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BERNSTEIN-VON MISES THEOREM FOR FRACTIONAL SPDES WITH SMALL VOLATILITY

JAYA P. N. BISHWAL

Abstract. The Bernstein-von Mises theorem, concerning the convergence of suitably normalized and centred posterior density to normal density, is proved for a certain class of linearly parametrized fractional stochastic partial differential equations (SPDEs) driven by space-time color noise as the volatility decreases to zero. As a consequence, the Bayes estimators of the drift parameter, for smooth loss functions and priors, are shown to be strongly consistent and asymptotically normal, asymptotically efficient and asymptotically equivalent to the maximum likelihood estimator as the volatility decreases to zero. Also computable quasi-posterior density and quasi-Bayes estimators based on finite dimensional projections are shown to have similar asymptotics as the volatility decreases to zero and the dimension of the projection remains fixed.

1. Introduction

Infinite dimensional diffusion models of Heath, Jarrow and Morton (HJM) type has been used for modeling forward interest rate, see Carmona and Tehranchi [10]. Bayesian inference for a structural discrete time credit risk model with stochastic volatility and stochastic interest rates was studied in Rodriguez *et al.* [27]. Batista and Laurini [1] studied Bayesian estimation of term structure models by the Hamiltonian Monte Carlo method. It is natural that interest rates have long memory. Loges [23] initiated the study of parameter estimation in infinite dimensional stochastic differential equations. When the length of the observation time becomes large, he obtained consistency and asymptotic normality of the maximum likelihood estimator (MLE) of a real valued drift parameter in a Hilbert space valued SDE. Koski and Loges [21] extended the work of Loges [23] to minimum contrast estimators. Koski and Loges [20] applied the work to a stochastic heat flow problem. Bishwal [5] obtained asymptotic statistical results for discretely sampled diffusions. See Bishwal [6] for recent results on likelihood asymptotics and Bayesian asymptotics for drift estimation of finite and infinite dimensional stochastic differential equations. Large time asymptotics for Bayes estimators for Hilbert valued SDEs is studied in Bishwal [6].

Huebner, Khasminskii and Rozovskii [14] started statistical investigation in SPDEs. They gave two contrast examples of parabolic SPDEs in one of which they obtained consistency,

Department of Mathematics and Statistics, University of North Carolina at Charlotte, 376 Fretwell Bldg, 9201 University City Blvd., Charlotte, NC 28223-0001, USA

E-mail address: J.Bishwal@uncc.edu.

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asymptotic normality and asymptotic efficiency of the MLE as the intensity of noise decreases to zero under the condition of absolute continuity of measures generated by the process for different parameters (the situation is similar to the classical finite dimensional case) and in the other they obtained these properties as the finite dimensional projection becomes large under the condition of singularity of the measures generated by the process for different parameters. The second example was extended by Huebner and Rozovskii [15] and the first example was extended by Huebner [13] to MLE for general parabolic SPDEs where the partial differential operators commute and satisfy different order conditions in the two cases.

Huebner [12] extended the problem to the ML estimation of multidimensional parameter. Lototsky and Rozovskii [24] studied the same problem without the commutativity condition. Small noise asymptotics of the nonparmetric estimation of the drift coefficient was studies by Ibragimov and Khasminskii [17].

The Bernstein-von Mises theorem (BVT, in short), concerning the convergence of suitably normalized and centered posterior distribution to normal distribution, plays a fundamental role in asymptotic Bayesian inference, see Le Cam and Yang [22]. Borwanker *et al.* [8] obtained the BVT for discrete time Markov processes. Bose [9] extended the BVT to the homogeneous nonlinear diffusions. As a further refinement in BVT, Bishwal [3] obtained sharp rates of convergence to normality of the posterior distribution and the Bayes estimators for the Ornstein-Uhlenbeck process.

All these above work on BVT are concerned with finite dimensional SDEs. Bishwal [2] proved the BVT and obtained asymptotic properties of regular Bayes estimator of the drift parameter in a Hilbert space valued SDE when the corresponding ergodic diffusion process is observed continuously over a time interval $[0, T]$. The asymptotics are studied as $T \to \infty$ under the condition of absolute continuity of measures generated by the process. Results are illustrated for the example of an SPDE.

Bishwal [4] obtained BVT and spectral asymptotics of Bayes estimators for parabolic SPDEs when the number of Fourier coefficients becomes large. In that case, the measures generated by the process for different parameters are singular. Here we treat the case when the measures generated by the process for different parameters are absolutely continuous under some conditions on the order of the partial differential operators. We study the asymptotic properties of the posterior distributions and Bayes estimators when we have either fully observed process or finite-dimensional projections. The asymptotic parameter is only the intensity of noise. In this paper we treat the more general model.

The rest of the paper is organized as follows : Section 2 contains model, assumptions and preliminaries. In Section 3 we prove the Bernstein-von Mises theorems and Section 4 contains the asymptotic properties of regular Bayes estimator and quasi Bayes estimator. Section 5 provides heat equation as an example of fSPDE.

2. Model and Preliminaries

Let G be a smooth bounded domain in $\mathbb{R}^d$. We assume that the boundary ∂G of this domain is a C^∞-manifold of dimension $(d-1)$ and locally G is totally on one side of ∂G. For a multi-index

$\gamma = (\gamma_1, \dots, \gamma_d)$ we write

$$D^\gamma f(x) := \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \dots \partial x_d^{\gamma_d}} f(x)$$

where $|\gamma| = \gamma_1 + \gamma_2 + \dots + \gamma_d$.

Let A_0 and A_1 be partial differential operators of order m_0 and m_1 (the order of the highest derivative in it) respectively, written in the form

$$A_i(x)u := -\sum_{|\alpha|,|\beta| \le m_i} (-1)^{|\alpha|} D^\alpha (a_i^{\alpha\beta}(x) D^\beta(u))$$

where $a_i^{\alpha\beta}(x) \in C^\infty(\overline{G})$. For $\theta \in \mathbb{R}$, write $A^\theta = \theta A_1 + A_0$ and $a^{\alpha\beta}(\theta, x) = \theta a_1^{\alpha\beta}(x) + a_0^{\alpha\beta}(x)$. Let us fix θ_0, the unknown true value of the parameter θ. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $W(t,x)$ be a cylindrical Brownian motion on on this space with values in the Schwarz space of distributions $\mathcal{D}'(G)$.

A cylindrical Brownian motion (C.B.M) is $W = W(t,x)$ is a distribution valued process such that for every such that for every $\phi \in C_0^\infty(G)$ with $\|\phi\|_{L^2(G)} = 1$ the inner product $\langle W(t,\cdot), \phi(\cdot)\rangle$ is a one dimensional Brownian motion and for every $\phi_1, \phi_2 \in C_0^\infty(G)$,

$$E(\langle W(s,\cdot), \phi_1(\cdot)\rangle \langle W(t,\cdot), \phi_2(\cdot)\rangle) = (s \wedge t)(\phi_1, \phi_2)_{L^2(G)}.$$

The C.B.M. W can be expanded in the series $W(t,x) = \sum_{i=1}^\infty W_i(t) h_i(x)$ where $\{W_i(t)\}_{i=1}^\infty$ are independent one dimensional Brownian motions and $\{h_i\}_{i=1}^\infty$ is complete orthonormal system in $L_2(G)$. The latter series converges P-a.s.

