lim_{t\to+\infty}\frac{R_H(t)}{t} \;=\; 0, \tag{26}$$

$$\lim_{t\to+\infty}\frac{S_m(t)}{t} \;=\; 0, \tag{27}$$

$$\lim_{t\to+\infty}\frac{I_m(t)}{t} \;=\; 0.\ a.s. \tag{28}$$

Proof. Our approach is inspired by the works of Yanan Zhao and Daqing Jiang (see [23]) and Yanli Zhou and Weiguo Zhang (see [25]).
Let $X(t) = S_H(t) + I_H(t) + R_H(t) + S_m(t) + I_m(t)$. Define $V(X(t)) = (1 + X(t))^\theta$ where, θ is a positive constant.
Applying Itô's formula to V, we get

$$dV(X(t)) = \theta(1 + X(t))^{\theta-1}dX + \frac{1}{2}\theta(\theta - 1)(1 + X(t))^{\theta-2}(dX(t))^2. \tag{29}$$

We have

$$\begin{aligned}
(dX(t))^2 &= \left[d\left(S_H(t) + I_H(t) + R_H(t) + S_m(t) + I_m(t)\right)\right]^2 && (30)\\
&= \left(dS_H(t) + dI_H(t) + dR_H(t) + dS_m(t) + dI_m(t)\right)^2 && (31)\\
&= (dS_H(t))^2 + (dI_H(t))^2 + (dR_H(t))^2 + (dS_m(t))^2 + (dI_m(t))^2 + d\varphi(t) && (32)
\end{aligned}$$

where,

$$\begin{aligned}
d\varphi(t) =\ & 2\left(dS_H(t)dI_H(t) + dS_H(t)dR_H(t) + dS_H(t)dS_m(t) + dS_H(t)dI_m(t)\right) && (33)\\
& +2\left(dI_H(t)dR_H(t) + dI_H(t)dS_m(t) + dI_H(t)dI_m(t) + dR_H(t)dS_m(t)\right) && (34)\\
& +2\left(dR_H(t)dI_m(t) + dS_m(t)dI_m(t)\right). && (35)
\end{aligned}$$

Let's calculate $(dX(t))^2$.
We get

$$dR_H(t)dI_m(t) = 0,\ dS_H(t)dI_H(t) = -\sigma_1^2 I_m^2 S_H^2 dt,\ dS_m(t)dI_m(t) = -\sigma_2^2 I_H^2 S_m^2 dt. \tag{36}$$

Then,

$$\begin{aligned}
(dS_m(t))^2 &= (dI_m(t))^2 = \sigma_2^2 I_H^2 S_m^2 dt,\\
(dS_H(t))^2 &= (dI_H(t))^2 = \sigma_1^2 I_m^2 S_H^2 dt,\\
(dR_H(t))^2 &= dS_H(t)dR_H(t) = 0.
\end{aligned}$$

Also,

$$\begin{aligned}
dS_H(t)dS_m(t) &= dS_H(t)dI_m(t) = 0, && (37)\\
dI_H(t)dR_H(t) &= dI_H(t)dS_m(t) = 0, && (38)\\
dI_H(t)dI_m(t) &= dR_H(t)dS_m(t) = 0. && (39)
\end{aligned}$$

Thus, we get

$$(dX(t))^2 = 2\sigma_1^2 I_m^2 S_H^2 dt + 2\sigma_2^2 I_H^2 S_m^2 dt - 2\left(\sigma_1^2 I_m^2 S_H^2 dt + \sigma_2^2 I_H^2 S_m^2 dt\right) = 0.$$

So,

$$\begin{aligned}
dV\left(X(t)\right) &= \theta(1 + X(t))^{\theta-1}dX\\
&= \mathcal{L}V\left(X(t)\right)dt.
\end{aligned}$$

Where,

$$\begin{aligned}
&\mathcal{L}V\left(X(t)\right) && (40)\\
&= \theta(1 + X(t))^{\theta-1}\left[\mu_H - \mu_H\left(S_H(t) + I_H(t) + R_H(t)\right) - \alpha_H I_H(t) + \mu_m - \mu_m\left(S_m(t) + I_m(t)\right)\right]\\
&\leq \theta(1 + X(t))^{\theta-1}\left[(\mu_H + \mu_m) - \mu_H\left(S_H(t) + I_H(t) + R_H(t)\right) - \mu_m\left(S_m(t) + I_m(t)\right)\right].
\end{aligned}$$

Set

$$\mu_1 = \max\left(\mu_H, \mu_m\right), \tag{41}$$

$$\mu_2 = \min\left(\mu_H, \mu_m\right). \tag{42}$$

The following mark-up is obtained

$$\begin{aligned}
\mathcal{L}V\left(X(t)\right) &\leq \theta(1+X(t))^{\theta-1}\left[2\mu_1 - \mu_2\left(S_H(t)+I_H(t)+R_H(t)\right) - \mu_m\left(S_m(t)+I_m(t)\right)\right] \\
&\leq \theta(1+X(t))^{\theta-1}\left[2\mu_1 - \mu_2 X(t)\right] \\
&\leq \theta(1+X(t))^{\theta-2}\left[(1+X(t))(2\mu_1 - \mu_2 X(t))\right] \\
&\leq \theta(1+X(t))^{\theta-2}\left[2\mu_1 + (2\mu_1-\mu_2)X(t) - \mu_2 X^2(t)\right].
\end{aligned}$$

It follows that

$$dV\left(X(t)\right) \leq \theta(1+X(t))^{\theta-2}\left[2\mu_1 + (2\mu_1-\mu_2)X(t) - \mu_2 X^2(t)\right] dt. \tag{43}$$

