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ON THE DYNAMICAL BEHAVIORS OF A QUADRATIC DIFFERENCE EQUATION OF ORDER THREE

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Abstract. In this paper provides proof of the existence of periodicity, asymptotic behavior, and boundedness of the following quadratic three order difference equation

$$w_{n+1} = \zeta w_{n-1} + \frac{\eta w_{n-1}^2 + \rho w_{n-1} w_{n-2} + \kappa w_{n-2}^2}{\alpha w_{n-1}^2 + \beta w_{n-1} w_{n-2} + \gamma w_{n-2}^2}, \ n = 0, 1, 2, \ldots,$$

constants $\zeta, \eta, \rho, \kappa, \alpha, \beta$ and γ are positive real numbers and the initial conditions w_{-2}, w_{-1} and w_0 are arbitrary non zero real numbers.

1. Introduction

Difference equations in mathematical models have become increasingly popular in recent years among researchers looking to explain problems in various sciences. Additionally, a variety of nonlinear difference equations can be explored, with rational nonlinear difference equations being one of the most popular. However, there are two main directions for difference equations research: the analysis of solution behavior comes first. In order to better understand the stability of the equilibrium points and the presence of periodic solutions for the nonlinear difference equations, a ton of publications have been published. The second approach is to obtain the solution's expressions if it is feasible since there are insufficient and explicit methods to find the solution of nonlinear difference equation.

Khaliq and Elsayed [24] studied the dynamics behavior and existence of the periodic solution of the difference equation:

$$w_{n+1} = \zeta_1 w_{n-2} + \frac{\zeta_2 w_{n-2}^2}{\beta_1 w_{n-2} + \beta_2 w_{n-5}}.$$

Sadiq and Kalim [32] get the solution behavior of the difference equation:

$$w_{n+1} = \zeta_1 w_{n-9} + \frac{\zeta_2 w_{n-19}^2}{\zeta_3 w_{n-9} + \zeta_4 w_{n-19}}.$$

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Also, the authors in [18] considered the stability, periodicity character of the following third order difference equation

$$w_{n+1} = \zeta w_n + \eta w_{n-1} + \frac{\rho + \kappa w_{n-2}}{\gamma + \beta w_{n-2}}.$$

In [3] Amleh considered some special cases of

$$w_{n+1} = \frac{(\eta w_n + \rho w_n w_{n-1} + \kappa w_{n-1})w_n}{B w_n + C w_n w_{n-1} + D w_{n-1}}.$$

Kostrov et al. [26] analyzed the existence of the boundedness, local and global stability of the following second order recursive equation

$$w_{n+1} = \frac{\eta + \kappa w_{n-1}}{\gamma w_n + \alpha w_n w_{n-1} + w_{n-1}}.$$

The purpose of this research paper is to study the following new rational difference equation

$$w_{n+1} = \zeta w_{n-1} + \frac{\eta w_{n-1}^2 + \rho w_{n-1} w_{n-2} + \kappa w_{n-2}^2}{\alpha w_{n-1}^2 + \beta w_{n-1} w_{n-2} + \gamma w_{n-2}^2}, \ n = 0, 1, 2, \dots, \tag{1}$$

constants $\zeta, \eta, \rho, \kappa, \alpha, \beta$ and γ are positive real numbers andthe initial conditions w_{-2}, w_{-1} and w_0 are arbitrary non zero real numbers.

2. Some Basic Theorems

In this part, we recall some basic theorems that we use in this paper.

Let Z be some interval of real numbers and the function ψ has continuous partial derivatives on Z^{k+1} where $Z^{k+1} = Z \times Z \times \cdots \times Z(k+1-times)$. Then, for initial conditions $w_{-k}, w_{-k+1}, \dots, w_0 \in Z$, it is clear to see that the difference equation

$$w_{n+1} = \psi(w_n, w_{n-1}, \dots, w_{n-k}), \ n = 0, 1, \dots, \tag{2}$$

has a unique solution $\{w_n\}_{n=-k}^{\infty}$.

