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ON THE NULLITY OF SOME FAMILIES OF R-PARTITE GRAPHS

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ABSTRACT. The nullity of a graph G, denoted by $\eta(G)$ is defined to be the multiplicity of the eigenvalue zero in the spectrum of a graph. The spectrum of a graph G is a two-row matrix, the first row elements are the distinct eigenvalues of its adjacency matrix $A(G)$ and the second row elements are its corresponding multiplicities. Furthermore, the rank of G, denoted by $rank(G)$ is also the rank of $A(G)$, that is $rank(G) = rank(A(G))$. In addition, given that G is of order n, it is known that $\eta(G) = n - rank(G)$. Thus, any result about rank can be stated in terms of nullity and vice versa. In this paper, we investigate some families of r-partite graphs of order n and we determine the nullity of these r-partite families using its rank. First, we consider the complete r-partite graphs denoted by $K_{n_1,n_2,n_3,...,n_r}$ where $n = n_1+n_2+n_3+...+n_r$ and $r \geq 4$. Second, we also consider a family of r-partite graphs where $n \geq 2r - 1$ and $r \geq 4$, which is an extension of a family of tripartite graphs introduced in the paper *"On the nullity of a family of tripartite graphs"* by Farooq, Malik, Pirzada and Naureen.

1. INTRODUCTION

Let $G = (V, E)$ be a simple graph where $V = \{v_1, v_2, ..., v_n\}$ is a finite set of vertices and E is a finite set of edges. The order of graph G is the number of its vertices denoted by n while the size of graph G is the number of its edges denoted by m. Throughout this paper, the order of G is n.

A square matrix that is used to represent a graph G is called its adjacency matrix. The adjacency matrix $A(G)$ of G of order n is the $n \times n$ symmetric matrix $[a_{ij}]$ such that $a_{ij} = 1$ if v_i and v_j are adjacent and 0 otherwise, for any pair $v_i, v_j \in V$. The main concern of this study is on multiplicity of one of the eigenvalues of the adjacency matrix of G. The nullity of a graph G, denoted by $\eta(G)$ is defined to be the multiplicity of the eigenvalue zero in the spectrum of a graph. The spectrum of a graph G is a two-row matrix, the first row elements are the distinct eigenvalues of its adjacency matrix $A(G)$ and the second row elements are its corresponding multiplicities. Moreover, the rank of G, denoted by $rank(G)$ is also the rank of $A(G)$, that is $rank(G) = rank(A(G))$. Recall that the rank of $A(G)$ is defined as the maximum number of linearly independent row/column vectors in $A(G)$. In addition, it is known that $\eta(G) = n - rank(G)$, thus any result about rank can be stated in terms of nullity and vice versa.

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A notion on graphs which is important is the isomorphism of graphs. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. The graph G and G' are said to be isomorphic and we write $G \cong G'$, if there exists a bijection $\phi : V \mapsto V'$ such that $xy \in E \iff \phi(x)\phi(y) \in E'$ for all $x, y \in V$.

Collatz and Sinogowitz [5], first posed the problem of characterizing all graphs which satisfy $\eta(G) > 0$. This question is of great interest in chemistry. As has been shown in [6], for a bipartite graph G corresponding to an alternant hydrocarbon, if $\eta(G) > 0$, then it indicates that the molecule which such a graph represents is unstable. The nullity of a graph is also important in mathematics, since it is related to the singularity of $A(G)$. Ashraf and Bamdad [7] considered the opposite problem where graphs have nullity zero.

Cheng and Liu [2] characterized the extremal graphs attaining the upper bound $n-2$ and the second upper bound $n-3$. They discussed the nullity of a complete bipartite and complete tripartite graphs. Fan and Qian [3] determined the nullity set of bipartite graphs of order n and characterized the bipartite graphs with nullity $n-4$ and the regular bipartite graphs with nullity $n-6$. Farooq et. al [1] obtained the nullity set of a class of n-vertex tripartite graphs and characterized these tripartite graphs with nullity $n-4$ and some tripartite graphs with nullity $n-6$ in this class. Moreover, Farooq et. al also mentioned that the nullity problem in tripartite graphs does not follow as an extension to that of the nullity of bipartite graphs.

In this paper, we considered and investigated some families of r-partite graphs of order n and the nullity of these graphs is going to be determine using its rank. First, the complete r-partite graphs denoted by $K_{n_1,n_2,n_3,...,n_r}$ where $n = n_1 + n_2 + n_3 + ... + n_r$ and $r \geq 4$ was found to have a nullity of $n-r$. Second, the family of r-partite graphs where $n \geq 2r-1$ and $r \geq 4$ with nullity $n - (r - 1 + 2rank(D))$, where D is a matrix defined on Section 3.2. This follow as an extension of family of tripartite graphs and as expansion of some results discussed in *On the nullity of a family of tripartite graphs* by Farooq, Malik, Pirzada and Naureen [1].

2. Preliminaries

Let $G = (V, E)$ be a graph of order n. Consider $S \subseteq V$, where S is nonempty. The neighbor set of S in G, denoted by $N(S)$ is a set containing those vertices of G that are adjacent to some vertex in S. The subgraph of G induced by S, denoted by $G[S]$ is defined to be the graph whose vertex set is S and whose edge set consists of all of the edges in E that have both endpoints in S. For any $v \in V$, the degree of vertex v, denoted by $d(v)$, is defined to be the number of edges incident to v. Now, the union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \cup G_2$, is defined to be the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. In this paper, we consider disjoint union of graphs where the union of vertex sets and the union of edge sets are disjoint.

The graph G is a bipartite if its vertex set can be partitioned into two subsets X and Y such that $G[X]$ and $G[Y]$ are empty graphs and the partition (X, Y) is called a bipartition. A complete bipartite graph is a bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y.

We also consider expanded path and expanded cycle in this study.

Definition 1. *The n-vertex graph G is said to be an expanded path of length k if its vertex set V can be partitioned into $V_1, V_2, ..., V_k$, $k \geq 2$ such that*

(1) $G[V_i]$ *is an empty graph for* $1 \leq i \leq k$*,*
(2) $G[V_i \cup V_{i+1}]$ *is a complete bipartite graph for* $1 \leq i \leq k-1$*,*
(3) $G[V_i \cup V_j]$ *is an empty graph for* $1 \leq i, j \leq k$ *with* $|i-j| > 1$*.*

In addition, the expanded path of length k is denoted by $\mathbb{P}_k(V_1, V_2, ..., V_k)$ or $\mathbb{P}_k$ and each V_i is called an expanded vertex of order $|V_i|$.