For $p > 0$, we get

$$\begin{aligned}
& d\left[e^{pt}V\left(X(t)\right)\right] \\
=\;& \mathcal{L}\left[e^{pt}V\left(X(t)\right)\right] dt \\
=\;& pe^{pt}V\left(X(t)\right) dt + e^{pt} dV\left(X(t)\right) dt \\
\leq\;& pe^{pt}(1+X(t))^{\theta} + \theta e^{pt}(1+X(t))^{\theta-2}\left[2\mu_1 + (2\mu_1-\mu_2)X(t) - \mu_2 X^2(t)\right] dt \\
\leq\;& \theta e^{pt}(1+X(t))^{\theta-2}\left[\frac{p}{\theta}(1+X(t))^2 - \mu_2 X^2(t) + (2\mu_1-\mu_2)X(t) + 2\mu_1\right] dt \\
\leq\;& \theta e^{pt}(1+X(t))^{\theta-2}\left[-\left(\mu_2 - \frac{p}{\theta}\right)X^2(t) + (2\mu_1 - \mu_2 + 2\frac{p}{\theta})X(t) + \left(2\mu_1 + \frac{p}{\theta}\right)\right] dt \\
\leq\;& \theta e^{pt} H dt, \qquad (44)
\end{aligned}$$

where,

$$H := \sup_{t\in\mathbb{R}_+}\left\{(1+X(t))^{\theta-2}\left[-\left(\mu_2 - \frac{p}{\theta}\right)X^2(t) + (2\mu_1 - \mu_2 + 2\frac{p}{\theta})X(t) + \left(2\mu_1 + \frac{p}{\theta}\right)\right]\right\}. \tag{45}$$

Since $(S_H(.), I_H(.), I_H(.), S_m(.), I_m(.)) \in (0,1)^5$, then $X(.) \in (0, 25)$. So, $0 < H < \infty$. Passing to the integral from 0 to t in (44), we get

$$\int_0^t d\left[e^{p\xi}V\left(X(\xi)\right) d\xi\right\} \leq \int_0^t \theta e^{p\xi} H d\xi, \tag{46}$$

$$e^{pt}V\left(X(t)\right) \leq V\left(X(0)\right) + \frac{\theta H e^{pt}}{p} - \frac{\theta H}{p}. \tag{47}$$

It can be deduced that

$$Ee^{pt}V\left(X(t)\right) \leq V\left(X(0)\right) + \frac{\theta H e^{pt}}{p} - \frac{\theta H}{p}.$$

That is to say,

$$\begin{aligned}
E\left[(1+X(t))^{\theta}\right] &\leq \frac{(1+X(0))^{\theta}}{e^{pt}} + \frac{\theta H}{P} \\
&\leq (1+X(0))^{\theta} + \theta H.
\end{aligned}$$

Set $C = (1 + X(0))^\theta + \theta H$.
Then,

$$E\left[(1 + X(t))^\theta\right] \leq C.$$

$\forall\ \delta > 0$ sufficiently small, $p = 1, 2, 3, ...$, by integrating (43) from $p\delta$ to t, we get

$$(1 + X(t))^\theta \leq (1 + X(p\delta))^\theta + \int_{p\delta}^{t} \theta(1 + X(\xi))^{\theta-2}\left[2\mu_1 + (2\mu_1 - \mu_2)X(\xi) - \mu_2 X^2(\xi)\right] d\xi.$$

It follows that

$$\begin{aligned}\sup_{p\delta \leq t \leq (p+1)\delta} (1 + X(t))^\theta \quad &\leq \quad (1 + X(p\delta))^\theta \\ &\quad + \sup_{p\delta \leq t \leq (p+1)\delta}\left|\int_{p\delta}^{t} \theta(1 + X(\xi))^{\theta-2}\left[2\mu_1 + (2\mu_1 - \mu_2)X(\xi) - \mu_2 X^2(\xi)\right] d\xi\right|\end{aligned}$$

Taking the mathematical expectation of both sides of the latter inequality we get

$$E\left[\sup_{p\delta \leq t \leq (p+1)\delta} (1 + X(t))^\theta\right] \leq E\left[(1 + X(p\delta))^\theta\right] + J, \tag{48}$$

where,

$$\begin{aligned} J \quad &= \quad E\left[\sup_{p\delta \leq t \leq (p+1)\delta}\left|\int_{p\delta}^{t} \theta(1 + X(\xi))^{\theta-2}\left[2\mu_1 + (2\mu_1 - \mu_2)X(\xi) - \mu_2 X^2(\xi)\right] d\xi\right|\right] && (49)\\ &= \quad E\left[\sup_{p\delta \leq t \leq (p+1)\delta}\left|\int_{p\delta}^{t} \theta(1 + X(\xi))^{\theta-2}(1 + X(\xi))\left(2\mu_1 - \mu_2 X(\xi)\right) d\xi\right|\right] && (50)\\ &= \quad E\left[\sup_{p\delta \leq t \leq (p+1)\delta}\left|\int_{p\delta}^{t} \theta(1 + X(\xi))^{\theta-1}\left(2\mu_1 - \mu_2 X(\xi)\right) d\xi\right|\right] && (51)\\ &= \quad E\left[\sup_{p\delta \leq t \leq (p+1)\delta}\left|\int_{p\delta}^{t} \theta(1 + X(\xi))^{\theta} \times \frac{2\mu_1 - \mu_2 X(\xi)}{(1 + X(\xi))} d\xi\right|\right]. && (52)\end{aligned}$$

Set

$$l = \theta \sup_{p\delta \leq t \leq (p+1)\delta}\left|\frac{2\mu_1 - \mu_2 X(\xi)}{(1 + X(\xi))}\right|. \tag{53}$$

It follows that

$$\begin{aligned} J \quad &\leq \quad lE\left[\sup_{p\delta \leq t \leq (p+1)\delta}\left|\int_{p\delta}^{t} (1 + X(\xi))^\theta d\xi\right|\right] && (54)\\ &\leq \quad lE\left[\int_{p\delta}^{(p+1)\delta} (1 + X(\xi))^\theta d\xi\right] && (55)\\ &\leq \quad lE\left[\delta \sup_{p\delta \leq \xi \leq (p+1)\delta} (1 + X(\xi))^\theta\right] && (56)\\ &\leq \quad l\delta E\left[\sup_{p\delta \leq \xi \leq (p+1)\delta} (1 + X(\xi))^\theta\right] && (57)\\ &\leq \quad l\delta E\left[\sup_{p\delta \leq t \leq (p+1)\delta} (1 + X(t))^\theta\right]. && (58)\end{aligned}$$