We will consider the Dirichlet problem for a parabolic SPDE associated with the operator A^θ, and driven by the C.B.M. W :

$$\frac{\partial u(t,x)}{\partial t} = A^\theta(x) u(t,x) + \frac{\partial}{\partial t} W(t,x) \tag{2.1}$$

$$u(0,x) = u_0(x) \tag{2.2}$$

$$D^\gamma u(t,x)|_{\partial G} = 0 \tag{2.3}$$

$$\text{for all multi-indices } \gamma \text{ with } |\gamma| \le m - 1.$$

The problem (2.1) - (2.3) is understood in the sense of distributions.

A cylindrical fractional Brownian motion (C.F.B.M) is $W = W^H(t,x)$ is a distribution valued process such that for every such that for every $\phi \in C_0^\infty(G)$ with $\|\phi\|_{L^2(G)} = 1$ the inner product $\langle W(t,\cdot), \phi(\cdot)\rangle$ is a one dimensional fractional Brownian motion and for every $\phi_1, \phi_2 \in C_0^\infty(G)$,

$$E(\langle W^H(s,\cdot), \phi_1(\cdot)\rangle \langle W^H(t,\cdot), \phi_2(\cdot)\rangle) = (s \wedge t)(\phi_1, \phi_2)_{L^2(G)}.$$

The C.F.B.M. W^H can be expanded in the series $W^H(t,x) = \sum_{i=1}^\infty W_i^H(t) h_i(x)$ where $\{W_i^H(t)\}_{i=1}^\infty$ are independent one dimensional fractional Brownian motions and $\{h_i\}_{i=1}^\infty$ is complete orthonormal system in $L_2(G)$. The latter series converges P-a.s.

Recall that a fractional Brownian motion (fBM) has the covariance

$$\widetilde{C}_H(s,t) = \frac{1}{2}\left[s^{2H} + t^{2H} - |s-t|^{2H}\right], \quad s, t > 0.$$

For $H > 0.5$ the process has long range dependence or long memory and the process is self-similar. For $H \neq 0.5$, the process is neither a Markov process nor a semimartingale. For $H = 0.5$, the process reduces to standard Brownian motion. Fractional Brownian motion can be represented as a Riemann-Liouville (fractional) derivative of Gaussian white noise, see

Decreusefond and Ustunel [11] and Jumarie [18]. For deterministic fractional calculus, see Samko *et al.* [28].

Let σ be the strength of noise. On the complete probability space $(\Omega, \mathcal{F}, P)$ define the parabolic SPDE

$$du^\sigma(t,x) = A^\theta u^\sigma(t,x)dt + \sigma dW^H(t,x),\ 0 \le t \le T,\ x \in G \tag{2.4}$$

with Dirichlet boundary conditions

$$u(0,x) = u_0(x) \tag{2.5}$$

$$D^\gamma u(t,x)|_{\partial G} = 0 \tag{2.6}$$

for all multi-indices γ with $|\gamma| \le m-1$

where $A^\theta = \theta A_1 + A_0$, A_1 and A_0 are partial differential operators of orders m_1 and m_2 respectively, A^θ has order $2m = \max(m_1, m_0)$, the process $W(t,x)$ is a cylindrical Brownian motion in $L^2([0,T]\times G)$ where G is a bounded domain in $\mathbb{R}^d$ and $u_0 \in L_2(G)$, and $\sigma > 0$ is the volatility which is assumed to be known. For estimation od σ, see Bishwal [7]. Here $\theta \in \Theta \subseteq \mathbb{R}$ is the unknown parameter to be estimated on the basis of the observations of the field $u^\theta(t,x), t \in [0,T], x \in G$. Let θ_0 be the true value of the unknown parameter.

The following regularity conditions are assumed:

(R1) $m_1 < m - d/2$ where d denotes the dimension of the $x-$space G.

(R2) The operators A_1 and A_0 are formally self-adjoint, i.e., for $i = 0, 1$,

$$\int_G A_i uv dx = \int_G uA_i v dx \quad \text{for all } u, v \in C_0^\infty(G).$$

(R3) There is a compact neighborhood Θ of θ_0 so that $\{A^\theta, \theta \in \Theta\}$ is a family of uniformly strongly elliptic operators of order $2m = \max(m_1, m_0)$.

The latter means that there exists a positive constant δ such that for all $x \in \overline{G}, \theta \in \Theta$ and $\xi \in \mathbb{R}^d$,

$$\sum_{|\alpha|,|\beta|=m} a^{\alpha\beta}(\theta,x)\xi^\alpha\xi^\beta \ge \delta|\xi|^{2m},$$

where $\xi^\gamma := \xi_i^{\gamma_1} \ldots \xi_d^{\gamma_d}$.

For $s > 0$, denote the closure of $C_0^\infty(G)$ in the Sobolev space $W^{s,2}(G)$ by $W_0^{s.2}$. It is well known from the theory of self-adjoint elliptic operators that the operator A^θ with boundary condition (2.6) can be extended to a closed, self-adjoint operator $\mathcal{L}_\theta$ on $L_2(G)$. The domain of $\mathcal{L}_\theta$, written $\mathcal{D}(\mathcal{L}_\theta)$, is the set of all functions $u \in W_0^{m,2}$ such that $\mathcal{L}_\theta u \in L_2(G)$. For all $v \in W_0^{m,2}$

$$a^\theta(u,v) \ := \ - \sum_{|\alpha|,|\beta|\le m} \int_G a^{\alpha\beta}(\theta,x)D^\beta u(x)D^\alpha v(x)dx = (\mathcal{L}_\theta u, v)_{L_2(G)}$$

and $\mathcal{L}_\theta u = A^\theta u$ in the sense of distribution. Under (R3), $\mathcal{L}_\theta$ is lower semibounded (i.e., there is a constant $k(\theta)$ so that $k(\theta)I - \mathcal{L}_\theta > 0$ and the resolvent $(k(\theta)I - \mathcal{L}_\theta)^{-1}$ is compact). Let $\Lambda_\theta := (k(\theta)I - \mathcal{L}_\theta)^{1/2m}$, the spectrum of this operator is a discrete set $\Sigma(\Lambda_\theta)$ consisting of eigenvalues of finite multiplicity. We enumerate them in order of magnitude,

$$\Sigma(\Lambda_\theta) = \{\lambda_i(\theta)\}_{i=1}^\infty,\ \ 0 < \lambda_i(\theta) < \lambda_2(\theta) < \ldots$$

where each one is counted repeatedly as many times as its multiplicity. Let $\{h_i(\theta)\}_{i=1}^{\infty}$ be an orthonormal system of eigenfunctions of Λ_θ. Then $\{h_i(\theta)\}_{i=1}^{\infty}$ is complete in $L_2(G)$ and $h_i(\theta) \in W_0^{m,2}(G) \cap C^\infty(\overline{G})$ for all i.

In general, the functions $h_i(\theta)$ might depend on θ. For the sake of simplicity we shall rule out this possibility in future. We assume :

(R4) There exists a complete orthonormal system $\{h_i\}_{i=1}^{\infty}$ in $L_2(G)$ such that for every $i = 1, 2, \ldots, h_i \in W_0^{m,2}(G) \cap C^\infty(\overline{G})$ and

$$\Lambda_\theta h_i = \lambda_i(\theta) h_i, \text{ and } \mathcal{L}_\theta h_i = \mu_i(\theta) h_i \text{ for all } \theta \in \Theta$$

where $\mathcal{L}_\theta$ is a closed self adjoint extension of A^θ, $\Lambda_\theta := (k(\theta)I - \mathcal{L}_\theta)^{1/2m}, k(\theta)$ is a constant and and the spectrum of the operator Λ_θ consists of eigenvalues $\{\lambda_i(\theta)\}_{i=1}^{\infty}$ of finite multiplicities and $\mu_i(\theta) = -\lambda_i^{2m}(\theta) + k(\theta)$.