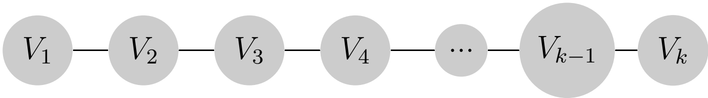


Figure 1: *Expanded Path of length* k

Definition 2. *An expanded cycle of length* k *where* $k \geq 3$*, denoted by* $\mathbb{C}_k(V_1, V_2, ..., V_k)$ *or* $\mathbb{C}_k$ *is obtained from the expanded path* $\mathbb{P}_k$ *by adding edges between each vertex of* V_1 *and each vertex of* V_k*.*

Definition 3. *An expanded decomposition of the graph* G *is a list of expanded subgraphs such that each edge of* G *appears in exactly one expanded subgraph in the list.*

The graph in Figure 2 has expanded decomposition $\mathbb{C}_5$,$\mathbb{C}_3$,$\mathbb{P}_2$.

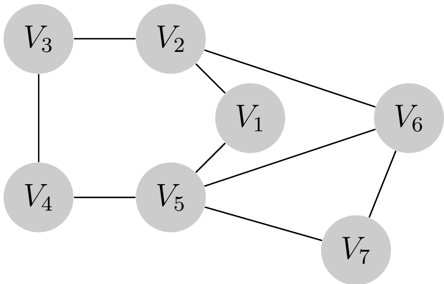


Figure 2: $\mathbb{C}_5(V_1, V_2, V_3, V_4, V_5)$,$\mathbb{C}_3(V_5, V_6, V_7)$ *and* $\mathbb{P}_2(V_2, V_6)$

3. Results and Discussion

In this section, we define two different families of r-partite graphs and determine its rank and nullity.

Remark 1. *All graphs considered in this section are expanded graphs.*

3.1. **Nullity of a complete r-partite graph.** Let $r \geq 2$ be an integer. A r-partite graph is a graph G in which vertex set V is partitioned into r nonempty subsets $P_1, P_2, ..., P_r$ in such a way that no edge joins two vertices in the same partite sets, that is, $G[P_i]$, $i = 1, 2, 3, ..., r$ are empty graphs. A complete r-partite graph denoted by $K_{n_1,n_2,n_3...,n_r}$ is a r-partite graph in which each vertex of P_i joined to each vertex of $G - P_i$ where $|P_i| = n_i$, $n = n_1 + n_2 + n_3 + ... + n_r$ and $n_1, n_2, ..., n_r > 0$. For an isolated vertex K_1, we denote by rK_1 the r copies of K_1.

In [2], Cheng and Liu discussed the nullity of a simple graph G such that G is isomorphic to a complete bipartite graph/complete tripartite graph.

The succeeding theorem is about the nullity of a simple graph G that is isomorphic to a complete r-partite graph, where $r \geq 4$.

Theorem 1. *Let G be a simple graph with n vertices and G has no isolated vertices. If G is isomorphic to a complete r-partite graph $K_{n_1,n_2,...,n_r}$ where $n = n_1 + n_2 + ... + n_r$, then $rank(G) = r$ and $\eta(G) = n - r$.*

Proof. Suppose G is isomorphic to a complete r-partite graph, that is $G \cong K_{n_1,n_2,...,n_r}$ and let $P_1, P_2, P_3, P_4, ..., P_r$ be the partite sets of G. Thus, the adjacency matrix $A(G)$ of G is

$$\begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ \vdots \\ P_r \end{array} \begin{array}{c} \begin{array}{cccccc} P_1 & P_2 & P_3 & P_4 & \cdots & P_r \end{array} \\ \begin{bmatrix} 0_{n_1\times n_1} & 1_{n_1\times n_2} & 1_{n_1\times n_3} & 1_{n_1\times n_4} & \cdots & 1_{n_1\times n_r} \\ 1_{n_2\times n_1} & 0_{n_2\times n_2} & 1_{n_2\times n_3} & 1_{n_2\times n_4} & \cdots & 1_{n_2\times n_r} \\ 1_{n_3\times n_1} & 1_{n_3\times n_2} & 0_{n_3\times n_3} & 1_{n_3\times n_4} & \cdots & 1_{n_3\times n_r} \\ 1_{n_4\times n_1} & 1_{n_4\times n_2} & 1_{n_4\times n_3} & 0_{n_4\times n_4} & \cdots & 1_{n_4\times n_r} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1_{n_r\times n_1} & 1_{n_r\times n_2} & 1_{n_r\times n_3} & 1_{n_r\times n_4} & \cdots & 0_{n_r\times n_r} \end{bmatrix} \end{array},$$

observe that the $A(G)$ have r sets of identical rows. Now, multiply -1 to row $(\sum_{i=1}^{j} n_i) + 1$, then add to row $(\sum_{i=1}^{j} n_i) + 2$, $(\sum_{i=1}^{j} n_i) + 3$, ... , $(\sum_{i=1}^{j} n_i) + n_{j+1}$, where $j = 0, 1, ..., r - 1$. By doing this, it will result to a matrix with r non-zero rows and all the other rows are zero rows. Then these r non-zero rows are linearly independent and the proof is straightforward. It follows that $rank(A(G)) = rank(G) = r$. Moreover, it is known that $\eta(G) = n - rank(G)$, therefore $\eta(G) = n - r$. □

3.2. **Nullity of the extension of family of tripartite graphs introduced in *"On the nullity of a family of tripartite graphs"*.** In [1] a special class of tripartite graphs was introduced and its nullity determined. We extend this family to a family of r-partite graphs, where $r \geq 4$. We also established the nullity of these graphs.