As a result

$$E\left[\sup_{p\delta \leq t \leq (p+1)\delta} (1 + X(t))^\theta\right] \leq E\left[(1 + X(p\delta))^\theta\right] + l\delta E\left[\sup_{p\delta \leq t \leq (p+1)\delta} (1 + X(t))^\theta\right]. \tag{59}$$

Choose $\delta > 0$ such as $l\delta \leq \frac{1}{2}$, then

$$E\left[\sup_{p\delta\leq t\leq(p+1)\delta}(1+X(t))^{\theta}\right] \leq 2E\left[(1+X(p\delta))^{\theta}\right]. \tag{60}$$

By using (48), we get

$$E\left[\sup_{p\delta\leq t\leq(p+1)\delta}(1+X(t))^{\theta}\right] \leq 2C. \tag{61}$$

Let ϵ_X an arbitrarily chosen positive constant. Applying Markov's inequality, we get

$$P\left\{\sup_{p\delta\leq t\leq(p+1)\delta}(1+X(t))^{\theta} > (p\delta)^{1+\epsilon_X}\right\} \leq \frac{E\left[\sup_{p\delta\leq t\leq(p+1)\delta}(1+X(t))^{\theta}\right]}{(p\delta)^{1+\epsilon_X}} \tag{62}$$

$$\leq \frac{2C}{(p\delta)^{1+\epsilon_X}}. \tag{63}$$

Let $U_p = \{\sup_{p\delta\leq t\leq(p+1)\delta}(1+X(t))^{\theta} > (p\delta)^{1+\epsilon_X}\}$ then $\sum_{p=1}^{\infty} P(U_p) < \sum_{p=1}^{\infty}\frac{2C}{(p\delta)^{1+\epsilon_X}}$. Since $1+\epsilon_X > 1$ then $\sum_{p=1}^{\infty}\frac{2C}{(p\delta)^{1+\epsilon_X}} < \infty$ the Borel-Cantelli lemma (see [19]) yields that for almost all $\omega \in \Omega$

$$\sup_{p\delta\leq t\leq(p+1)\delta}(1+X(t))^{\theta} \leq (p\delta)^{1+\epsilon_X}, \quad p = 1, 2, 3, \ldots \tag{64}$$

Since this inequality holds for all p, then there exists a positive inter $p_0 = p_0(\omega)$ for almost all $\omega \in \Omega$ such that (64) remains true, $\forall p \geq p_0$. Therefore, for almost all $\omega \in \Omega$, if $p \geq p_0$ and $p\delta \leq t \leq (p+1)\delta$,

$$\frac{\ln(1+X(t))^{\theta}}{\ln t} \leq \frac{(1+\epsilon_X)\ln(p\delta)}{\ln(p\delta)} \tag{65}$$

$$= 1+\epsilon_X. \tag{66}$$

So,

$$\limsup_{t\to\infty}\frac{\ln(1+X(t))^{\theta}}{\ln t} \leq 1+\epsilon_X, \ a.s. \tag{67}$$

Let's make $\epsilon_X \longrightarrow 0$, we get

$$\limsup_{t\to\infty}\frac{\ln(1+X(t))^{\theta}}{\ln t} \leq 1, \ a.s. \tag{68}$$

For $\theta > 1$, we get

$$\limsup_{t\to\infty}\frac{\ln(X(t))}{\ln t} \leq \limsup_{t\to\infty}\frac{\ln(1+X(t))}{\ln t} \leq \frac{1}{\theta}, \ a.s. \tag{69}$$

That is to say, for $0 < \gamma < 1 - \frac{1}{\theta}$, there exists a constant $T = T(\omega)$ such as, $\forall t \geq T$

$$\ln(1+X(t)) \leq \left(\frac{1}{\theta}+\gamma\right)\ln t. \tag{70}$$

That is to say, for $0 < \gamma < 1 - \frac{1}{\theta}$, there exists a constant $T = T(\omega)$ and a set Ω_γ such as $P(\Omega_\gamma) \geq 1 - \gamma$ and, $\forall t \geq T$, $\omega \in \Omega_\gamma$,

$$\ln(X(t)) \leq \left(\frac{1}{\theta} + \gamma\right) \ln t. \tag{71}$$

As a result

$$\limsup_{t\to\infty} \frac{X(t)}{t} \leq \limsup_{t\to\infty} \frac{t^{\frac{1}{\theta}+\gamma}}{t} = 0. \tag{72}$$

This leads to

$$\lim_{t\to\infty} \frac{X(t)}{t} = \lim_{t\to\infty} \frac{S_H(t) + I_H(t) + R_H(t) + S_m(t) + I_m(t)}{t} = 0. \ a.s. \tag{73}$$

Thanks to the positivity of S_H, I_H, R_H, S_m and I_m. So, we get

$$\lim_{t\to\infty} \frac{S_H(t)}{t} = \lim_{t\to\infty} \frac{I_H(t)}{t} = \lim_{t\to\infty} \frac{R_H(t)}{t} = \lim_{t\to\infty} \frac{S_m(t)}{t} = \lim_{t\to\infty} \frac{I_m(t)}{t} = 0 \ a.s. \tag{74}$$

This completes the proof. □

Lemma 3.3. *Let $(S_H(.), I_H(.), R_H(.), S_m(.), I_m(.))$ a solution of (11) with initial conditions $(S_H(0), I_H(0), I_H(0), S_m(0), I_m(0)) \in (0;1)^5$. Then*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t I_m(\xi) S_H(\xi) I_H^{-1}(\xi) dB_1(\xi) = 0, \tag{75}$$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t I_H(\xi) S_m(\xi) I_m^{-1}(\xi) dB_2(\xi) = 0 \ .a.s. \tag{76}$$

Proof. Let

$$\mathcal{M}_1(t) = \int_0^t I_m(\xi) S_H(\xi) I_H^{-1}(\xi) dB_1(\xi), \tag{77}$$

$$\mathcal{M}_2(t) = \int_0^t I_H(\xi) S_m(\xi) I_m^{-1}(\xi) dB_2(\xi). \tag{78}$$