A point $\bar{w} \in Z$ is called an equilibrium point of Eq.(2) if

$$\bar{w} = \psi(\bar{w}, \bar{w}, \dots, \bar{w}).$$

That is, $w_n = \bar{w}$ for $n \geq 0$ is a solution of Eq.(2), or equivalently, $\bar{w}$ is a fixed point of ψ.

Definition 1.(Stability). (i) The equilibrium point $\bar{w}$ of Eq.(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $w_{-k}, w_{-k+1}, \dots, w_0 \in Z$ with

$$|w_{-k} - \bar{w}| + |w_{-k+1} - \bar{w}| + \cdots + |w_0 - \bar{w}| < \delta,$$

we have

$$|w_n - \bar{w}| < \epsilon \ for \ all \ n \geq -k.$$

(ii) The equilibrium point $\bar{w}$ of Eq.(2) is locally asymptotically stable if $\bar{w}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $w_{-k}, w_{-k+1}, \dots, w_0 \in Z$ with

$$|w_{-k} - \bar{w}| + |w_{-k+1} - \bar{w}| + \cdots + |w_0 - \bar{w}| < \gamma,$$

we have

$$\lim_{n \longrightarrow \infty} w_n = \bar{w}.$$

(iii) The equilibrium point $\bar{w}$ of Eq.(2) is global attractor if for all $w_{-k}, w_{-k+1}, \ldots, w_0 \in Z$, we have

$$\lim_{n\longrightarrow\infty} w_n = \bar{w}.$$

(iv) The equilibrium point $\bar{w}$ of Eq.(2) is globally asymptotically stable if $barw$is locally stable, and $\bar{w}$ is also a global attractor of Eq.(2).
(v) The equilibrium point $\bar{w}$ of Eq.(2) is unstable if $\bar{w}$ is not locally stable. The linearized equation of Eq.(2) about the equilibrium $\bar{w}$ is the linear difference equation

$$s_{n+1} = \Sigma_{i=0}^{k} \frac{\partial\psi(\bar{w}, \bar{w}, \ldots, \bar{w})}{\partial w_{n-i}} s_{n-i}. \tag{3}$$

Now, assume that the characteristic equation associated with Eq.(3) is

$$p(\lambda) = p_0\lambda^k + p_1\lambda^{k-1} + \cdots + p_{k-1}\lambda + p_k = 0.$$

where $p_i = \frac{\partial\psi(\bar{w},\bar{w},\ldots,\bar{w})}{\partial w_{n-i}}$.

Theorem A [15]. Assume that $p_i \in R$,$i = 1, 2 \ldots$ and $k \in \{0, 1, 2, \ldots\}$. Then

$$\Sigma_{i=1}^{k}|p_i| < 1,$$

is a suffcient condition for the asymptotic stability of the difference equation

$$s_{n+k} + p_1 s_{n+k+1} + \cdots + p_k s_n = 0, \ n = 0, 1, \ldots.$$

Theorem B [35]. Let $g : [a, b]^{k+1} \longrightarrow [a, b]$, be a continuous function, where k is a positive integer, and where $[a, b]$ is an interval of real numbers. Consider the difference equation

$$w_{n+1} = g(w_n, w_{n-1}, \ldots, w_{n-k}), \ n = 0, 1, \ldots. \tag{4}$$

Suppose that g satisfies the following conditions.
(1) For each integer i with $1 \leq i \leq k+1$; the function $g(z_1, z_2, \ldots, z_{k+1})$ is weakly monotonic in z_i for fixed $z_1, z_2, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{k+1}$.
(2) If m, M is a solution of the system

$$m = g(m_1, m_2, \ldots, m_{k+1}), \ M = g(M_1, M_2, \ldots, M_{k+1}),$$

then $m = M$, where for each $i = 1, 2, \ldots, k+1$, we set

$$m_i = \{{}^{m,\ if\ g\ is\ non-decreasing\ in\ z_i}_{M,\ if\ g\ is\ non-increasing\ in\ z_i}\}, \ M_i = \{{}^{M,\ if\ g\ is\ non-decreasing\ in\ z_i}_{m,\ if\ g\ is\ non-increasing\ in\ z_i}\}.$$

Then there exists exactly one equilibrium point $\bar{w}$ of Equation (4), and every solution of Equation (4) converges to $\bar{w}$.