3.2.1. *For $r = 4$, 4-partite graphs.* Let $G = (V, E)$ be a graph of order n. Suppose G is a 4-partite with vertex set V partitioned into four subsets P_1,P_2,P_3 and P_4 such that $G[P_i]$, $i = 1, 2, 3, 4$ are empty graphs and the partition $\{P_1, P_2, P_3, P_4\}$ is called a 4-partition. We consider a special class of 4-partite graphs defined as follows. Let $\mathcal{F}_{4_n}$ be the family of those n - vertex 4-partite graphs G, $n \geq 7$, whose 4-partition $\{P_1, P_2, P_3, P_4\}$ satisfies the following:

$$G[P_2 \cup P_3 \cup P_4] \text{ is a complete tripartite.} \tag{1}$$

$$N_{P_2}(P_1') \neq P_2,\ N_{P_3}(P_1') \neq P_3 \text{ and } N_{P_4}(P_1') \neq P_4,\ \forall P_1' \subseteq P_1. \tag{2}$$

Consider $G \in \mathcal{F}_{4_n}$ with 4-partition $\{P_1, P_2, P_3, P_4\}$. Since $G \in \mathcal{F}_{4_n}$, G satisfies property (1) and (2). So, we can define the adjacency matrix $A(G)$ of G as

$$A(G) = \begin{array}{c} \\ P_2 \\ P_3 \\ P_4 \\ P_1 \end{array} \begin{array}{c} \begin{array}{cccc} P_2 & P_3 & P_4 & P_1 \end{array} \\ \begin{bmatrix} 0 & J & C & D_1 \\ J^t & 0 & M & D_2 \\ C^t & M^t & 0 & D_3 \\ D_1^t & D_2^t & D_3^t & 0 \end{bmatrix} \end{array}$$

such that J, C and M denote the matrices with all entries 1 while 0 denote zero matrix. Furthermore, D_1, D_2, D_3 denote the matrices that shows the relationship of P_1 to P_2, P_3, P_4, respectively.
Let B and D be defined as follow,

$$B = \begin{bmatrix} 0 & J & C \\ J^t & 0 & M \\ C^t & M^t & 0 \end{bmatrix}$$

and

$$D = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}$$

Thus, the matrix $A(G)$ can be written as

$$A(G) = \begin{bmatrix} B & D \\ D^t & 0 \end{bmatrix}. \tag{3}$$

Let

$$U = \begin{bmatrix} B & D \end{bmatrix}, \; L = \begin{bmatrix} D^t & 0 \end{bmatrix}. \tag{4}$$

Then $A(G)$ can be written as

$$A(G) = \begin{bmatrix} U \\ L \end{bmatrix}.$$

For vertex $v \in V$, denote by U_v the row of $A(G)$ corresponding to the vertex v if $v \in P_2 \cup P_3 \cup P_4$, and by L_v if $v \in P_1$.
Now, consider $S \subseteq P_2 \cup P_3 \cup P_4$. Thus from the matrix $A(G)$, we have

$$\sum_{v \in S} q_v U_v = [b_1 \; b_2 \; b_3 \; d] \tag{5}$$

where b_1, b_2, b_3 are constant row matrices respectively of dimension $1 \times |P_2|$, $1 \times |P_3|$ and $1 \times |P_4|$, while d is a row vector of dimension $1 \times |P_1|$, and q_v's are real constants. Equivalently, for any $P_1' \subseteq P_1$, we can write

$$\sum_{v \in P_1'} q_v' L_v = [d_1 \; d_2 \; d_3 \; 0] \tag{6}$$

where d_1, d_2, d_3 and 0 are row vectors respectively of dimension $1 \times |P_2|$, $1 \times |P_3|$, $1 \times |P_4|$ and $1 \times |P_1|$, and q_v''s are real constants.

The following results gives information about the rank of a 4-partite graph in $\mathcal{F}_{4_n}$.

Lemma 2. *Let $G \in \mathcal{F}_{4_n}$ with 4-partition $\{P_1, P_2, P_3, P_4\}$ and the adjacency matrix $A(G)$ defined by (3). Then $rank(G) = rank(U) + rank(L)$ where U and L are defined by (4).*

Proof. To prove $rank(G) = rank(U) + rank(L)$, it is enough to show that if $\sum_{v \in S} q_v U_v \neq 0$ and $\sum_{v \in P_1'} q_v' L_v \neq 0$ where q_v's and q_v''s are real constants, then $\sum_{v \in S} q_v U_v \neq \sum_{v \in P_1'} q_v' L_v$.
Let S and P_1' be an arbitrary subsets of $P_2 \cup P_3 \cup P_4$ and P_1, respectively. Now, we have $\sum_{v \in S} q_v U_v = [b_1 \; b_2 \; b_3 \; d]$ and $\sum_{v \in P_1'} q_v' L_v = [d_1 \; d_2 \; d_3 \; 0]$ such that $b_1, b_2, b_3, d, d_1, d_2, d_3,$ and 0 are defined in (5) and (6).

Suppose $\sum_{v\in S} q_v U_v = \sum_{v\in P_1'} q_v' L_v$, then $[b_1\ b_2\ b_3\ d] = [d_1\ d_2\ d_3\ 0]$. By condition (2), there exists a vertex in P_2, a vertex in P_3 and a vertex in P_4 which are not adjacent to any vertex in P_1. This implies that each of D_1, D_2, D_3 has at least one zero row which also implies that there is at least one zero column in each D_1^t, D_2^t, D_3^t. It follows that there are at least three zero columns in D^t corresponding to a vertex in each P_2, P_3 and P_4. Thus, there are zero entries in vectors d_1, d_2 and d_3.
In addition, since $[b_1\ b_2\ b_3\ d] = [d_1\ d_2\ d_3\ 0]$ and as b_1, b_2 and b_3 are constant vectors, thus vectors $b_1, b_2, b_3, d, d_1, d_2, d_3$ are all zero vectors. Therefore, $\sum_{v\in S} q_v U_v = 0$ and $\sum_{v\in P_1'} q_v' L_v = 0$.
This completes the proof. □

Theorem 3. *Let $G \in \mathcal{F}_{4_n}$ with 4-partition $\{P_1, P_2, P_3, P_4\}$ and the adjacency matrix $A(G)$ defined by (3). Then $rank(G) = 3 + 2rank(D)$.*

Proof. Consider $G \in \mathcal{F}_{4_n}$ and let $A(G)$ be the adjacency matrix defined in (3). Now, by similar arguments applied in Lemma 1, then we have $rank(U) = rank(B) + rank(D)$ and $rank(L) = rank(D^t) = rank(D)$. Since matrix B is an adjacency matrix of complete tripartite graph, it follows that $rank(B) = 3$. Thus by Lemma 1, we can get $rank(G) = rank(U) + rank(L) = (rank(B) + rank(D)) + rank(D)$ and it implies that $rank(G) = 3 + 2rank(D)$. □

Corollary 4. *If $G \in \mathcal{F}_{4_n}$ with 4-partition $\{P_1, P_2, P_3, P_4\}$, then $\eta(G) = n - (3 + 2rank(D))$.*

Let $\mathbb{C}_k(\bar{e})$ denote an expanded cycle of length k with an expanded chord $\bar{e}$ joining two non-adjacent expanded vertices of the cycle $\mathbb{C}_k$ such that the expanded vertices joined by $\bar{e}$ form a complete bipartite.

Now, we have the following observations.