As the maps I_H, S_H and I_m are continous then by using the Weierstrass theorem, we get

$$\sup_{0\leq\xi\leq t} \{I_m(\xi) S_H(\xi) I_H^{-1}(\xi)\} = C_1 < \infty. \tag{79}$$

Thus

$$\langle \mathcal{M}_1(t), \mathcal{M}_1(t) \rangle < C_1 t. \ a.s. \text{ and } \limsup_{t\to\infty} \frac{\langle \mathcal{M}_1(t), \mathcal{M}_1(t) \rangle}{t} < C_1. \ a.s. \tag{80}$$

By using the strong law of large numbers for local martingales, we conclude that

$$\lim_{t\to\infty} \frac{\mathcal{M}_1(t)}{t} = 0. \ a.s. \tag{81}$$

In the same way we get

$$\lim_{t\to\infty} \frac{\mathcal{M}_2(t)}{t} = 0. \ a.s. \tag{82}$$

Hence the lemma has been established. □

Lemma 3.4. *Let $f \in \mathcal{C}([0;\infty) \times \Omega, (0,\infty))$. If there are positive constants λ_0, λ and T such that*

$$\ln f(t) \geq \lambda t - \lambda_0 \int_0^t f(\xi) d\xi + F(t), \tag{83}$$

$\forall t \geq T$ *with* $F \in \mathcal{C}([0;\infty) \times \Omega, \mathbb{R})$, $\lim_{t\to\infty} \frac{F(t)}{t} = l$ *a.s. and* $\lambda + l > 0$. *Then*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t f(\xi)d\xi \geq \frac{\lambda + l}{\lambda_0} \ a.s. \tag{84}$$

Proof

Our approach is inspired by the work of Zhaoa, Daqing Jiang and Donal O'Regancc (see [24]) and Liu Huaping and Ma Zhien (see [12]) .

Note that $\lim_{t\to\infty} \frac{F(t)}{t} = l$ a.s. then for arbitrary $0 < \epsilon < \lambda + l$ there exists a $T_0 = T_0(\omega) > 0$ and a set Ω_r such that $P(\Omega_r) > 0$ and $\left|\frac{F(t)}{t} - l\right| \leq \epsilon$ for all $t \geq T_0$, $\omega \in \Omega_\epsilon$. Let $\overline{T} = T \vee T_0$ and $\psi(t) = \int_0^t f(\zeta)d\zeta$ for $t \geq \overline{T}$, $\omega \in \Omega_r$.

Since $f \in C([0,\infty) \times \Omega, (0,\infty))$, then ψ is differentiable on $[\overline{T}, \infty)$ a.s. and

$$d\psi(t) = f(t) > 0 \text{ for } t \geq \overline{T}, \omega \in \Omega_r. \tag{85}$$

Substituting $\frac{d\psi(t)}{dt}$ and $\psi(t)$ into (83), we have

$$\ln\left(\frac{d\psi(t)}{dt}\right) \geq \lambda t - \lambda_0\psi(t) + F(t) \tag{86}$$

$$\geq (\lambda - \epsilon + l)t - \lambda_0\psi(t), \text{ for } t \geq \overline{T}, \omega \in \Omega_r. \tag{87}$$

So

$$\exp(\lambda_0\psi(t))\frac{d\psi(t)}{dt} \geq \exp(\lambda - \epsilon + l)\,t, \text{ for } t \geq \overline{T}, \omega \in \Omega_r. \tag{88}$$

Integrating this inequality from $\overline{T}$ to t results in

$$\lambda_0^{-1}\left[\exp(\lambda_0\psi(t)) - \exp(\lambda_0\psi(\overline{T}))\right] \geq \left[\exp((\lambda + l - \epsilon)t) - \exp((\lambda + l - \epsilon)\overline{T})\right] \tag{89}$$

This inequality can be rewritten into

$$\exp(\lambda_0\psi(t)) \geq \lambda_0(\lambda + l - \epsilon)^{-1}\left[\exp((\lambda + l - \epsilon)t) - \exp((\lambda + l - \epsilon)\overline{T})\right] + \exp(\lambda_0\psi(\overline{T}). \tag{90}$$

Taking the logarithm of both sides yields

$$\psi(t) \geq \lambda_0^{-1}\ln\left[\lambda_0(\lambda + l - \epsilon)^{-1}\exp((\lambda + l - \epsilon)t) + \lambda_{\overline{T}}\right], \tag{91}$$

where,

$$\lambda_{\overline{T}} = \exp(\lambda_0\psi(\overline{T})) - \lambda_0(\lambda + l - \epsilon)^{-1}\exp((\lambda + l - \epsilon)\overline{T}) \tag{92}$$

or

$$\int_0^t f(\zeta)d\zeta \geq \lambda_0^{-1}\ln\left[\lambda_0(\lambda + l - \epsilon)^{-1}\exp((\lambda + l - \epsilon)t) + \lambda_{\overline{T}}\right], \text{ for } t \geq \overline{T}, \omega \in \Omega_r. \tag{93}$$

Dividing both sides by $t \geq \overline{T} > 0$ gives

$$t^{-1}\int_0^t f(\zeta)d\zeta \geq \lambda_0^{-1}t^{-1}\ln\left[\lambda_0(\lambda + l - \epsilon)^{-1}\exp((\lambda + l - \epsilon)t) + \lambda_{\overline{T}}\right], \text{ for } t \geq \overline{T}, \omega \in \Omega_r. \tag{94}$$

Taking the limit superior of both sides and applying L'Hospital's rule on the right-hand side of this inequality, we obtain

(95) $$\limsup_{t\to\infty} t^{-1}\int_0^t f(\zeta)d\zeta \geq \frac{\lambda + l - \epsilon}{\lambda_0} \text{ for } \omega \in \Omega_r.$$

Letting $\epsilon \to 0$ yields

(96) $$\limsup_{t\to\infty} t^{-1}\int_0^t f(\zeta)d\zeta \geq \frac{\lambda + l}{\lambda_0} \text{ for } a.s.$$

This finishes the proof of the Lemma. □

We now turn to the study of the persistence in the mean of the system (11). To this end, we present a definition of persistence in the mean that can be found in [5, 16].