3. Linearized Stability of Equation (1)

This section proves that Eq.(1) has a unique equilibrium point which is asymptotically stable under a certain condition. The fixed point of Eq.(1) is given by

$$\bar{w} = \zeta\bar{w} + \frac{\eta\bar{w}^2 + \rho\bar{w}\bar{w} + \kappa\bar{w}^2}{\alpha\bar{w}^2 + \beta\bar{w}\bar{w} + \gamma\bar{w}^2},$$

from which we can obtain the following unique equilibrium point:

$$\bar{w} = \frac{\eta + \rho + \kappa}{(1-\zeta)(\alpha + \beta + \gamma)}.$$

Next, we deifne a function $\psi : (0,\infty)^2 \longrightarrow (0,\infty)$ as follows:

$$\psi(\chi,\nu) = \zeta\chi + \frac{\eta\chi^2 + \rho\chi\nu + \kappa\nu^2}{\alpha\chi^2 + \beta\chi\nu + \gamma\nu^2}. \tag{5}$$

We now turn to find the following partial derivatives:

$$\frac{\partial\psi(\chi,\nu)}{\partial\chi} = \zeta + \frac{(2\eta\chi + \rho\nu)(\alpha\chi^2 + \beta\chi\nu + \gamma\nu^2) - (\eta\chi^2 + \rho\chi\nu + \kappa\nu^2)(2\alpha\chi + \beta\nu)}{(\alpha\chi^2 + \beta\chi\nu + \gamma\nu^2)^2}$$

$$= \zeta + \frac{(\eta\beta - \rho\alpha)\chi^2\nu + 2(\eta\gamma - \kappa\alpha)\chi\nu^2 + (\rho\gamma - \kappa\beta)\nu^3}{(\alpha\chi^2 + \beta\chi\nu + \gamma\nu^2)^2}.$$

$$\frac{\partial\psi(\chi,\nu)}{\partial\nu} = \frac{(\rho\chi + 2\kappa\nu)(\alpha\chi^2 + \beta\chi\nu + \gamma\nu^2) - (\eta\chi^2 + \rho\chi\nu + \kappa\nu^2)(\beta\chi + 2\gamma\nu)}{(\alpha\chi^2 + \beta\chi\nu + \gamma\nu^2)^2}$$

$$= \frac{2(\kappa\alpha - \eta\gamma)\chi^2\nu + 2(\kappa\beta - \rho\gamma)\chi\nu^2 + (\rho\alpha - \eta\beta)\chi^3}{(\alpha\chi^2 + \beta\chi\nu + \gamma\nu^2)^2}.$$

Next, evaluating these partial derivatives at the fixed point gives

$$\frac{\partial\psi(\bar{w},\bar{w})}{\partial\chi} = \zeta + \frac{(\eta\beta - \rho\alpha)\bar{w}^3 + 2(\eta\gamma - \kappa\alpha)\bar{w}^3 + (\rho\gamma - \kappa\beta)\bar{w}^3}{(\alpha\bar{w}^2 + \beta\bar{w} + \gamma\bar{w}^2)^2}$$

$$= \zeta + \frac{((\eta - \kappa)\beta - (2\kappa + \rho)\alpha + (2\eta + \rho)\gamma)(1-\zeta)}{(\alpha + \beta + \gamma)(\eta + \rho + \kappa)} = -p_1,$$

$$\frac{\partial\psi(\bar{w},\bar{w})}{\partial\nu} = \frac{2(\kappa\alpha - \eta\gamma)\bar{w}^3 + 2(\kappa\beta - \rho\gamma)\bar{w}^3 + (\rho\alpha - \eta\beta)\bar{w}^3}{(\alpha\bar{w}^2 + \beta\bar{w}^2 + \gamma\bar{w}^2)^2}$$

$$= \zeta + \frac{((\kappa - \eta)\beta + (2\kappa + \rho)\alpha - (2\eta + \rho)\gamma)(1-\zeta)}{(\alpha + \beta + \gamma)(\eta + \rho + \kappa)} = -p_2.$$