Theorem 5. *If G is a graph of order n such that G has expanded decomposition $\mathbb{C}_7(\bar{e}) \cup kK_1$, $2\mathbb{C}_3$, $\mathbb{P}_2$ shown in Figure 3, $k \geq 0$, then $G \in \mathcal{F}_{4_n}$ and $\eta(G) = n - 5$.*

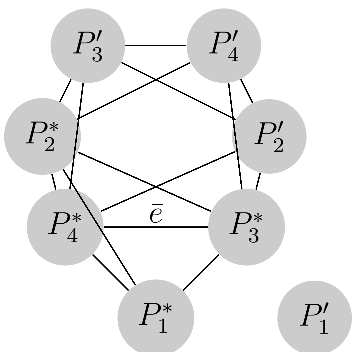


Figure 3

Proof. We need to show that (i) $G \in \mathcal{F}_{4_n}$ and (ii) $\eta(G) = n - 5$.
(i). Suppose $P_1 = P_1^* \cup P_1'$, $P_2 = P_2^* \cup P_2'$, $P_3 = P_3^* \cup P_3'$ and $P_4 = P_4^* \cup P_4'$ where P_1' is possibly empty and $k = |P_1'|$. Thus, we notice that G is a 4-partite graph with 4-partition $\{P_1, P_2, P_3, P_4\}$. Furthermore, G satisfies property (2) since $N_{P_2}(P_1) \neq P_2$, $N_{P_3}(P_1) \neq P_3$ and $N_{P_4}(P_1) \neq P_4$. In addition, see that $G[P_2 \cup P_3 \cup P_4] = \mathbf{C}_3(P_2, P_3, P_4)$ which is a complete

tripartite graph, it follows that G satisfies property (1). Therefore $G \in \mathcal{F}4_n$.

(ii). Since $G \in \mathcal{F}_{4_n}$, it follows from (3) that the adjacency matrix of G is given by

$$A(G) = \begin{array}{c} P_2^* \\ P_2' \\ P_3^* \\ P_3' \\ P_4^* \\ P_4' \\ P_1^* \\ P_1' \end{array} \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

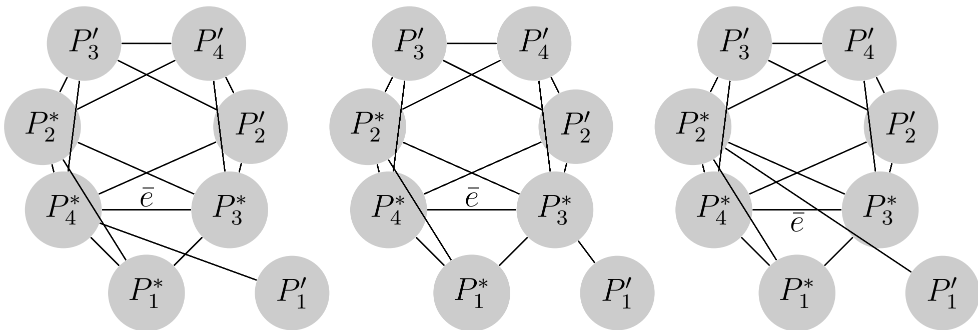
where 1 denotes all-ones matrix and 0 denotes the zero matrix with appropriate sizes. Observe the adjacency matrix of G, the sub matrix D where its columns correspond to P_1^*, P_1' and the sub matrix D^t with rows correspond to P_1^*, P_1'. Then, we see that rows of P_1^* are identical while the rows of P_1' are all zero rows, it implies that $rank(D) = 1$. By using Corollary 1, it follows that $\eta(G) = n - (3 + 2rank(D)) = n - (3 + 2(1))$. Therefore, $\eta(G) = n - 5$. □

Theorem 6. *Let G be a graph of order n. If G has one of the following expanded decomposition,*

(1) $\mathbb{C}_7(\bar{e})$, $2\mathbb{C}_3$, $2\mathbb{P}_2$ shown in Figure 4 (a,b,c)

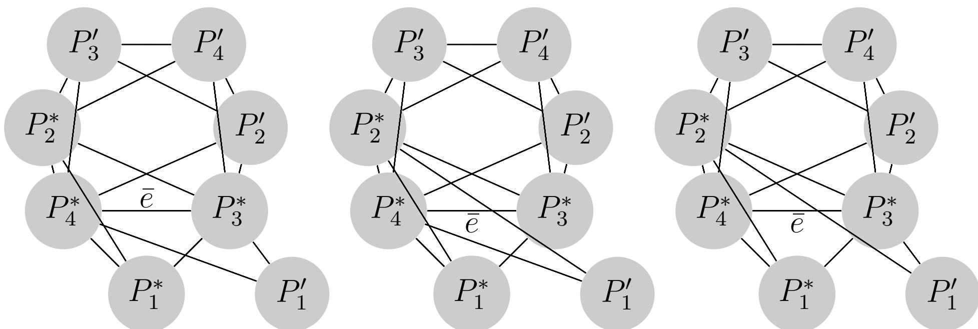
(2) $\mathbb{C}_7(\bar{e})$, $2\mathbb{C}_3$, $3\mathbb{P}_2$ shown in Figure 5 (d,e,f),

then $G \in \mathcal{F}_{4_n}$ and $\eta(G) = n - 7$.



(a) **(b)** **(c)**

Figure 4



(d) **(e)** **(f)**

Figure 5

Proof. We want to show that (i) $G \in \mathcal{F}_{4_n}$ and (ii) $\eta(G) = n - 7$.

(i). For both decomposition, let $P_1 = P_1^* \cup P_1'$, $P_2 = P_2^* \cup P_2'$, $P_3 = P_3^* \cup P_3'$ and $P_4 = P_4^* \cup P_4'$.

We see that G is a 4-partite graph with 4-partition $\{P_1, P_2, P_3, P_4\}$. Now, observe that G satisfies Property (2), because $N_{P_2}(P_1) \neq P_2$, $N_{P_3}(P_1) \neq P_3$ and $N_{P_4}(P_1) \neq P_4$. Similar to Theorem 12, G satisfies Property (1) because $G[P_2 \cup P_3 \cup P_4] = \mathbf{C}_3(P_2, P_3, P_4)$ is a complete tripartite graph. Since G satisfies Property (1) and (2), $G \in \mathcal{F}_{4_n}$.