Definition 3.1. *We say that the system (11) is persistent in mean if*

(97) $$\liminf_{t\to\infty} \langle I_H(t)\rangle > 0 \text{ or } \liminf_{t\to\infty} \langle I_m(t)\rangle > 0,$$

where $\langle z(t)\rangle = \frac{1}{t}\int_0^t z(\xi)d\xi$.

For future needs, define the following threshold parameters

(98) $$\mathcal{R}_0^H = \frac{\mu_H\left(\lambda_H - M_H - \frac{1}{2}\sigma_1^2 c_1\right)}{\lambda_H M_H}, \text{ whith } c_1 = \sup_{\xi\in\mathbb{R}_+}\{I_H^{-1}(\xi)\},$$

(99) $$\mathcal{R}_0^m = \frac{\left(\lambda_m - \mu_m - \frac{1}{2}\sigma_2^2 c_2\right)}{\lambda_m}, \text{ whith } c_2 = \sup_{\xi\in\mathbb{R}_+}\{I_m^{-1}(\xi)\}$$

and formulate the following hypotheses

(100) $$(\mathcal{H})_1 \quad I_H(t)I_m^{-1}(t) \leq 1, \ \forall t \geq 0, \text{ and } \mathcal{R}_0^H > 0,$$

(101) $$(\mathcal{H})_2 \quad I_H(t)I_m^{-1}(t) > 1, \ \forall t \geq 0, \text{ and } \mathcal{R}_0^m > 0.$$

Theorem 3.4. *Let* $(S_H(.), I_H(.), I_H(.), S_m(.), I_m(.))$ *a solution of system (11) with the initial conditions* $(S_H(0), I_H(0), I_H(0), S_m(0), I_m(0)) \in (0;1)^5$.

(i) *If the assumption* $(\mathcal{H})_1$ *is verified then* $\liminf_{t\to\infty} \langle I_H(t)\rangle > 0$ *a.s.*

(ii) *If the assumption* $(\mathcal{H})_2$ *is verified then* $\liminf_{t\to\infty} \langle I_m(t)\rangle > 0$ *a.s.*

Proof. Applying the integral between 0 and t the two sides of the two first equation of system (11), we get

(102) $$\frac{S_H(t) - S_H(0)}{t} = \mu_H - \lambda_H\langle S_H(t)I_m(t)\rangle - \mu_H\langle S_H(t)\rangle - \frac{\sigma_1}{t}\int_0^t S_H(\xi)I_m(\xi)dB_1(\xi),$$

(103) $$\frac{I_H(t) - I_H(0)}{t} = -M_H\langle I_H(t)\rangle + \lambda_H\langle S_H(t)I_m(t)\rangle + \frac{\sigma_1}{t}\int_0^t S_H(\xi)I_m(\xi)dB_1(\xi).$$

By member by member sum of (102) and (103), we get

$$\frac{S_H(t) - S_H(0)}{t} + \frac{I_H(t) - I_H(0)}{t} = \mu_H - \mu_H\langle S_H(t)\rangle - M_H\langle I_H(t)\rangle.$$

Which yields

(104) $$\langle S_H(t)\rangle = 1 - \frac{M_H}{\mu_H}\langle I_H(t)\rangle + \phi(t)$$

$$\text{where, } \phi(t) = \frac{-1}{\mu_H}\left[\frac{S_H(t) - S_H(0)}{t} + \frac{I_H(t) - I_H(0)}{t}\right].$$

Using the results of the lemma 3.2, we get $\lim_{t\to\infty}\phi(t)=0$. By applying the Itô formula to the second equation of the system (11), we obtain

$$d\ln(I_H(t)) = \frac{1}{I_H(t)}dI_H - \frac{1}{2}\frac{1}{I_H^2(t)}(dI_H(t))^2 \tag{105}$$

$$= \left[\lambda_H\frac{S_H(t)I_m(t)}{I_H(t)} - M_H - \frac{1}{2}\sigma_1^2\frac{S_H^2(t)I_m^2(t)}{I_H^2(t)}\right]dt + \sigma_1\frac{S_H(t)I_m(t)}{I_H(t)}dB_1(t). \tag{106}$$

Passing to the integral between 0 and t of this last equality, it follows that

$$\frac{\ln(I_H(t)) - \ln(I_H(0))}{t} = \lambda_H\left\langle S_H(t)I_m(t)I_H^{-1}(t)\right\rangle - M_H \tag{107}$$

$$+ \sigma_1\frac{1}{t}\int_0^t\frac{S_H(\xi)I_m(\xi)}{I_H(\xi)}dB_1(\xi) - \frac{1}{2}\sigma_1^2\left\langle S_H^2(t)I_m^2(t)I_H^{-2}(t)\right\rangle. \tag{108}$$

By applying the integral from 0 to t of the two sides of the last two equations of system (11), we get

$$\frac{S_m(t)-S_m(0)}{t} = \mu_m - \lambda_m\langle S_m(t)I_H(t)\rangle - \mu_m\langle S_m(t)\rangle - \frac{\sigma_2}{t}\int_0^t S_m(\xi)I_H(\xi)dB_2(\xi), \tag{109}$$

$$\frac{I_m(t)-I_m(0)}{t} = -\mu_m\langle I_m(t)\rangle + \lambda_m\langle S_m(t)I_H(t)\rangle + \frac{\sigma_2}{t}\int_0^t S_m(\xi)I_H(\xi)dB_2(\xi). \tag{110}$$

By member to member sum of (109) and (110), it follows that

$$\frac{S_m(t)-S_m(0)}{t} + \frac{I_m(t)-I_m(0)}{t} = \mu_m - \mu_m\langle S_m(t)\rangle - \mu_m\langle I_m(t)\rangle. \tag{111}$$