The corresponding linearized difference equation of Eq.(1) about the equilibrium point is given by

$$s_{n+1} + p_1 s_n + p_2 s_{n-1} = 0.$$

Theorem 1. Suppose that

$$2|E| < (\alpha + \beta + \gamma)(\eta + \rho + \kappa),\ \zeta < 1,$$

where

$$E = (\eta - \kappa)\beta - (2\kappa + \rho)\alpha + (2\eta + \rho)\gamma.$$

Then, the equilibrium point of Eq.(1) is locally asymptotically stable.

Proof. As stated in Theorem A the fixed point of Eq.(1) is asymptotically stable if

$$|p_1| + |p_2| < 1.$$

This can be written as

$$|\zeta + \frac{E(1-\zeta)}{(\alpha + \beta + \gamma)(\eta + \rho + \kappa)}| + | - \frac{E(1-\zeta)}{(\alpha + \beta + \gamma)(\eta + \rho + \kappa)}| < 1,$$

$$|\zeta(\alpha+\beta+\gamma)(\eta+\rho+\kappa)+E(1-\zeta+E(1-\zeta)|<(\alpha+\beta+\gamma)(\eta+\rho+\kappa).$$

Thus,

$$2|E(1-\zeta)|<(1-\zeta)(\alpha+\beta+\gamma)(\eta+\rho+\kappa).$$

If $\zeta<1$, we have

$$2|E|<(\alpha+\beta+\gamma)(\eta+\rho+\kappa).$$

The proof is complete.

4. Global Attractivity Results

In this section, we will study the global stability of the equilibrium point.

Theorem 2. Let Eq.(5) be increasing in the first and the second variable. Then, the fixed point of Eq.(1) is a global attractor if $\zeta \neq 1$.

Proof. Assume that Eq.(5) is increasing in the first and the second variable, and let (m, M) be a solution of the following system:

$$m=\psi(m,m)=\zeta m+\frac{\eta m^2+\rho m^2+\kappa m^2}{\alpha m^2+\beta m^2+\gamma m^2},$$

$$M=\psi(M,M)=\zeta M+\frac{\eta M^2+\rho M^2+\kappa M^2}{\alpha M^2+\beta M^2+\gamma M^2}.$$

Simplifying this gives

$$m(\alpha+\beta+\gamma)=am(\alpha+\beta+\gamma)\eta+\rho+\kappa, \tag{6}$$

$$M(\alpha+\beta+\gamma)=aM(\alpha+\beta+\gamma)\eta+\rho+\kappa. \tag{7}$$

Subtracting Eq.(7) from Eq.(6) yields

$$(1-\zeta)(m-M)=0.$$

If $\zeta\neq 1$, we have

$$m=M.$$

As claimed by Theorem B, the equilibrium point of Eq.(1) is a global attractor.

Theorem 3. Let Eq.(5) be decreasing in the first and the second variable. Then, the equilibrium point of Eq.(1) is a global attractor.

Proof. The proof is similar to the previous one and it will be omitted.

Theorem 4. Let Eq.(5) be increasing in the first variable and decreasing in the second variable. Then, the fixed point of Eq.(1) is a global attractor if $\zeta<1, \gamma<\alpha+\beta$ and $\eta<\kappa$.