(ii). By (3), the adjacency matrix of G can be written as follow since we already established that $G \in \mathcal{F}_{4_n}$,

$$
\text{(a).}\ \begin{array}{c} P_2^* \\ P_2' \\ P_3^* \\ P_3' \\ P_4^* \\ P_4' \\ P_1^* \\ P_1' \end{array}
\begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\qquad
\text{(b).}\ \begin{array}{c} P_2^* \\ P_2' \\ P_3^* \\ P_3' \\ P_4^* \\ P_4' \\ P_1^* \\ P_1' \end{array}
\begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\text{(c).}\ \begin{array}{c} P_2^* \\ P_2' \\ P_3^* \\ P_3' \\ P_4^* \\ P_4' \\ P_1^* \\ P_1' \end{array}
\begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\qquad
\text{(d).}\ \begin{array}{c} P_2^* \\ P_2' \\ P_3^* \\ P_3' \\ P_4^* \\ P_4' \\ P_1^* \\ P_1' \end{array}
\begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\text{(e).}\ \begin{array}{c} P_2^* \\ P_2' \\ P_3^* \\ P_3' \\ P_4^* \\ P_4' \\ P_1^* \\ P_1' \end{array}
\begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\qquad
\text{(f).}\ \begin{array}{c} P_2^* \\ P_2' \\ P_3^* \\ P_3' \\ P_4^* \\ P_4' \\ P_1^* \\ P_1' \end{array}
\begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
$$

where 1 denotes all-ones matrix and 0 denotes zero matrix with appropriate sizes. Through the adjacency matrix of G, the sub matrix D where its columns correspond to P_1^*, P_1' and the sub matrix D^t with rows correspond to P_1^*, P_1'. Thus, we notice that in both decomposition, rows of P_1^* are identical and rows of P_1' are also identical, it follows that $rank(D) = 2$. Now, by Corollary 1, it implies that $\eta(G) = n - (3 + 2rank(D)) = n - (3 + 2(2))$. Therefore, $\eta(G) = n - 7$. □

Theorem 7. *For graph $G \in \mathcal{F}_{4_n}$ with 4-partition $\{P_1, P_2, P_3, P_4\}$, $\eta(G) = n - 3$ if and only if $G = \mathbb{C}_3(P_2, P_3, P_4) \bigcup |P_1|K_1$.*

Proof. ($\Longrightarrow$) Suppose $G \in \mathcal{F}_{4_n}$ with 4-partition $\{P_1, P_2, P_3, P_4\}$. Using Corollary 1, we have the equation $\eta(G) = n-(3+2rank(D))$ and since $\eta(G) = n-3$ from our assumption, it implies that $rank(D) = 0$. Thus, the degree of v or $d(v)$ is zero, for all $v \in P_1$ which implies that all vertices in P_1 are isolated vertex. Hence, $G = \mathbb{C}_3(P_2, P_3, P_4) \bigcup |P_1|K_1$.
($\Longleftarrow$) Suppose $G = \mathbb{C}_3(P_2, P_3, P_4) \bigcup |P_1|K_1$. Then by Theorem 2.2 from the paper *On the nullity of bipartite graphs [3]*, its follows that $\eta(G) = n-3$.
This completes the proof. □

3.2.2. *For $r \geq 5$.* This is an extension of family of 4-partite graphs discussed in Section 3.2.1.

Let $G = (V, E)$ be a graph of order n. Now, consider a special class of r-partite graphs defined as follows. Let $\mathcal{F}_{r_n}$ be the family of those n - vertex r-partite graphs G with $n \geq 2r-1$, whose r-partition $\{P_1, P_2, P_3, ..., P_r\}$ satisfies the following:

$$G[P_2 \cup P_3 \cup ... \cup P_r] \text{ is complete } (r-1)\text{-partite.} \tag{7}$$

$$N_{P_j}(P_1') \neq P_j, \text{ where j} = 2, 3, ..., r \ \forall \ P_1' \subseteq P_1. \tag{8}$$

Let $G \in \mathcal{F}_{r_n}$ with r-partition $\{P_1, P_2, P_3, ..., P_r\}$. The adjacency matrix $A(G)$ of G is defined by

$$A(G) = \begin{array}{c} \\ P_2 \\ P_3 \\ P_4 \\ \vdots \\ P_r \\ P_1 \end{array} \begin{array}{c} \begin{array}{cccccc} P_2 & P_3 & P_4 & \cdots & P_r & P_1 \end{array} \\ \begin{bmatrix} 0_{n_2\times n_2} & 1_{n_2\times n_3} & 1_{n_2\times n_4} & \cdots & 1_{n_2\times n_r} & D_1 \\ 1_{n_3\times n_3} & 0_{n_3\times n_3} & 1_{n_3\times n_4} & \cdots & 1_{n_3\times n_r} & D_2 \\ 1_{n_4\times n_2} & 1_{n_4\times n_3} & 0_{n_4\times n_4} & \cdots & 1_{n_4\times n_r} & D_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1_{n_r\times n_2} & 1_{n_r\times n_3} & 1_{n_r\times n_4} & \cdots & 0_{n_r\times n_r} & D_{r-1} \\ D_1^t & D_2^t & D_3^t & \cdots & D_{r-1}^t & 0 \end{bmatrix} \end{array}$$

such that D_k, $k = 1, 2, 3..., r-1$ denote the matrices that shows the relationship of P_1 to P_j, $j = 2, 3, 4, ..., r$, respectively.
Let B and D denote the matrices defined as follows,

$$B = \begin{array}{c} \\ P_2 \\ P_3 \\ P_4 \\ \vdots \\ P_r \end{array} \begin{array}{c} \begin{array}{ccccc} P_2 & P_3 & P_4 & \cdots & P_r \end{array} \\ \begin{bmatrix} 0_{n_2\times n_2} & 1_{n_2\times n_3} & 1_{n_2\times n_4} & \cdots & 1_{n_2\times n_r} \\ 1_{n_3\times n_2} & 0_{n_3\times n_3} & 1_{n_3\times n_4} & \cdots & 1_{n_3\times n_r} \\ 1_{n_4\times n_2} & 1_{n_4\times n_3} & 0_{n_4\times n_4} & \cdots & 1_{n_4\times n_r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1_{n_r\times n_2} & 1_{n_r\times n_3} & 1_{n_r\times n_4} & \cdots & 0_{n_r\times n_r} \end{bmatrix} \end{array} \text{ and } D = \begin{array}{c} \\ P_2 \\ P_3 \\ P_4 \\ \vdots \\ P_r \end{array} \begin{array}{c} P_1 \\ \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ \vdots \\ D_{r-1} \end{bmatrix} \end{array}.$$

The matrix $A(G)$ can be viewed as

$$A(G) = \begin{bmatrix} B & D \\ D^t & 0 \end{bmatrix}. \tag{9}$$

Let

$$U = \begin{bmatrix} B & D \end{bmatrix}, L = \begin{bmatrix} D^t & 0 \end{bmatrix}. \tag{10}$$

Then $A(G)$ can be written as

$$A(G) = \begin{bmatrix} U \\ L \end{bmatrix}.$$

For vertex $v \in V$, denote by U_v the row of $A(G)$ corresponding to the vertex v if $v \in P_2 \cup P_3 \cup P_4 \cup P_5 \cup ... \cup P_r$, and by L_v if $v \in P_1$.