Which yields

$$\langle S_m(t)\rangle = 1 - \langle I_m(t)\rangle + \psi(t), \tag{112}$$

$$\text{where, } \psi(t) = \frac{-1}{\mu_m}\left[\frac{S_m(t)-S_m(0)}{t} + \frac{I_m(t)-I_m(0)}{t}\right]. \tag{113}$$

Using the result of the lemma 3.2, we get $\lim_{t\to\infty}\psi(t)=0$. By applying the Itô formula to the fourth equation of system (11), we obtain

$$d\ln(I_m(t)) = \frac{1}{I_m(t)}dI_m - \frac{1}{2}\frac{1}{I_m^2(t)}(dI_m(t))^2 \tag{114}$$

$$= \left[\lambda_m\frac{S_m(t)I_H(t)}{I_m(t)} - \mu_m - \frac{1}{2}\sigma_2^2\frac{S_m^2(t)I_H^2(t)}{I_m^2(t)}\right]dt + \sigma_2\frac{S_m(t)I_H(t)}{I_m(t)}dB_2(t). \tag{115}$$

Passing to the integral between 0 and t of this last equality, we get

$$\frac{\ln(I_m(t)) - \ln(I_m(0))}{t} = \lambda_m\left\langle S_m(t)I_H(t)I_m^{-1}(t)\right\rangle - \mu_m \tag{116}$$

$$+\sigma_2\frac{1}{t}\int_0^t\frac{S_m(\xi)I_H(\xi)}{I_m(\xi)}dB_2(\xi) - \frac{1}{2}\sigma_2^2\left\langle S_m^2(t)I_H^2(t)I_m^{-2}(t)\right\rangle. \tag{117}$$

We distinguish two cases:
1^{rt} case: Suppose that $(\mathcal{H})_1$ is verified. From the equality (107)-(108), we get the following

minoration:

$$\frac{\ln(I_H(t))-\ln(I_H(0))}{t} \geq \lambda_H\langle S_H(t)\rangle - M_H \tag{118}$$

$$+\sigma_1\frac{1}{t}\int_0^t \frac{S_H(\xi)I_m(\xi)}{I_H(\xi)}dB_1(\xi)-\frac{1}{2}\sigma_1^2\left\langle S_H^2(t)I_m^2(t)I_H^{-2}(t)\right\rangle, \tag{119}$$

$$\geq \lambda_H\langle S_H(t)\rangle - M_H \tag{120}$$

$$+\sigma_1\frac{1}{t}\int_0^t \frac{S_H(\xi)I_m(\xi)}{I_H(\xi)}dB_1(\xi)-\frac{1}{2}\sigma_1^2\left\langle I_H^{-2}(t)\right\rangle,$$

because S_H and I_m are in $(0,1)$. By using (104), to replace $\langle S_H(t)\rangle$ in (120), give

$$\frac{\ln(I_H(t))-\ln(I_H(0))}{t} \geq \lambda_H\left(1-\frac{M_H}{\mu_H}\langle I_H(t)\rangle+\phi(t)\right)-M_H$$

$$+\sigma_1\frac{1}{t}\int_0^t \frac{S_H(\xi)I_m(\xi)}{I_H(\xi)}dB_1(\xi)-\frac{1}{2}\sigma_1^2\left\langle I_H^{-2}(t)\right\rangle \tag{121}$$

$$\geq M_H\left(\frac{\lambda_H}{M_H}-1\right)-\frac{\lambda_H M_H}{\mu_H}\langle I_H(t)\rangle+\lambda_H\phi(t)+G(t). \tag{122}$$

where

$$G(t)=\sigma_1\frac{1}{t}\int_0^t \frac{S_H(\xi)I_m(\xi)}{I_H(\xi)}dB_1(\xi)-\frac{1}{2}\sigma_1^2 c_1. \tag{123}$$

Clearly $\lim_{t\to\infty} G(t) = -\frac{1}{2}\sigma_1^2 c_1$.
It follows that

$$\ln(I_H(t)) \geq M_H\left(\frac{\lambda_H}{M_H}-1\right)t-\frac{\lambda_H M_H}{\mu_H}\int_0^t I_H(\xi)d\xi+F_1(t). \tag{124}$$

where,

$$F_1(t)=tG(t)+\lambda_H t\phi(t)+\ln(I_H(0)). \tag{125}$$

Using the result of the lemma 3.2 and lemma 3.3 we get $\lim_{t\to\infty}\frac{F_1(t)}{t} = -\frac{1}{2}\sigma_1^2 c_1$. Thus applying the lemma 3.4 we get

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I_H(\xi)d\xi \geq \mathcal{R}_0^H. \tag{126}$$

2^{th} case: Suppose that $(\mathcal{H})_2$ is verified. From the equality (116)-(117), we get the following minoration

$$\frac{\ln(I_m(t))-\ln(I_m(0))}{t} \geq \lambda_m\langle S_m(t)\rangle - \mu_m \tag{127}$$

$$+\sigma_2\frac{1}{t}\int_0^t \frac{S_m(\xi)I_H(\xi)}{I_m(\xi)}dB_2(\xi)-\frac{1}{2}\sigma_2^2\left\langle S_m^2(t)I_H^2(t)I_m^{-2}(t)\right\rangle,$$

$$\geq \lambda_m\langle S_m(t)\rangle - \mu_m \tag{128}$$

$$+\sigma_2\frac{1}{t}\int_0^t \frac{S_m(\xi)I_H(\xi)}{I_m(\xi)}dB_2(\xi)-\frac{1}{2}\sigma_2^2\left\langle I_m^{-2}(t)\right\rangle,$$

thanks to the fact that S_m and I_H are in $(0,1)$. By using (112), to replace $\langle S_m(t)\rangle$ in (128), give

$$\begin{aligned}\frac{\ln(I_m(t)) - \ln(I_m(0))}{t} &\geq \lambda_m\left(1 - \langle I_m(t)\rangle + \psi(t)\right) - \mu_m \\ &\quad +\sigma_2 \frac{1}{t}\int_0^t \frac{S_m(\xi)I_H(\xi)}{I_m(\xi)} dB_2(\xi) - \frac{1}{2}\sigma_2^2 \left\langle I_m^{-2}(t)\right\rangle \\ &\geq \mu_m\left(\frac{\lambda_m}{\mu_m} - 1\right) - \lambda_m\langle I_m(t)\rangle + \lambda_m\psi(t) + G_2(t).\end{aligned}$$

where

$$G_2(t) = \sigma_2 \frac{1}{t}\int_0^t \frac{S_m(\xi)I_H(\xi)}{I_m(\xi)} dB_2(\xi) - \frac{1}{2}\sigma_2^2 c_2. \tag{129}$$

Clearly $\lim_{t\to\infty} G_2(t) = -\frac{1}{2}\sigma_2^2 c_2$.