Proof. Let Eq.(5) be increasing in χ and decreasing in ν, and assume that (m, M) is a solution of the system

$$m = \psi(m, M) = \zeta m + \frac{\eta m^2 + \rho m M + \kappa M^2}{\alpha m^2 + \beta m M + \gamma M^2},$$

$$M = \psi(M, m) = \zeta M + \frac{\eta M^2 + \rho M m + \kappa m^2}{\alpha M^2 + \beta M m + \gamma m^2},$$

from which we obtain

$$\alpha m^3 + \beta m^2 M + \gamma m M^2 = \zeta\alpha m^3 + \zeta\beta m^2 M + \zeta\gamma m M^2 + bm^2 + \rho M m + \kappa M^2, \tag{8}$$

$$\alpha M^3 + \beta M^2 m + \gamma M m^2 = \zeta\alpha M^3 + \zeta\beta M^2 m + \zeta\gamma M m^2 + bM^2 + \rho M m + \kappa m^2. \tag{9}$$

Subtracting Eq.(9) from Eq.(8) and simplifying the result give

$$(m - M)\{(1 - \zeta)[\alpha(m^2 + M^2) + (\alpha + \beta - \gamma)mM] + (\kappa - \eta)(m + M)\} = 0.$$

Hence, if $\zeta < 1$, $\gamma < \alpha + \beta$ and $\eta < \kappa$, then

$$m = M.$$

The equilibrium point of Eq. (1) is a global attractor, as by Theorem B.

Theorem 5. Let Eq.(5) be decreasing in the first variable and increasing in the second variable. Then, the equilibrium point of Eq.(1) is a global attractor if $\zeta < 1$,$\alpha < \gamma$ and $\kappa < \eta$.

Proof. Suppose that Eq.(5) is decreasing in χ and increasing in ν, and let (m, M) be a solution of the following system:

$$m = \psi(M, m) = \zeta M + \frac{\eta M^2 + \rho M m + \kappa m^2}{\alpha M^2 + \beta M m + \gamma m^2},$$

$$M = \psi(m, M) = \zeta m + \frac{\eta m^2 + \rho m M + \kappa M^2}{\alpha m^2 + \beta m M + \gamma M^2},$$

which can be written as

$$\alpha m M^2 + \beta M m^2 + \gamma m^3 = \zeta\alpha M^3 + \zeta\beta M^2 m + \zeta\gamma m^2 M + bM^2 + \rho M m + \kappa m^2, \tag{10}$$

$$\alpha m^2 M + \beta m M^2 + \gamma M^3 = \zeta\alpha m^3 + \zeta\beta m^2 M + \zeta\gamma m M^2 + bm^2 + \rho m M + \kappa M^2. \tag{11}$$

Subtract Eq.(11) from Eq.(10) and simplify the result to have

$$(m - M)\{[\beta - \alpha + \zeta(\beta - \gamma) + \gamma]mM + (\gamma + \zeta\alpha)(m^2 + M^2) + (\eta - \kappa)(m + M)\} = 0.$$

Or,

$$(m - M)\{[\beta(1 + \zeta) - (1 - \zeta)(\gamma - \alpha)]mM + (\gamma + \zeta\alpha)(m^2 + M^2) + (\eta - \kappa)(m + M)\} = 0.$$

Hence, if $\zeta < 1$, $\alpha < \gamma$ and $\kappa < \eta$, then

$$m = M.$$

The fixed point in Eq. (1) is a global attractor as a result, according to Theorem B.

5. Periodic Solution

In this section, we will present a principal theorem that proves the existence of periodic two solutions of Eq.(1).

Theorem 6. Eq.(1) has positive prime periodic two solutions if

$$\frac{\eta + \rho r + \kappa r^2}{(r-\zeta)(\alpha + \beta r + \gamma r^2)} = \frac{\eta r^2 + \rho r + \kappa}{(1-\zeta r)(\alpha r^2 + \beta r + \gamma)}, \ r \neq 0, \pm 1, \ r \in R.$$