Let $S \subseteq P_2 \cup P_3 \cup P_4 \cup P_5 \cup ... \cup P_r$. Then from the matrix $A(G)$, we see that

$$\sum_{v \in S} q_v U_v = [b_1 \ b_2 \ b_3 \ b_4 \ ...b_{r-1} \ d] \tag{11}$$

where $b_1, \ b_2, \ b_3, \ b_4, \ ..., \ b_{r-1}$ are constant row matrices respectively of dimension $1 \times |P_2|$, $1 \times |P_3|$, $1 \times |P_4|$, $1 \times |P_5|$, ... , $1 \times |P_r|$ and d is row vector of dimension $1 \times |P_1|$ and q_v's are real constants. Similarly, for any $P_1' \subseteq P_1$, we can write

$$\sum_{v \in P_1'} q_v' L_v = [d_1 \ d_2 \ d_3 \ d_4 \ ...d_{r-1} \ 0] \tag{12}$$

where $d_1, \ d_2, \ d_3, \ d_4, \ ..., \ d_{r-1}$ and 0 are row vectors respectively of dimension $1 \times |P_2|$, $1 \times |P_3|$, $1 \times |P_4|$, $1 \times |P_5|$, ... , $1 \times |P_r|$ and $1 \times |P_1|$, and q_v''s are real constants.

The following result gives information about the rank of a r-partite graph in $\mathcal{F}_{r_n}$.

Lemma 8. *Let $G \in \mathcal{F}_{r_n}$ with r-partition $\{P_1, P_2, P_3, ..., P_r\}$ and the adjacency matrix $A(G)$ defined by (9). Then $rank(G) = rank(U) + rank(L)$ where U and L are defined by (10).*

Proof. Similar to Lemma 1, to prove $rank(G) = rank(U) + rank(L)$, it is enough to show that if $\sum_{v \in S} q_v U_v \neq 0$ and $\sum_{v \in P_1'} q_v' L_v \neq 0$ where q_v's and q_v''s are real constants, then $\sum_{v \in S} q_v U_v \neq \sum_{v \in P_1'} q_v' L_v$.

Suppose S and P_1' be an arbitrary subsets of $P_2 \cup P_3 \cup P_4 \cup ... \cup P_r$ and P_1, respectively. We can write $\sum_{v \in S} q_v U_v = [b_1 \ b_2 \ b_3 \ ... \ b_{r-1} \ d]$ and $\sum_{v \in P_1'} q_v' L_v = [d_1 \ d_2 \ d_3 \ ... \ d_{r-1} \ 0]$ such that $b_1, b_2, ..., b_{r-1}, d, d_1, d_2, ..., d_{r-1},$ and 0 are defined in (11) and (12).

Assume that $\sum_{v \in S} q_v U_v = \sum_{v \in P_1'} q_v' L_v$, it implies that $[b_1 \ b_2 \ b_3 \ ... \ b_{r-1} \ d] = [d_1 \ d_2 \ d_3 \ ... \ d_{r-1} \ 0]$. Because of condition (8), there exists a vertex in each P_j, $j = 2, 3, 4, ..., r$, which are not adjacent to any vertex in P_1. It follows that each D_k, $k = 1, 2, 3..., r-1$, has at least one zero row which also means that there is at least one zero columns in each D_k^t. Thus, there are at least $r-1$ zero columns in D^t corresponding to a vertex in each P_j. That is, there are zero entries in vectors d_k. Futhermore, since $[b_1 \ b_2 \ b_3 \ ... \ b_{r-1} \ d] = [d_1 \ d_2 \ d_3 \ ... \ d_{r-1} \ 0]$ and as $b_1, b_2, b_3, ..., b_{r-1}$ are constant vectors, then vectors $b_1, b_2, b_3, ..., b_{r-1}, d, d_1, d_2, ..., d_{r-1}$ are all zero vectors. Therefore, $\sum_{v \in S} q_v U_v = 0$ and $\sum_{v \in P_1'} q_v' L_v = 0$.

This completes the proof. ∎

Farooq et. al already established in [1] that the $r(G) = 2 + 2rank(C)$ for $G \in \mathcal{T}_n$ and in section 3.2.1, we have proved that $rank(G) = 3 + 2rank(D)$ for $G \in \mathcal{F}_{4_n}$. We now give a generalization of the rank of a graph G belonging to the family $\mathcal{F}_{r_n}$ of r-partite graph satisfying condition (7) and (8)

Theorem 9. *Let $G \in \mathcal{F}_{r_n}$ with r-partition $\{P_1, P_2, P_3, ..., P_r\}$ and the adjacency matrix $A(G)$ defined by (9). Then $rank(G) = (r-1) + 2rank(D)$.*

Proof. We prove Theorem 6 using induction.
i.) Let $r = 5$, show that $rank(G) = 4+2r(D)$. Let $\mathcal{F}_{5_n}$ be the family of those n-vertex 5-partite graphs, $n \geq 9$, whose 5-partition $\{P_1, P_2, P_3, P_4, P_5\}$ satisfies the following:
$NP_2(P_1') \neq P_2,\ N_{P_3}(P_1') \neq P_3,\ N_{P_4}(P_1') \neq P_4,\ N_{P_5}(P_1') \neq P_5,\ \forall\ P_1' \subseteq P_1.$
$G[P_2 \cup P_3 \cup P_4 \cup P_5]$ is complete 4 - partite.
Thus, for $G \in \mathcal{F}_{5_n}$, the adjacency matrix $A(G)$ can be defined by

$$A(G) = \begin{bmatrix} B & D \\ D^t & 0 \end{bmatrix}$$

where

$$B = \begin{array}{c} \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{array} \begin{array}{c} \begin{array}{cccc} P_2 & P_3 & P_4 & P_5 \end{array} \\ \begin{bmatrix} 0_{n_2\times n_2} & 1_{n_2\times n_3} & 1_{n_2\times n_4} & 1_{n_2\times n_5} \\ 1_{n_3\times n_2} & 0_{n_3\times n_3} & 1_{n_3\times n_4} & 1_{n_3\times n_5} \\ 1_{n_4\times n_2} & 1_{n_4\times n_3} & 0_{n_4\times n_4} & 1_{n_4\times n_5} \\ 1_{n_5\times n_2} & 1_{n_5\times n_3} & 1_{n_5\times n_4} & 0_{n_5\times n_5} \end{bmatrix} \end{array}$$

and

$$D = \begin{array}{c} \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{array} \begin{array}{c} P_1 \\ \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix} \end{array}$$