(R5) The operator A_1 is uniformly strongly elliptic and has the same system of eigenfunctions $\{h_i\}_{i=1}^{\infty}$ as $\mathcal{L}_\theta$.

Now we focus on the fundamental semimartingale behind the SPDE model. Define

$$\begin{aligned}
\kappa_H &:= 2H\Gamma(3/2 - H)\Gamma(H + 1/2), \quad k_H(t,s) := \kappa_H^{-1}(s(t-s))^{\frac{1}{2}-H},\\
\lambda_H &:= \frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(3/2-H)}, \quad v_t \equiv v_t^H := \lambda_H^{-1} t^{2-2H}, \quad \mathcal{M}_t^H := \int_0^t k_H(t,s) dW_s^H.
\end{aligned}$$

From Norros *et al.* [25] it is well known that $\mathcal{M}_{k,t}^H$ is a Gaussian martingale, called the fundamental martingale whose variance function $\langle \mathcal{M}_k^H \rangle_t$ is v_t^H. Moreover, the natural filtration of the martingale $\mathcal{M}^H$ coincides with the natural filtration of the fBm W^H since

$$W_{k,t}^H := \int_0^t K(t,s) d\mathcal{M}_{k,s}^H$$

holds for $H \in (1/2, 1)$ where

$$K_H(t,s) := H(2H-1) \int_s^t r^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}} dr, \quad 0 \le s \le t$$

and for $H = 1/2$, the convention $K_{1/2} \equiv 1$ is used.

Define

$$Q_{k,t} := \frac{d}{dv_t} \int_0^t k_H(t,s) u_{k,s} ds.$$

Define the process $Z_k = (Z_{k,t}, t \in [0,T], k \ge 1)$ by

$$Z_{k,t} := \int_0^t k_H(t,s) du_{k,s}.$$

It is easy to see that

$$Q_{k,t} = \frac{\lambda_H}{2(2-2H)} \left\{ t^{2H-1} Z_{k,t} + \int_0^t r^{2H-1} dZ_{k,s} \right\}.$$

The following facts are known from Kleptsyna and Le Breton [19]:

(i) Z_k is the fundamental semimartingale associated with the process u_k.

(ii) Z_k is a $(\mathcal{F}_t)$ -semimartingale with the decomposition

$$Z_{k,t} = \theta \int_0^t Q_{k,s} dv_s + \mathcal{M}_{k,t}^H.$$

(iii) u_k admits the representation

$$u_{k,t} = \int_0^t K_H(t,s)dZ_{k,s}.$$

(iv) The natural filtration $(\mathcal{Z}_t)$ of Z_k and $(\mathcal{U}_t)$ of u_k coincide.

We have

$$\begin{aligned} Q_{k,t} &= \frac{d}{dv_t}\int_0^t k_H(t,s)u_{k,s}ds \\ &= \kappa_H^{-1}\frac{d}{dv_t}\int_0^t s^{1/2-H}(t-s)^{1/2-H}u_{k,s}ds \\ &= \kappa_H^{-1}\lambda_H t^{2H-1}\frac{d}{dt}\int_0^t s^{1/2-H}(t-s)^{1/2-H}u_{k,s}ds \\ &= \kappa_H^{-1}\lambda_H t^{2H-1}\int_0^t \frac{d}{dt}s^{1/2-H}(t-s)^{1/2-H}u_{k,s}ds \\ &= \kappa_H^{-1}\lambda_H t^{2H-1}\int_0^t s^{1/2-H}(t-s)^{-1/2-H}u_{k,s}ds. \end{aligned}$$

The process Q_k depends continuously on u_k and therefore, the discrete observations of u_k does not allow one to obtain the discrete observations of Q. The process Q_k can be approximated by

$$\widetilde{Q}_{k,n} = \kappa_H^{-1}\lambda_H n^{2H-1}\sum_{j=0}^{n-1} j^{1/2-H}(n-j)^{-1/2-H}u_{k,j}.$$

It is easy to show that $\widetilde{Q}_n \to Q_t$ almost surely as $n\to\infty$, see Tudor and Viens [29].

Define a new partition $0 \le r_1 < r_2 < r_3 < \cdots < r_{m_k} = t_k,\quad k = 1,2,\cdots,n$ and

$$\widetilde{Q}_i(t_k) := \kappa_H^{-1}\eta_H t_k^{2H-1}\sum_{j=1}^{m_k} r_j^{1/2-H}(r_{m_k}-r_j)^{-1/2-H}u_i(r_j)(r_j-r_{j-1}),$$

$k = 1,2,\cdots,n$. It is easy to show that $\widetilde{Q}_i(t_k) \to Q_i(t)$ almost surely as $m_k \to \infty$ for each $k = 1,2,\cdots,n$ and $i \ge 1$, see Tudor and Viens [29].

We use this approximate observation in the calculation of our estimators. Thus our observations are

$$u_i(t) \approx \int_0^t K_H(t,s)d\widetilde{Z}_i(s) \quad \text{where} \quad \widetilde{Z}_i(t) = \theta\int_0^t \widetilde{Q}_i(s)dv_s + \mathcal{M}_{i,t}^H$$

observed at $t_1, t_2, \ldots, t_n$.

Note that for equally spaced data

$$\Delta v_{t_i} := v_{t_i} - v_{t_{i-1}} = \lambda_H^{-1}\left(\frac{T}{n}\right)^{2-2H}[i^{2-2H} - (i-1)^{2-2H}].$$

For $H = 0.5$,

$$v_{t_i} - v_{t_{i-1}} = \lambda_H^{-1}\left(\frac{T}{n}\right)^{2-2H}[i^{2-2H} - (i-1)^{2-2H}] = \frac{T}{n},\quad i = 1,2,\ldots,n$$

the standard equispaced partition. In this paper we do not need to assume $T/n \to 0$ unlike the finite dimensional diffusion models as we take advantage of the decreasing volatility $\sigma \to 0$ in this paper.

For $\alpha > d/2$, define the Hilbert space $H^{-\alpha}$ with norm $\|\cdot\|$ as in Huebner and Rozovskii [15]. Let $P_\theta^{T,\sigma}$ the measure generated by the solution $\{u^\sigma(t,x), t\in[0,T], x\in G\}$ to the problem (2.4) – (2.6) on the space $\mathcal{C}([0,T];H^{-\alpha})$ with the associated Borel σ-algebra $\mathcal{B}_T$. Note that condition (R1) is equivalent to

$$\int_0^T \| A_1 Q^\sigma(s) \|^2\, ds < \infty \text{ a. s. for fixed } \sigma.$$

Thus under (R1), for different θ, the measures $P_\theta^{T,\sigma}$ are mutually absolutely continuous.