It follows that

$$\ln(I_m(t)) \geq \mu_m\left(\frac{\lambda_m}{\mu_m} - 1\right)t - \lambda_m\int_0^t I_m(\xi)d\xi + F_2(t) \tag{130}$$

where

$$F_2(t) = tG_2(t) + \lambda_m t\psi(t) + \ln(I_m(0)). \tag{131}$$

Using the result of the lemma 3.2 and lemma 3.3 we get $\lim_{t\to\infty} \frac{F_2(t)}{t} = -\frac{1}{2}\sigma_2^2 c_2$. Thus, applying the lemma 3.4, we find:

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t I_m(\xi)d\xi \geq \mathcal{R}_0^m. \tag{132}$$

This completes the proof of the theorem. □

Remark 3.1. *The average persistence of the model means that there are almost certainly a number of infectious individuals in the human or mosquito population. That is, dengue persists in mean with a probability one.*

4. Numerical simulations

In this section, we perform numerical simulations of the deterministic model as well as the stochastic model in order to show forth our results.

4.1. Numerical simulations of the deterministic model.

We use the software MATLAB as the simulation environment. The figure we present in this section give the dynamics of the different compartments in the case where, $\mathcal{R}_0$ is less than one. The values of the parameters used are given by: $\mu_H = 0,3$, $\mu_m = 0,2$, $\gamma_H = 0,4$, $\alpha_H = 0,001$, $\lambda_H = 0,0005$ and $\lambda_m = 0,0021$. With these given parameters values, we find $\mathcal{R}_0 = 0.002 < 1$. The curves in figure 1 show respectively the variation of S_H, I_H, I_H, S_m, and I_m over time. The deterministic model stabilises at the free equilibrium point when $\mathcal{R}_0$ is less than one as illustrated by the proposition 2.4.

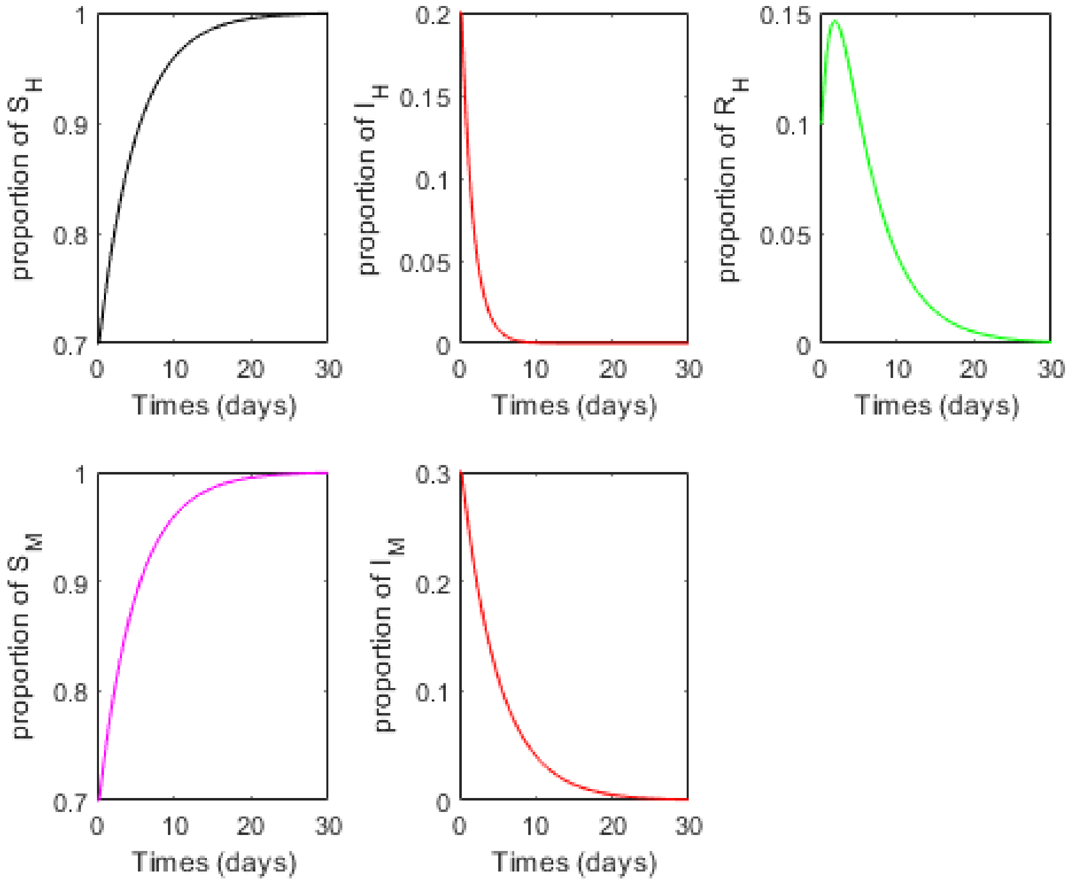

FIGURE 1. Graphs showing the behavior of trajectories S_H, I_H, I_H, S_m, and I_m of the deterministic model for $\mathcal{R}_0$ less than one.