Proof. . Assume that there exist a prime period two solution

$$\ldots p, q, p, q, \ldots,$$

of Eq.(1). Then, it can be observed from Eq.(1) that

$$p = \zeta q + \frac{\eta q^2 + \rho qp + \kappa p^2}{\alpha q^2 + \beta qp + \gamma p^2},$$

$$q = \zeta p + \frac{\eta p^2 + \rho pq + \kappa q^2}{\alpha p^2 + \beta pq + \gamma q^2}.$$

Clearing the denominator gives

$$\alpha pq^2 + \beta p^2 q + \gamma p^3 = \zeta\alpha q^3 + \zeta\beta q^2 p + \zeta\gamma p^2 q + \eta q^2 + \rho pq + \kappa p^2, \tag{12}$$

$$\alpha p^2 q + \beta pq^2 + \gamma q^3 = \zeta\alpha p^3 + \zeta\beta p^2 q + \zeta\gamma pq^2 + \eta p^2 + \rho pq + \kappa q^2, \tag{13}$$

Dividing Eq.(12) and Eq.(13) by $p^3$and q^3, respectively, yields

$$\alpha(\frac{q}{p})^2 + \beta(\frac{q}{p}) + \gamma = \zeta\alpha(\frac{q}{p})^3 + \zeta\beta(\frac{q}{p})^2 + \zeta\gamma(\frac{q}{p}) + \eta(\frac{q^2}{p^3}) + \rho(\frac{q}{p^2}) + \frac{\kappa}{p}, \tag{14}$$

$$\alpha(\frac{p}{q})^2 + \beta(\frac{p}{q}) + \gamma = \zeta\alpha(\frac{p}{q})^3 + \zeta\beta(\frac{p}{q})^2 + \zeta\gamma(\frac{p}{q}) + \eta(\frac{p^2}{q^3}) + \rho(\frac{p}{q^2}) + \frac{\kappa}{q}. \tag{15}$$

Now, we suppose that $p = rq, r \pm 0, \pm 1, r \in$ R. Then, Eq.(14) and Eq.(15)become

$$\frac{\alpha}{r^2} + \frac{\beta}{r} + \gamma = \frac{\zeta\alpha}{r^3} + \frac{\zeta\beta}{r^2} + \frac{\zeta\gamma}{r} + \frac{\eta}{r^2 p} + \frac{\rho}{rp}) + \frac{\kappa}{p}, \tag{16}$$

$$\alpha r^2 + \beta r + \gamma = \zeta\alpha r^3 + \zeta\beta r^2 + \zeta\gamma r + \frac{\eta r^2}{q} + \frac{\rho r}{q}) + \frac{\kappa}{q}. \tag{17}$$

Multiplying Eq.(16) by r^3 and simplifying yields

$$p = \frac{r(\eta + \rho r + \kappa r^2)}{(r - \zeta)(\alpha + \beta r + \gamma r^2)}. \tag{18}$$

Now, we can obtain from Eq.(17) that

$$q = \frac{\eta r^2 + \rho r + \kappa}{(1 - \zeta r)(\alpha r^2 + \beta r + \gamma r^2)}. \tag{19}$$

Since $p = rq$, it is easy to see from Eq.(18) and Eq.(19) that

$$\frac{r(\eta + \rho r + \kappa r^2)}{(r - \zeta)(\alpha + \beta r + \gamma r^2)} = \frac{r(\eta r^2 + \rho r + \kappa)}{(1 - \zeta r)(\alpha r^2 + \beta r + \gamma r^2)}.$$

The proof is complete.

6. Existence of Bounded Solution

In this section, we will study the existence of the boundedness of Eq.(1)