The Radon-Nikodym derivative (likelihood) of $P_\theta^{T,\sigma}$ with respect to $P_{\theta_0}^{T,\sigma}$ is given by

$$\begin{aligned}\Lambda_{T,\sigma}^{\theta}(Q) := \frac{dP_\theta^{T,\sigma}}{dP_{\theta_0}^{T,\sigma}}(Q^\sigma) \;=\; & \exp\Big\{\sigma^{-1}(\theta-\theta_0)\int_0^T (A_1Q^\sigma(s), dZ^\sigma(s))_0 \\ & -\frac{1}{2}\sigma^{-2}(\theta^2-\theta_0^2)\int_0^T \|A_1u^\sigma(s)\|_0^2 dv_s \\ & -\sigma^{-1}(\theta-\theta_0)\int_0^T (A_1Q^\sigma(s), A_0Q^\sigma(s))_0 dv_s\Big\}.\end{aligned} \tag{2.7}$$

Maximizing $\Lambda_{T,\sigma}^{\theta}(Q)$ with respect to θ provides the maximum likelihood estimator (MLE) given by

$$\hat{\theta}^\sigma = \frac{\int_0^T (A_1Q^\sigma(s), dZ^\sigma(s) - A_0Q^\sigma(s)dv_s)_0}{\int_0^T \|A_1Q^\sigma(s)\|_0^2 dv_s}. \tag{2.8}$$

The Fisher information $I(\theta_0)$ related to $\frac{dP_\theta^{T,\sigma}}{dP_{\theta_0}^{T,\sigma}}$ is given by

$$I(\theta_0) \;:=\; E_{\theta_0}\int_0^T \|A_1Q^\sigma(s)\|_0^2 dv_s.$$

Note that $u^\sigma(t,x)$ is the observation at time t at point x. In practice, it is impossible to observe the field $Q^\sigma(t,x)$ at all points t and x. Hence, only a finite dimensional projection $Q^{n,\sigma} := (Q_1^\sigma(t),\ldots,Q_n^\sigma(t))$, $t\in[0,T]$ of the solution of the equation (2.4) are observable. In other words, we can observe the first n highest nodes in the Fourier expansion

$$Q^\sigma(t,x) = \sum_{t=1}^{\infty} Q_i^\sigma(t)\phi_i(x)$$

corresponding to some orthogonal basis $\{\phi_i(x)\}_{i=1}^\infty$. We consider observation continuous in time $t\in[0,T]$. Note that $Q_i^\theta(t), i\geq 1$ are independent one dimensional Ornstein-Uhlenbeck processes (see Huebner and Rozovskii [15]).

Consider the projection of $H^{-\alpha}$ on to the subspace $\mathbb{R}^n$. Let $P_\theta^{T,n,\sigma}$ be the measure generated by $u^{n,\sigma}$ on $\mathcal{C}[(0,T];\mathbb{R}^n)$ with the associated Borel σ-algebra $\mathcal{B}_T^n$.

For $\theta\in\Theta$, the measures $P_\theta^{T,n,\sigma}$and $P_{\theta_0}^{T,n,\sigma}$ are mutually absolutely continuous with Radon-Nikodym derivative (likelihood ratio) given by

$$\begin{aligned}\Lambda_{T,n,\sigma}^{\theta}(Q) := \frac{dP_\theta^{T,n,\sigma}}{dP_{\theta_0}^{T,n,\sigma}}(Q^{n,\sigma}) \;=\; & \exp\Big\{\sigma^{-1}(\theta-\theta_0)\int_0^T (A_1Q^{n,\sigma}(s), dZ^{n,\sigma}(s))_0 \\ & -\frac{1}{2}\sigma^{-2}(\theta^2-\theta_0^2)\int_0^T \|A_1Q^{n,\sigma}(s)\|_0^2 dv_s \\ & -\sigma^{-1}(\theta-\theta_0)\int_0^T (A_1Q^{n,\sigma}(s), A_0Q^{n,\sigma}(s))_0 dv_s\Big\}.\end{aligned} \tag{2.9}$$

Maximizing $\Lambda^{\theta}_{n,\sigma}(u)$ with respect to θ provides the approximate maximum likelihood estimator (AMLE) given by

$$\hat{\theta}^{n,\sigma} = \frac{\int_0^T (A_1 Q^{n,\sigma}(s), dZ^{n,\sigma}(s) - A_0 Q^{n,\sigma}(s) dv_s)_0}{\int_0^T \|A_1 Q^{n,\sigma}(s)\|_0^2 dv_s}. \tag{2.10}$$

Assumption (R5) implies in particular that for every i, $\mu_i := \mu_i(\theta_0) = \theta_0 \nu_i + k_i$ and $A_1 h_i = \nu_i h_i$ and $A_0 h_i = k_i h_i$.

Thus

$$\hat{\theta}^{\sigma} = \frac{\sum_{i=1}^{\infty} \lambda_i^{2\alpha} \nu_i \int_0^T Q_i^{\sigma}(t)(dQ_i^{\sigma}(t) - k_i Q_i^{\sigma}(t) dv_t)}{\sum_{i=1}^{\infty} \lambda_i^{2\alpha} \nu_i^2 \int_0^T |Q_i^{\sigma}|^2(t) dv_t}$$

and

$$\hat{\theta}^{n,\sigma} = \frac{\sum_{i=1}^{n} \lambda_i^{2\alpha} \nu_i \int_0^T Q_i^{\sigma}(t)(dQ_i^{\sigma}(t) - k_i Q_i^{\sigma}(t) dv_t)}{\sum_{i=1}^{n} \lambda_i^{2\alpha} \nu_i^2 \int_0^T |Q_i^{\sigma}|^2(t) dv_t}.$$

The normalized errors are given by

$$\sigma^{-1}(\hat{\theta}^{\sigma} - \theta_0) = \frac{\sum_{i=1}^{\infty} \lambda_i^{2\alpha} \nu_i \int_0^T Q_i^{\sigma}(t) dW_i^H(t)}{\sum_{i=1}^{\infty} \lambda_i^{2\alpha} \nu_i^2 \int_0^T |Q_i^{\sigma}|^2(t) dv_t}$$

and

$$\sigma^{-1}(\hat{\theta}^{n,\sigma} - \theta_0) = \frac{\sum_{i=1}^{n} \lambda_i^{2\alpha} \nu_i \int_0^T Q_i^{\sigma}(t) dW_i^H(t)}{\sum_{i=1}^{n} \lambda_i^{2\alpha} \nu_i^2 \int_0^T |Q_i^{\sigma}|^2(t) dv_t}.$$

Recall that the Fisher information is given by

$$I(\theta_0) = E_{\theta_0} \int_0^T \|A_1 Q^{\sigma}(s)\|_0^2 dv_s.$$

By the central limit theorem for stochastic integrals (see Nourdin and Peccati [26]), $\sigma^{-1}(\hat{\theta}^{\sigma} - \theta_0) \to \mathcal{N}(0, I(\theta_0)^{-1})$ as $\sigma \to 0$ and $\sigma^{-1}(\hat{\theta}^{n,\sigma} - \theta_0) \to \mathcal{N}(0, I_n(\theta_0)^{-1})$ as $\sigma \to 0$.