Since the $A(G)$ can be viewed as (9), by the same arguments stated in Lemma 2, we can have $rank(U) = rank(B) + rank(D)$ and $rank(L) = rank(D^t) = rank(D)$. Now, observe that B is an adjacency matrix of complete 4-partite graphs. It follows that the $rank(B) = 4$. Thus, by using $rank(G) = rank(U) + rank(L)$, we get $rank(G) = (rank(B) + rank(D)) + rank(D^t) = (rank(B)+rank(D))+rank(D) = rank(B)+2rank(D)$. Therefore, $rank(G) = 4+2rank(D)$.
ii.) Let $r = k$, assume that it is true for k. That is, $rank(G) = (k-1) + 2rank(D)$ where $G \in \mathcal{F}_{k_n}$ and $\mathcal{F}_{k_n}$ is the family of those n-vertex k-partite graphs, $n \geq 2k-1$, whose k-partition $\{P_1, P_2, P_3, ..., P_k\}$ satisfies the following:
$N_{P_j}(P_1') \neq P_j$, where j $= 2, 3, ..., k\ \forall\ P_1' \subseteq P_1$.
$G[P_2 \cup P_3 \cup ... \cup P_k]$ is complete $(k-1)$ - partite.
In addition, for $G \in \mathcal{F}_{k_n}$

$$A(G) = \begin{bmatrix} B & D \\ D^t & 0 \end{bmatrix}$$

where

$$B = \begin{array}{c} \\ P_2 \\ P_3 \\ P_4 \\ \vdots \\ P_k \end{array} \begin{array}{c} \begin{array}{ccccc} P_2 & P_3 & P_4 & \cdots & P_k \end{array} \\ \begin{bmatrix} 0_{n_2\times n_2} & 1_{n_2\times n_3} & 1_{n_2\times n_4} & \cdots & 1_{n_2\times n_k} \\ 1_{n_3\times n_2} & 0_{n_3\times n_3} & 1_{n_3\times n_4} & \cdots & 1_{n_3\times n_k} \\ 1_{n_4\times n_2} & 1_{n_4\times n_3} & 0_{n_4\times n_4} & \cdots & 1_{n_4\times n_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1_{n_k\times n_2} & 1_{n_k\times n_3} & 1_{n_k\times n_4} & \cdots & 0_{n_k\times n_k} \end{bmatrix} \end{array}$$

and

$$D = \begin{array}{c} \\ P_2 \\ P_3 \\ P_4 \\ \vdots \\ P_k \end{array} \begin{array}{c} P_1 \\ \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ \vdots \\ D_{k-1} \end{bmatrix} \end{array}.$$

Moreover, since the matrix B is an adjacency matrix of a complete $(k-1)$-partite graphs, it implies that $rank(B) = k - 1$.

Next, we need to show that $rank(G) = (r-1) + 2rank(D)$ is also true for $r = k+1$, that is $rank(G) = (k+1-1) + 2rank(D) = (k + 2rank(D)$.

Now, add one partition P_{k+1} with n_{k+1} vertices to $G \in \mathcal{F}_{k_n}$ to form $G \in \mathcal{F}_{(k+1)_n}$ such that $G[P_{k+1}]$ is an empty graph.

Thus $\mathcal{F}_{(k+1)_n}$ is the family of those n - vertex $(k+1)$-partite graphs, $n \geq 2(k+1)-1 = 2k+1$, whose $(k+1)$-partition $\{P_1, P_2, P_3, ..., P_k, P_{k+1}\}$ satisfies the following:

$N_{P_j}(P_1') \neq P_j$, where j $= 2, 3, ..., k+1$, $\forall\ P_1' \subseteq P_1$.

$G[P_2 \cup P_3 \cup ... \cup P_k \cup P_{k+1}]$ is complete k-partite.

Since the $A(G)$ where $G \in \mathcal{F}_{k_n}$ can be defined as (9), thus $A(G)$ such that $G \in \mathcal{F}_{(k+1)_n}$ can also be formed as (9), that is $\begin{bmatrix} B & D \\ D^t & 0 \end{bmatrix}$.

But by adding partite set P_{k+1}, the matrix B and the matrix D in $A(G)$ for which $G \in \mathcal{F}_{(k+1)_n}$ will result to

$$B = \begin{array}{c} \\ P_2 \\ P_3 \\ P_4 \\ \vdots \\ P_k \\ P_{k+1} \end{array} \begin{array}{c} \begin{array}{cccccc} P_2 & P_3 & P_4 & \cdots & P_k & P_{k+1} \end{array} \\ \begin{bmatrix} 0_{n_2\times n_2} & 1_{n_2\times n_3} & 1_{n_2\times n_4} & \cdots & 1_{n_2\times n_k} & 1_{n_2\times n_{k+1}} \\ 1_{n_3\times n_2} & 0_{n_3\times n_3} & 1_{n_3\times n_4} & \cdots & 1_{n_3\times n_k} & 1_{n_3\times n_{k+1}} \\ 1_{n_4\times n_2} & 1_{n_4\times n_3} & 0_{n_4\times n_4} & \cdots & 1_{n_4\times n_k} & 1_{n_4\times n_{k+1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1_{n_k\times n_2} & 1_{n_k\times n_3} & 1_{n_k\times n_4} & \cdots & 0_{n_k\times n_k} & 1_{n_k\times n_{k+1}} \\ 1_{n_{k+1}\times n_2} & 1_{n_{k+1}\times n_3} & 1_{n_{k+1}\times n_4} & \cdots & 1_{n_{k+1}\times n_k} & 0_{n_{k+1}\times n_{k+1}} \end{bmatrix} \end{array}$$

and

$$D = \begin{array}{c} \\ P_2 \\ P_3 \\ P_4 \\ \vdots \\ P_k \\ P_{k+1} \end{array} \begin{array}{c} P_1 \\ \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ \vdots \\ D_{k-1} \\ D_k \end{bmatrix} \end{array}.$$