Theorem 7. Every solution of Eq.(1) is bounded if $\zeta < 1$.
Proof. Let $\{w_n\}_{n=-2}^{\infty}$ be a solution of Eq.(1). Then, it follows from Eq.(1) that

$$\begin{aligned}
w_{n+1} &= \zeta w_{n-1} + \frac{\eta w_{n-1}^2 + \rho w_{n-1}w_{n-2} + \kappa w_{n-2}^2}{\alpha w_{n-1}^2 + \beta w_{n-1}w_{n-2} + \gamma w_{n-2}^2} \\
&= \zeta w_{n-1} + \frac{\eta w_{n-1}^2}{\alpha w_{n-1}^2 + \beta w_{n-1}w_{n-2} + \gamma w_{n-2}^2} + \frac{\rho w_{n-1}w_{n-2}}{\alpha w_{n-1}^2 + \beta w_{n-1}w_{n-2} + \gamma w_{n-2}^2} \\
&+ \frac{\kappa w_{n-2}^2}{\alpha w_{n-1}^2 + \beta w_{n-1}w_{n-2} + \gamma w_{n-2}^2} \\
&\leq \zeta w_{n-1} + \frac{\eta}{\alpha} + \frac{\rho}{\beta} + \frac{\kappa}{\gamma}.
\end{aligned}$$

By using comparison, we have

$$s_{n+1} = \zeta s_{n-1} + \frac{\eta}{\alpha} + \frac{\rho}{\beta} + \frac{\kappa}{\gamma}.$$

This difference equation has the following solution:

$$s_{n+1} = \zeta^n s_0 + constant,$$

which is asymptotically stable if $\zeta < 1$; and converges to the equilibrium point

$$\bar{s} = \frac{\eta\beta\gamma + \rho\alpha\gamma + \kappa\alpha\beta}{\alpha\beta\gamma(1 - \zeta)}.$$

Therefore,

$$\lim_{n \longrightarrow \infty} \sup w_n \leq \frac{\eta\beta\gamma + \rho\alpha\gamma + \kappa\alpha\beta}{\alpha\beta\gamma(1 - \zeta)}.$$

Theorem 8. Every solution of Eq.(1) is unbounded if $\zeta > 1$.
Proof. Let $\{w_n\}_{n=-2}^{\infty}$ be a solution of Eq.(1). Then, it follows from Eq.(1) that

$$w_{n+1} = \zeta w_{n-1} + \frac{\eta w_{n-1}^2 + \rho w_{n-1}w_{n-2} + \kappa w_{n-2}^2}{\alpha w_{n-1}^2 + \beta w_{n-1}w_{n-2} + \gamma w_{n-2}^2} > \zeta w_{n-1}, \ \ for\ all\ n \geq 1.$$

Hence, the right hand side can be written as follows $s_{n+1} = as_n$, which has the following solution

$$s_n = \zeta^n s_0 + constant.$$

Since $\zeta > 1$; $\lim_{n\longrightarrow\infty} s_n = \infty$. Then, by using ratio test $\{w_n\}_{n=-2}^{\infty}$ is unbounded from above.

7. Numerical Examples

This section aims to validate our theoretical work from the earlier sections.

Example 1. This example demonstrates how the behavior of the solution in Eq.(1) when we assume that $\zeta = 0.2,\ \eta = 1,\ \rho = 0.3,\ \kappa = 1,\ \alpha = 4,\ \beta = 2.8,\ \gamma = 5.1,\ w_{-2} = 0.1,\ w_{-1} = 0.4$ and $w_0 = 0.56$. See Figure 1.

Example 2. In figure 2 illustrates the stability of the solution to Equation (1) when we take the supposition that $\zeta = 0.2,\ \eta = 0.13,\ \rho = 0.4,\ \kappa = 0.1,\ \alpha = 1,\ \beta = 2,\ \gamma = 3,\ w_{-2} = 0.1,\ w_{-1} = 0.31$ and $w_0 = 0.2$.

Example 3. Figure 3 We present that Eq. (1) is unbounded if we let $\zeta = 4,\ \eta = 1,\ \rho = 0.3,\ \kappa = 5,\ \alpha = 8,\ \beta = 4,\ \gamma = 2,\ w_{-2} = 8,\ w_{-1} = 5$ and $w_0 = 2$.