Now we will derive the Fisher information $I(\theta_0)$. The observations $Q_i^{\sigma}(t), Q_2^{\sigma}(t), \ldots$ where $Q_i^{\sigma}(t), i \geq 1$ are the Fourier coefficients of the $Q^{\sigma}(t,x)$ satisfy the system of fractional stochastic differential equations (fractional Vasicek models)

$$dQ_i^{\sigma}(t) = \mu_i(\theta) Q_i^{\sigma}(t) dt + \sigma \lambda_i^{-\alpha} W_i^H(t),$$
$$Q_i^{\sigma}(0) = Q_{0i}$$

where $\mu_i(\theta_0) = k_i + \theta_0 \nu_i$. The solution of the above fSDE is

$$Q_i^{\sigma}(t) = Q_{0i} e^{\mu_i(\theta_0)t} + \sigma \lambda_i^{-\alpha} \int_0^t e^{\mu_i(\theta_0)(t-s)} dW_i^H(s).$$

The likelihood $\Lambda^{\theta}_{T,n,\sigma}(u)$ can be written as

$$\begin{aligned} \Lambda^{\theta}_{T,n,\sigma}(u) := \frac{dP_{\theta}^{T,n,\sigma}}{dP_{\theta_0}^{T,n,\sigma}}(Q^{n,\sigma}) \quad &= \quad \exp\Bigg\{ \sigma^{-1}(\theta - \theta_0) \sum_{i=1}^{n} \lambda_i^{2\alpha} \nu_i \int_0^T Q_i^{\sigma}(t) dW_i^H(t) \\ &\quad - \frac{1}{2} \sigma^{-2} (\theta - \theta_0)^2 \sum_{i=1}^{n} \lambda_i^{2\alpha} \nu_i^2 \int_0^T |Q_i^{\sigma}|^2(t) dv_t \Bigg\}. \end{aligned} \tag{2.11}$$

The Fisher information corresponding to the likelihood $\Lambda^{\theta}_{T,n,\sigma}(u)$ is given by

$$I_{n,\sigma}(\theta_0) = \sigma^{-2} E \sum_{i=1}^{n} \lambda_i^{2\alpha} \nu_i^2 \int_0^T |Q_i^{\sigma}(t)|^2 dv_t$$

$$= \sigma^{-2} \sum_{i=1}^{n} \frac{\lambda_i^{2\alpha} \nu_i^2}{2\mu_i} Q_{0i}^2 (e^{2\mu_i T} - 1) - T \sum_{i=1}^{n} \frac{\nu_i^2}{2\mu_i} + \sum_{i=1}^{n} \nu_i^2 \left(\frac{e^{2\mu_i T} - 1}{4\mu_i^2} \right).$$

For smooth initial conditions, i.e., $\sum_{i=1}^{\infty} i^{2s/d} Q_{0i}^2 < \infty$ for some s, the first sum converges as $n \to \infty$. The second sum dominates the third.

Similar to the operator A^θ, the operator A_1 supplemented by the Dirichlet boundary conditions $D^\gamma u(t,x)|_{\partial G} = 0$ for all $|\gamma| \le r-1$ can be extended to a closed self-adjoint operator on $L_2(G)$. We will denote this operator by $\mathcal{L}_1$. Its domain $\mathcal{D}(\mathcal{L}_1)$ consists of all functions $u \in W_0^{r,2}$ such that $\mathcal{L}_1 \in L_2(G)$. Thus $A_1 h_i = \nu_i h_i$ for all $i = 1, 2, \ldots$. According to the spectral theory of self-adjoint operators, the asymptotics of the eigenvalues μ_i and ν_i are given by $|\nu_i| \sim i^{m_1/d}$ and $\mu_i \sim -i^{2m/d}$, $2m = \max(m_0, m_1)$.

Due to the asymptotics of the eigenvalues we have

$$-\sum_{i=1}^{\infty} \frac{\nu_i^2}{\mu_i} = \sum_{i=1}^{\infty} i^{2(m_1 - m)/d} < \infty$$

since $2(ord(A_1) - ord(A_0 + \theta A_1))/d = 2(m_1 - m)/d < -1$ by (R1).

Hence

$$\lim_{n\to\infty} \lim_{\sigma\to 0} \sigma^2 I_{n,\sigma}(\theta_0) = \lim_{\sigma\to 0} \lim_{n\to\infty} \sigma^2 I_{n,\sigma}(\theta_0) = \sum_{i=1}^{\infty} \frac{\lambda_i^{2\alpha} \nu_i^2}{2\mu_i} Q_{0i}^2 (e^{2\mu_i T} - 1) =: I(\theta_0).$$

With smooth initial condition this sum converges and the Fisher information is finite $I(\theta_0) < \infty$ and if $u_{0i} \neq 0$, then $I(\theta_0) > 0$.

The Fisher information $I_n(\theta_0)$ related to $\frac{dP_\theta^{T,n,\sigma}}{dP_{\theta_0}^{T,n,\sigma}}$ is given by

$$\lim_{\sigma\to 0} \sigma^2 I_{n,\sigma}(\theta_0) = \sum_{i=1}^{n} \frac{\lambda_i^2}{2\mu_i} Q_{0i}^2 (e^{2\mu_i T} - 1) =: I_n(\theta_0).$$

Let ω be a real valued, non-negative loss function of polynomial majorant defined on $\mathbb{R}$, which are symmetric, $\omega(0) = 0$ and monotone on the positive real line.

Under the conditions (R1) – (R5), Huebner [13] showed that $\hat{\theta}_\sigma$ and $\hat{\theta}_{\sigma,n}$ are strongly consistent, asymptotically normally distributed with normalization σ^{-1} and asymptotically efficient with respect to the loss function ω as $\sigma \to 0$ and n and T are fixed.

3. Bernstein-von Mises Theorem

In this section, we show the convergence of the posterior distributions to normal distribution, which is called the Bernsten-von Mises theorem or Bayesian central limit theorem. Suppose that Π is a prior probability measure on $(\Theta, \mathcal{D})$, where $\mathcal{D}$ is the σ-algebra of Borel subsets of Θ. Assume that Π has a density $\pi(\cdot)$ with respect to the Lebesgue measure and the density is continuous and positive in an open neighborhood of θ_0.

The posterior density of θ given Q^σ is given by

$$p(\theta|u^\sigma) := \frac{\Lambda_{T,\sigma}^\theta(u)\pi(\theta)}{\int_\Theta \Lambda_{T,\sigma}^\theta(u)\pi(\theta)d\theta}. \tag{3.1}$$

Let $\tau := \sigma^{-1}(\theta - \hat{\theta}^{\sigma})$. Then the posterior density of $\sigma^{-1}(\theta - \hat{\theta}^{\sigma})$ is given by

$$p^*(\tau|u^{\sigma}) := \sigma^{-1}p(\hat{\theta}^{\sigma} + \sigma\tau|Q^{\sigma}).$$

Let

$$\nu_{T,\sigma}(\tau) := \frac{dP^{T,\sigma}_{\hat{\theta}^{\sigma}+\sigma\tau}/dP^{T,\sigma}_{\theta_0}}{dP^{T,\sigma}_{\hat{\theta}^{\sigma}}/dP^{T,\sigma}_{\theta_0}} = \frac{dP^{T,\sigma}_{\hat{\theta}^{\sigma}+\sigma\tau}}{dP^{T,\sigma}_{\hat{\theta}^{\sigma}}}, \quad C_{\sigma} := \int_{-\infty}^{\infty} \nu_{\sigma}(\tau)\pi(\hat{\theta}^{\sigma} + \sigma\tau)d\tau.$$

Clearly

$$p^*(\tau|u^{\sigma}) = C_{\sigma}^{-1}\nu_{T,\sigma}(\tau)\pi(\hat{\theta}^{\sigma} + \sigma\tau).$$

The quasi-posterior density of θ given in $Q^{n,\sigma}$ is given by

$$q(\theta|Q^{n,\sigma}) := \frac{\Lambda^{\theta}_{T,n,\sigma}(u)\pi(\theta)}{\int_{\Theta}\Lambda^{\theta}_{T,n,\sigma}(u)\pi(\theta)d\theta}. \tag{3.2}$$

The idea behind quasi-posterior density is that while a regular posterior density uses the full exact likelihood, quasi-posterior uses the partial likelihood based on the finite number of Fourier coefficients $Q^{n,\sigma} := (Q_1^{\sigma}(t), \ldots, Q_n^{\sigma}(t))$, $t \in [0,T]$. Because the complete observation can not be observed in practice, quasi-posterior density has computational advantage.