4.2. **Numerical simulations of the stochastic model.** For the simulation of the stochastic model, we use the MATLAB software and the technique described in [11]. The graphs we present in this section give the dynamics of the different compartments in the case where, $\mathcal{R}_0$ is less than one. The values of the parameters used are given by: $\mu_H = 0, 3$, $\mu_m = 0, 2$, $\gamma_H = 0, 4$, $\alpha_H = 0, 001$, $\lambda_H = 0, 0005$, and $\lambda_m = 0, 0021$. With these given parameters values, we find $\mathcal{R}_0 = 0.002 < 1$. The stochastic model stabilises at the free equilibrium point when $\mathcal{R}_0$ is less than one as illustrated by the theorem 3.2.

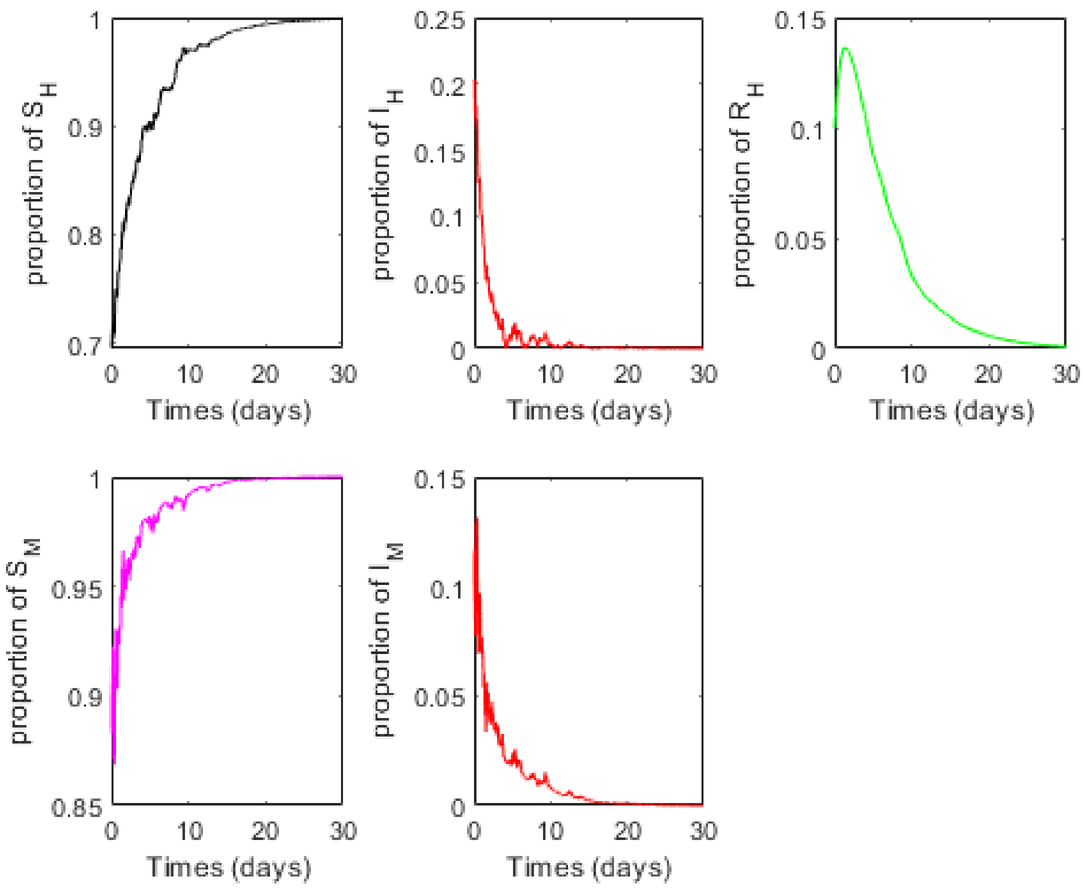

FIGURE 2. Graphs illustrate the behavior of trajectories S_H, I_H, I_H, S_m, and I_m of the stochastic case for $\mathcal{R}_0$ less than one.

5. Numerical example and remarks

Let's judiciously choose values for the parameters λ_m, λ_H, M_H et μ_m for which, the deterministic model is in extinction yet there is persistence in the mean for the stochastic model.
Consider the following table containing data when dengue fever is spreading:

Parameters	λ_H	λ_m	μ_m	M_H	μ_H	σ_1	c_1
Values	0.8	0.021	0.2	0.6	0.3	0.6	1

With these values, we get the following threshold values:

$\mathcal{R}_0$	$\mathcal{R}_0^H$
0.14	0.0125

We can notice that $\mathcal{R}_0 < 1$ thus according to the proposition 2.4 the equilibrium point E_0 of the system is globally asymptotically stable that is to say that the dengue stops propagating. However and $\mathcal{R}_0^H$ being greater than zero shows that dengue persists in the mean according to the theorem 3.4 under the assumption $(\mathcal{H})_1$. Hence the importance of taking the randomness aspect into account when modelling the spread of dengue.

6. Conclusion

In this paper, we focused on the comparative mathematical analysis of a deterministic and a stochastic epidemic model of dengue. First, we built a deterministic model of dengue fever. We showed the local stability of the disease-free equilibrium point by using a resolution method developed by Van den Driessche, P., Wathmough J and then showed the global stability of this point by constructing an appropriate Lyapunov function. Then we developed a stochastic model by adding two white noises at the contact rates. This addition is done in order to take into account the fluctuations in the transmission of dengue. We have shown the existence and uniqueness of a positive solution using a Lyapunov function and the îto formula. To analyse the extinction of dengue, we established that the disease-free equilibrium point is n-exponentially stable, and then the almost certain convergence of the solution to the disease-free equilibrium point when $\mathcal{R}_0$ is less than one, by successively constructing a Lyapunov function and applying the Itô's formula. We have also, established a persistence condition in mean of the stochastic differential system by constructing an appropriate Lyapunov function followed by an application of the Itô's formula and by using many other methods of stochastic analysis. In the last section we performed simulations to evaluate our results and then compared the two models. However, challenges remain in this work. We intend to conceive and analyse a discrete stochastic model of dengue. We also wish to analyse the transmission dynamics of other vector-borne diseases such as lymphatic filariasis, yellow fever and Zika. l'évenement

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