Example 4. This illustration shows how Eq is periodic (1)if we assume that $\zeta = 0.2111,\ \eta = 5.5,\ \rho = 0.3,\ \kappa = 1,\ \alpha = 3,\ \beta = 2,\ \gamma = 5.2,\ r = 2,$ and let us assume that the initial conditions are $w_{-2} = p,\ w_{-1} = q$ Eqs. (19) and (18) describe this, respectively. The behavior of the Eq. (1) solution is then seen in Figure 4.

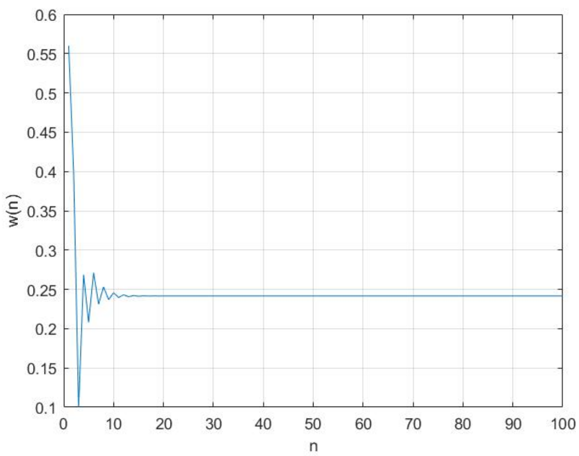

Figure 1.

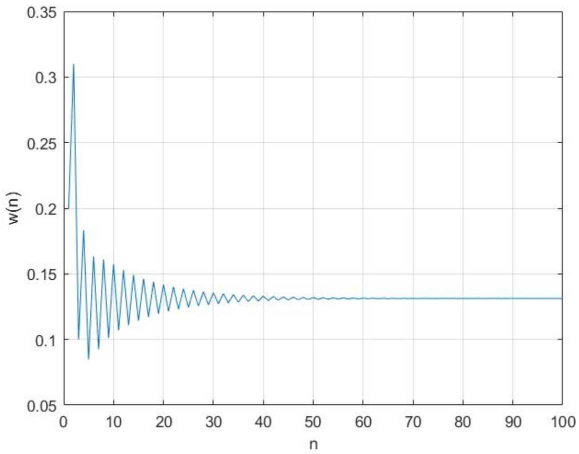

Figure 2.

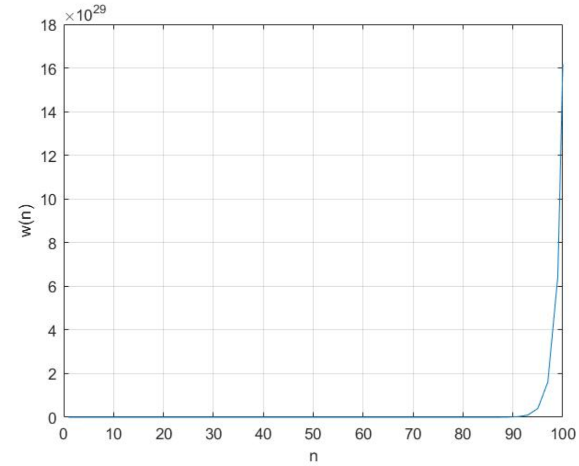

Figure 3.

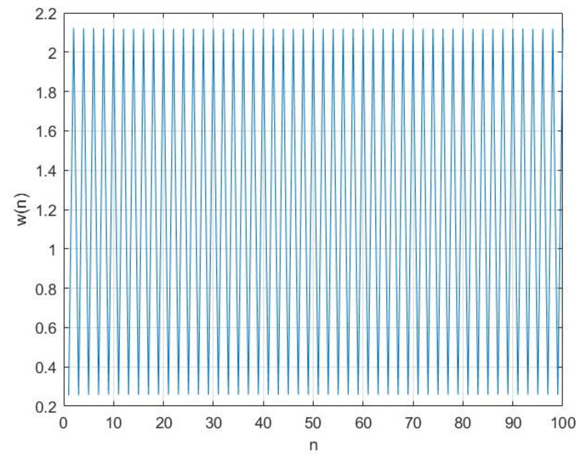

Figure 4.

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