Let $\phi := \sigma^{-1}(\theta - \hat{\theta}^{n,\sigma})$. Then the quasi-posterior density of $\sigma^{-1}(\theta - \hat{\theta}^{n,\sigma})$ is given by

$$q^*(\phi|Q^{n,\sigma}) := \sigma^{-1}q(\hat{\theta}^{n,\sigma} + \sigma\phi|Q^{n,\sigma}).$$

Let

$$\nu_{T,n,\sigma}(\phi) := \frac{dP^{T,n,\sigma}_{\hat{\theta}^{n,\sigma}+\sigma\phi}/dP^{T,n,\sigma}_{\theta_0}}{dP^{T,n,\sigma}_{\hat{\theta}^{n,\sigma}/dP^{T,n,\sigma}_{\theta_0}}} = \frac{dP^{T,n,\sigma}_{\hat{\theta}^{n,\sigma}+\sigma\phi}}{dP^{T,n,\sigma}_{\hat{\theta}^{n,\sigma}}}, \quad D_{n,\sigma} := \int_{-\infty}^{\infty} \nu_{T,n,\sigma}(\phi)\pi(\hat{\theta}^{\sigma} + \sigma\phi)d\phi.$$

Clearly

$$q^*(\phi|Q^{n,\sigma}) = D_{n,\sigma}^{-1}\nu_{T,n,\sigma}(\phi)\pi(\hat{\theta}^{n,\sigma} + \sigma\phi).$$

Let $K(\cdot)$ be a non-negative measurable function satisfying the following two conditions :

(K1) There exists a number $\eta, 0 < \eta < 1$, for which

$$\int_{-\infty}^{\infty} K(\tau)\exp\{-\frac{1}{2}\tau^2(1-\eta)\}d\tau < \infty.$$

(K2) For every $\lambda > 0$ and $\delta > 0$

$$e^{-\lambda\sigma^{-2}}\int_{|\tau|>\delta} K(\sigma^{-1}\tau)\pi(\hat{\theta}^{\sigma} + \tau)d\tau \to 0 \ \text{ a.s. } [P_{\theta_0}] \text{ as } \sigma \to 0.$$

We need the following Lemma to prove the Bernstein-von Mises theorem.

Lemma 3.1 Under the assumptions (R1) – (R5) and (K1) – (K2),

(i) There exists a $\delta_0 > 0$ such that

$$\lim_{\sigma\to 0}\int_{|\tau|\le\delta_0\sigma^{-1}} K(\tau)\left|\nu_{\sigma}(\tau)\pi(\hat{\theta}^{\sigma} + \sigma^{-1}\tau) - \pi(\theta_0)\exp(-\frac{1}{2}I(\theta_0)\tau^2)\right|d\tau = 0 \ \text{ a.s. } [P_{\theta_0}].$$

(ii) For every $\delta > 0$,

$$\lim_{\sigma\to 0}\int_{|\tau|\ge\delta\sigma^{-1}} K(\tau)\left|\nu_{\sigma}(\tau)\pi(\hat{\theta}^{\sigma} + \sigma^{-1}\tau) - \pi(\theta_0)\exp(-\frac{1}{2}I(\theta_0)\tau^2)\right|d\tau = 0 \ \text{ a.s. } [P_{\theta_0}].$$

Proof. From (3.1) and (3.2), it is easy to check that

$$\log\nu_{\sigma}(\tau) = -\frac{1}{2}\tau^2\sigma^2 \int_0^T \|A_1Q^{\sigma}(s)\|_0^2 dv_s.$$

Now (i) follows by an application of dominated convergence theorem.

For every $\delta > 0$, there exists $\lambda > 0$ depending on δ and β such that

$$\begin{aligned} &\lim_{\sigma\to 0}\int_{|\tau|\geq\delta\sigma^{-1}} K(\tau)\left|\nu_\sigma(\tau)\pi(\hat{\theta}^n+\sigma\tau)-\pi(\theta_0)\exp(-\frac{1}{2}\tau^2)\right| d\tau \\ \leq\ & \int_{|\tau|\geq\delta\sigma^{-1}} K(\tau)\nu_\sigma(\tau)\pi(\hat{\theta}^n+\sigma\tau)d\tau + \int_{|\tau|\geq\delta\sigma^{-1}} \pi(\theta_0)\exp(-\frac{1}{2}\tau^2)d\tau \\ \leq\ & e^{-\lambda\sigma^{-2}}\int_{|\tau|\geq\delta\sigma^{-1}} K(\tau)\pi(\hat{\theta}^n+\sigma\tau)d\tau + \pi(\theta_0)\int_{|\tau|\geq\delta\sigma^{-1}}\exp(-\frac{1}{2}\tau^2)d\tau \\ =:\ & F_\sigma + G_\sigma \end{aligned}$$

By condition (K2), it follows that $F_\sigma \to 0$ a.s. $[P_{\theta_0}]$ as $\sigma \to 0$ for every $\delta > 0$. Condition K(1) implies that $G_\sigma \to 0$ as $\sigma \to 0$. This completes the proof of the Lemma. □

Now we are ready to prove the generalized version of the Bernstein-von Mises theorem for parabolic fractional SPDEs.

Theorem 3.1 Under the assumptions (R1) – (R5) and (K1) – (K2), we have

$$\lim_{\sigma\to 0}\int_{-\infty}^{\infty} K(\tau)\left|p^*(\tau|Q^\sigma)-(\frac{I(\theta_0)}{2\pi})^{1/2}\exp(-\frac{1}{2}I(\theta_0)\tau^2)\right| d\tau = 0 \ \text{ a.s. } [P_{\theta_0}].$$

Proof. From Lemma 3.1, we have

$$\lim_{\sigma\to 0}\int_{-\infty}^{\infty} K(\tau)\left|\nu_\sigma(\tau)\pi(\hat{\theta}^\sigma+\sigma\tau)-\pi(\theta_0)\exp(-\frac{1}{2}I(\theta)\tau^2)\right| d\tau = 0 \ \text{ a.s. } [P_{\theta_0}]. \tag{3.3}$$

Substituting $K(\tau) = 1$ which trivially satisfies (K1) and (K2), we have

$$C_\sigma = \int_{-\infty}^{\infty}\nu_n(\tau)\pi(\hat{\theta}^\sigma+\sigma\tau) \to \pi(\theta_0)\int_{-\infty}^{\infty}\exp(-\frac{1}{2}I(\theta)\tau^2)d\tau \ \text{ a.s. } [P_{\theta_0}]. \tag{3.4}$$