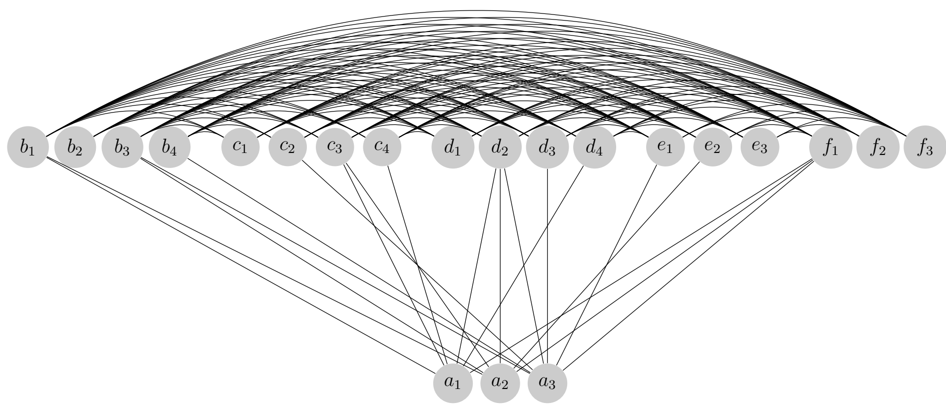
From our assumption, the $rank(B)$ for $G \in \mathcal{F}_{k_n}$ is $k-1$.

But, by adding P_{k+1}, the $rank(B)$ will increase by 1, that is $rank(B) = (k-1)+1 = k$. Moreover, B is an adjacency matrix of a complete k-partite graphs for $G \in \mathcal{F}_{(k+1)_n}$. It follows that $rank(G) = rank(B) + 2rank(D) = k + 2rank(D)$. □

Corollary 10. *If $G \in \mathcal{F}_{r_n}$ with r-partition $\{P_1, P_2, P_3, ..., P_r\}$, then $\eta(G) = n - ((r-1) + 2rank(D))$.*

This is an illustration of theorems in Section 3.2.

Illustration: Let $G \in \mathcal{F}_{6_n}$ with 6-partition $\{P_1, P_2, P_3, P_4, P_5, P_6\}$ where $P_1 = \{a_1, a_2, a_3\}$,$P_2 = \{b_1, b_2, b_3, b_4\}$,$P_3 = \{c_1, c_2, c_3, c_4\}$,$P_4 = \{d_1, d_2, d_3, d_4\}$,$P_5 = \{e_1, e_2, e_3\}$ and $P_6 = \{f_1, f_2, f_3\}$. The 6-partition satisfies the following:
$G[P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6]$ is complete 5-partite and $N_{P_2}(P_1') \neq P_2, N_{P_3}(P_1') \neq P_3, N_{P_4}(P_1') \neq P_4, N_{P_5}(P_1') \neq P_5, N_{P_6}(P_1') \neq P_6$ $\forall P_1' \subseteq P_1$



Now, we have the adjacency matrix of G,

$$\begin{array}{c} b_1\\ b_2\\ b_3\\ b_4\\ c_1\\ c_2\\ c_3\\ c_4\\ d_1\\ d_2\\ d_3\\ d_4\\ e_1\\ e_2\\ e_3\\ f_1\\ f_2\\ f_3\\ a_1\\ a_2\\ a_3 \end{array}
\begin{bmatrix}
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0\\
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0\\
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1\\
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&1\\
1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1&1&0&0&0\\
1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1&1&0&0&1\\
1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&0\\
1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&0&0&1\\
1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&1&1&1&0&0&1\\
1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&1&1&1&0&1&0\\
1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&1&1&1&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0\\
1&0&0&0&0&0&1&1&0&1&0&1&0&0&0&1&0&0&0&0&0\\
1&0&1&0&0&0&1&0&0&1&0&0&0&1&0&1&0&0&0&0&0\\
0&0&1&1&0&1&0&0&0&1&1&0&1&0&0&1&0&0&0&0&0
\end{bmatrix}$$

where $U = \begin{bmatrix} B & D \end{bmatrix} =$

$$\begin{array}{c} b_1\\ b_2\\ b_3\\ b_4\\ c_1\\ c_2\\ c_3\\ c_4\\ d_1\\ d_2\\ d_3\\ d_4\\ e_1\\ e_2\\ e_3\\ f_1\\ f_2\\ f_3 \end{array}
\begin{bmatrix}
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0\\
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0\\
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1\\
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&1\\
1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1&1&0&0&0\\
1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1&1&0&0&1\\
1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&0\\
1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&0&0&1\\
1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&1&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&1&1&1&0&0&1\\
1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&1&1&1&0&1&0\\
1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&1&1&1&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0
\end{bmatrix}$$

and $L = \begin{bmatrix} D^t & 0 \end{bmatrix} =$

$$\begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In addition,

$$B = \begin{array}{c} b_1 \\ b_2 \\ b_3 \\ b_4 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \\ e_1 \\ e_2 \\ e_3 \\ f_1 \\ f_2 \\ f_3 \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and

$$D = \begin{array}{c} b_1 \\ b_2 \\ b_3 \\ b_4 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \\ e_1 \\ e_2 \\ e_3 \\ f_1 \\ f_2 \\ f_3 \end{array} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By computation, we get the following, $rank(G) = 11$, $rank(U) = 8$, $rank(L) = 3$, $rank(B) = 5$ and $rank(D) = 3$. From Lemma 2, $rank(G) = 11 = 8 + 3 = rank(U) + rank(L)$ and from Theorem 6, $rank(G) = 11 = 5 + 2(3) = rank(B) + 2rank(D)$. Now by Corollary 2, since the $rank(G) = 11$ and $n = 21$, it follows that $\eta(G) = 10$.

4. Conclusion

In this paper, we studied and investigated some families of r-partite graphs where $r \geq 4$, these are the complete r-partite graphs of order n and the n-vertex r-partite graphs satisfying (7) and (8). We were able to established that the rank of complete r-partite graphs is r and the rank of n-vertex r-partite graphs satisfying (7) and (8) is $(r-1)+2rank(D)$. We also obtained the nullity of these r-partite graphs by using its rank. As a special type, the complete r-partite graphs denoted by $K_{n_1,n_2,n_3,\ldots,n_r}$ where $n = n_1 + n_2 + n_3 + \ldots + n_r$ and $r \geq 4$ was found to have a nullity of $n - r$. The n-vertex r-partite graphs satisfying (7) and (8) and its nullity follows as an extension of family of tripartite graphs introduced in the paper *"On the nullity of a family of tripartite graphs"* by Farooq, Malik, Pirzada and Naureen.

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