Therefore, by (3.3) and (3.4), we have

$$\begin{aligned} &\int_{-\infty}^{\infty} K(\tau)\left|p^*(\tau|Q^\sigma)-(\frac{\beta}{2\pi})^{1/2}\exp(-\frac{1}{2}\tau^2)\right| d\tau \\ \leq\ & \int_{-\infty}^{\infty} K(\tau)\left|C_\sigma^{-1}\nu_n(\tau)\pi(\hat{\theta}^\sigma+\sigma\tau)-C_\sigma^{-1}\pi(\theta_0)\exp(-\frac{1}{2}I(\theta_0)\tau^2)\right| d\tau \\ & +\int_{-\infty}^{\infty} K(\tau)\left|C_\sigma^{-1}\pi(\theta_0)\exp(-\frac{1}{2}\tau^2)-(\frac{I(\theta_0)}{2\pi})^{1/2}\exp(-\frac{1}{2}I(\theta_0)\tau^2)\right| d\tau \\ \longrightarrow\ & 0 \ \text{ a.s. } [P_{\theta_0}] \text{ as } \sigma\to 0. \end{aligned}$$

Theorem 3.2 Suppose (R1)-(R5) and $\int_{-\infty}^{\infty}|\theta|^r\pi(\theta)d\theta < \infty$ for some non-negative integer r hold. Then

$$\lim_{\sigma\to 0}\int_{-\infty}^{\infty} |\tau|^r\left|p^*(\tau|Q^\sigma)-(\frac{I(\theta_0)}{2\pi})^{1/2}\exp(-\frac{1}{2}I(\theta_0)\tau^2)\right| d\tau = 0 \ \text{ a.s. } [P_{\theta_0}].$$

Proof. For $r = 0$, the verification of (K1) and (K2) is easy and the theorem follows from Theorem 3.1. Suppose $r \geq 1$. Let $K(\tau) = |\tau|^r, \delta > 0$ and $\sigma > 0$. Using $|a+b|^r \leq 2^{r-1}(|a|^r+|b|^r)$, we have

$$\begin{aligned} & e^{-\lambda\sigma^{-2}} \int_{|\tau|>\delta} K(\tau\sigma^{-1})\pi(\hat{\theta}^{\sigma}+\tau)d\tau \\ \leq \quad & \sigma^{-r/2} e^{-\lambda\sigma^{-1}} \int_{|\tau-\hat{\theta}^{\sigma}|>\delta} \pi(\tau)|\tau-\hat{\theta}^{\sigma}|^{r} d\tau \\ \leq \quad & 2^{r-1}\sigma^{-r}e^{-\lambda\sigma^{-2}}[\int_{|\tau-\hat{\theta}^{\sigma}|>\delta} \pi(\tau)|\tau|^{r}d\tau + \int_{|\tau-\hat{\theta}^{\sigma}|>\delta} \pi(\tau)|\hat{\theta}^{\sigma}|^{r}d\tau] \\ \leq \quad & 2^{r-1}\sigma^{-r}e^{-\lambda\sigma^{-1}}[\int_{-\infty}^{\infty} \pi(\tau)|\tau|^{r}d\tau + |\hat{\theta}^{\sigma}|^{r}] \\ \longrightarrow \quad & 0 \text{ a.s. } [P_{\theta_0}] \text{ as } \sigma \to 0 \end{aligned}$$

from the strong consistency of $\hat{\theta}^{\sigma}$ (see Huebner [13]) and hypothesis of the theorem. Thus the theorem follows from Theorem 3.1. □

Results similar to Theorems 3.1 and 3.2 hold when the posterior density is replaced by the quqasi-posterior density, the MLE by the AMLE and the Fisher information by $I_n(\theta_0)$.

Theorem 3.3 Under the assumptions (R1) – (R5) and (K1) – (K2), we have

$$\lim_{\sigma\to 0}\int_{-\infty}^{\infty} K(\tau)\left| q^*(\phi|Q^{n,\sigma}) - (\frac{I_n(\theta_0)}{2\pi})^{1/2}\exp(-\frac{1}{2}I_n(\theta_0)\tau^2)\right| d\tau = 0 \ \text{ a.s. } [P_{\theta_0}].$$

Theorem 3.4 Suppose (R1)–(R5) and $\int_{-\infty}^{\infty}|\theta|^r\pi(\theta)d\theta < \infty$ for some non-negative integer r hold. Then

$$\lim_{\sigma\to 0}\int_{-\infty}^{\infty} |\phi|^r\left| q^*(\phi|Q^{n,\sigma}) - (\frac{I_n(\theta_0)}{2\pi})^{1/2}\exp(-\frac{1}{2}I_n(\theta_0)\tau^2)\right| d\phi = 0 \ \text{ a.s. } [P_{\theta_0}].$$

Remark 3.1 For $r = 0$ in Theorem 3.2, we have

$$\lim_{\sigma\to 0}\int_{-\infty}^{\infty} \left| p^*(\tau|Q^{\sigma}) - (\frac{I(\theta_0)}{2\pi})^{1/2}\exp(-\frac{1}{2}I(\theta_0)\tau^2)\right| d\tau = 0 \ \text{ a.s. } [P_{\theta_0}].$$

For $r = 0$ in Theorem 3.4, we have

$$\lim_{\sigma\to 0}\int_{-\infty}^{\infty} \left| q^*(\tau|Q^{\sigma}) - (\frac{I_n(\theta_0)}{2\pi})^{1/2}\exp(-\frac{1}{2}I_n(\theta_0)\tau^2)\right| d\tau = 0 \ \text{ a.s. } [P_{\theta_0}].$$

These are the classical forms of Bernstein-von Mises theorem for parabolic fSPDEs in its simplest form.

As a special case of Theorem 3.2, we obtain for all $r \geq 1$,

$$E_{\theta_0}[\sigma^{-1}(\hat{\theta}^{\sigma} - \theta_0)]^r \to E[\xi^r]$$

as $\sigma \to 0$ where $\xi \sim \mathcal{N}(0, I(\theta_0))$.

As a special case of Theorem 3.4, we obtain for all $r \geq 1$,

$$E_{\theta_0}[\sigma^{-1}(\hat{\theta}^{n,\sigma} - \theta_0)]^r \to E[\zeta^r]$$

as $\sigma \to 0$ where $\zeta \sim \mathcal{N}(0, I_n(\theta_0))$.

4. Bayes Estimation

As an application of Theorem 3.1, we obtain the asymptotic properties of a regular Bayes estimator of θ. Suppose $l(\theta,\phi)$ is a loss function defined on $\Theta\times\Theta$. Assume that $l(\theta,\phi)=l(|\theta-\phi|)\ge 0$ and $l(\cdot)$ is non decreasing. Suppose that J is a non-negative function on $\mathbb{R}^+$ and $K(\cdot)$ and $G(\cdot)$ are functions on $\mathbb{R}$ such that

(B1) $J(\sigma)l(\tau\sigma)\le G(\tau)$ for all $\sigma>0$,

(B2) $J(\sigma)l(\tau\sigma)\to K(\tau)$ as $\sigma\to 0$ uniformly on bounded subsets of $\mathbb{R}$.

(B3) $\int_{-\infty}^{\infty}K(\tau+s)\exp\{-\frac{1}{2}\tau^2\}d\tau$ has a strict minimum at $s=0$.

(B4) $G(\cdot)$ satisfies (K1) and (K2).

Let

$$B_\sigma(\psi):=\int_\Theta l(\theta,\psi)p(\theta|Q^\sigma)d\theta.$$

A regular Bayes estimator $\tilde{\theta}^\sigma$ based on Q^σ is defined as

$$\tilde{\theta}^\sigma:=\arg\inf_{\psi\in\Theta}B_\sigma(\psi).$$

Assume that such an estimator exists.

Further assume that $\tilde{J}$ is a non-negative function on $\mathbb{N}\times\mathbb{R}^+$ and $K(\cdot)$ and $G(\cdot)$ are functions on $\mathbb{R}$ such that

(M1) $\tilde{J}(n,\sigma)l(\tau\sigma)\le G(\tau)$ for all n and $\sigma>0$,

(M2) $\tilde{J}(n,\sigma)l(\tau\sigma)\to K(\tau)$ as $\sigma\to 0$ uniformly on bounded subsets of $\mathbb{R}$.

(M3) $\int_{-\infty}^{\infty}K(\tau+s)\exp\{-\frac{1}{2}\tau^2\}d\tau$ has a strict minimum at $s=0$.

(M4) $G(\cdot)$ satisfies (K1) and (K2).

Let

$$M_{n,\sigma}(\psi)=\int_\Theta l(\theta,\psi)q(\theta|Q^{n,\sigma})d\theta.$$

A quasi-Bayes estimator $\tilde{\theta}^{n,\sigma}$ based on $Q^{n,\sigma}$ is defined as

$$\tilde{\theta}^{n,\sigma}:=\arg\inf_{\psi\in\Theta}M_{n,\sigma}(\psi).$$

Assume that such an estimator exists.

The following Theorem shows that MLE and Bayes estimators are asymptotically equivalent as $\sigma\to 0$.

Theorem 4.1 Assume that (R1) – (R5), (K1) – (K2) and (B1) – (B4) hold. Then we have

(i) $\sigma^{-1}(\tilde{\theta}^\sigma-\hat{\theta}^\sigma)\to 0$ a.s. $[P_{\theta_0}]$ as $\sigma\to 0$,

(ii) $$\lim_{\sigma\to 0}J(\sigma)B_\sigma(\tilde{\theta}^\sigma)=\lim_{\sigma\to 0}J(\sigma)B_\sigma(\hat{\theta}^\sigma)=(\frac{1}{2\pi})^{1/2}\int_{-\infty}^{\infty}K(\tau)\exp(-\frac{1}{2}I^{-1}(\theta_0)\tau^2)d\tau \quad \text{a.s. } [P_{\theta_0}].$$

Proof. The proof is analogous to Theorem 4.1 in Borwanker *et al.* [8]. We omit the details.

Corollary 4.1 Under the assumptions of Theorem 4.1, we have

(i) $\tilde{\theta}^\sigma\to\theta_0$ a.s. $[P_{\theta_0}]$ as $\sigma\to 0$.

(ii) $\sigma^{-1}(\tilde{\theta}^\sigma-\theta_0)\xrightarrow{\mathcal{L}}\mathcal{N}(0,I^{-1}(\theta_0))$ as $\sigma\to 0$.

Proof. (i) and (ii) follow easily by combining Theorem 4.1 and the strong consistency and asymptotic normality results of the MLE in Huebner [13]. □

Theorem 4.2 Under the assumptions of Theorem 4.1, we have

$$\lim_{\delta\to\infty}\lim_{\sigma\to 0}\sup_{|\theta-\theta_0|<\delta} E\omega\left(\sigma^{-1}(\tilde{\theta}^{\sigma}-\theta_0)\right) = E\omega(\xi), \quad \mathcal{L}(\xi) = \mathcal{N}(0, I^{-1}(\theta_0)),$$

where $\omega(\cdot)$ is a loss function as defined at the end of Section 2.

Proof. The Theorem follows from Theorem III.2.1 in Ibragimov-Has'minskii [16] since here conditions (N1) – (N4) of the said theorem are satisfied using Lemmas 3.1 – 3.3 and local asymptotic normality (LAN) property. □

The following theorem shows that the AMLE and quasi-Bayes estimators are asymptotically equivalent.

Theorem 4.3 Assume that (R1) – (R5), (K1) – (K2) and (M1) – (M4) hold. Then we have
(i) $\sigma^{-1}(\tilde{\theta}^{n,\sigma}-\hat{\theta}^{n,\sigma}) \to 0$ a.s. $[P_{\theta_0}]$ as $\sigma \to 0$,

(ii) $\lim_{\sigma\to 0} R(n,\sigma)M_{n,\sigma}(\tilde{\theta}^{n,\sigma}) = \lim_{\sigma\to 0} R(n,\sigma)M_{n,\sigma}(\hat{\theta}^{n,\sigma})$

$$= (\frac{1}{2\pi})^{1/2}\int_{-\infty}^{\infty} K(\phi)\exp(-\frac{1}{2}I_n^{-1}(\theta_0)\phi^2)d\phi \quad \text{a.s. } [P_{\theta_0}].$$

Corollary 4.2 Under the assumptions of Theorem 4.3, we have
(i) $\tilde{\theta}^{n,\sigma} \to \theta_0$ a.s. $[P_{\theta_0}]$ as $\sigma \to 0$.
(ii) $\sigma^{-1}(\tilde{\theta}^{n,\sigma}-\theta_0)\xrightarrow{\mathcal{L}}\mathcal{N}(0, I_n^{-1}(\theta_0))$ as $\sigma \to 0$.

Theorem 4.4 Under the assumptions of Theorem 4.3, we have

$$\lim_{\delta\to\infty}\lim_{\sigma\to 0}\sup_{|\theta-\theta_0|<\delta} E\omega\left(\sigma^{-1}(\tilde{\theta}^{n,\sigma}-\theta_0)\right) = E\omega(\zeta), \quad \mathcal{L}(\zeta) = \mathcal{N}(0, I_n^{-1}(\theta_0)),$$

where $\omega(\cdot)$ is a loss function as defined at the end of Section 2.

5. Example

Here we give an example where the conditions of the previous theorems are satisfied. Consider the parabolic SPDE

$$du^{\sigma}(t,x) = \theta u^{\sigma}(t,x) + \frac{\partial^2}{\partial x^2}u^{\sigma}(t,x)dt + \sigma dW^H(t,x),\ 0 \le t \le T,\ x \in [0,1] \tag{5.1}$$

$$u(0,x) = u_0(x) \in L_2([0,1]) \tag{5.2}$$

$$u^{\sigma}(t,0) = u^{\sigma}(t,1) \tag{5.3}$$

Here $A_0 = \frac{\partial^2}{\partial x^2}$, $A_1 = I$. Thus $m_1 = ord(A_1) = ord(I) = 0$, $m_0 = ord(A_0) = ord(\frac{\partial^2}{\partial x^2}) = 2$. Recall that $2m = ord(A^{\theta}) = \max(m_1, m_0)$. Hence $m = ord(\frac{\partial^2}{\partial x^2} + \theta I)/2 = 1$. The dimension of the x-space $d = 1$ since $x \in [0,1]$. Hence $m - \frac{d}{2} = 1 - \frac{1}{2} = \frac{1}{2} > 0$. So (R1) is satisfied. Other conditions are trivially satisfied. Thus al the results of the previous sections hold for this case.

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