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STABILIZATION OF THE SCHRÖDINGER EQUATION WITH DISTRIBUTED DELAY IN BOUNDARY FEEDBACK

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Abstract. In this paper, we investigate the effect of a distributed time-delay in boundary stabilization of the Schrödinger equation. Under suitable assumptions, we establish sufficient conditions on the distributed delay term that guarantee the exponential stability of the solution using the frequency domain approach and a duality argument.

1. Introduction

In recent years, time-delay appears in many areas (in biology, electrical engineering and mechanics, ...), and in many cases of them, instabilities appear due to the effect of time delays for some internal or boundary control system. However the stability issue of these control systems with delay is of theoretical and practical importance. For it, mathematical tools have been recently developed in order to prove exponential or polynomial decay of the energy of wave type equations with delay. We refer readers to [17] for a list of early works, and to [3–5, 9–12, 18–20] and the references therein, for some other relevant results.

In this paper, inspired by [13, 15, 21], we investigate the effect of a distributed time-delay on a Schrödinger equation.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of $\mathbb{R}^n$ with smooth boundary Γ of class C^2 such that $\Gamma = \Gamma_D \cup \Gamma_N$, $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$ and $\Gamma_D \neq \emptyset$, $\Gamma_N \neq \emptyset$. The aim of this article is to study the mixed Dirichlet and Neumann following system :

$$(1) \qquad \begin{cases} u_t(x,t) - i\Delta u(x,t) = 0 \text{ in } \Omega \times (0,+\infty) \\ u(x,t) = 0 \text{ on } \Gamma_D \times (0,+\infty) \\ \dfrac{\partial u}{\partial \nu}(x,t) - i\beta_1 u(x,t) + i\displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s) u(x,t-s)ds = 0 \text{ on } \Gamma_N \times (0,+\infty) \\ u(x,0) = u_0(x) \text{ in } \Omega \\ u(x,-t) = f_0(x,-t) \text{ on } \Gamma_N \times (0,\tau_2), \end{cases}$$

where $\dfrac{\partial u}{\partial \nu}$ is the normal derivative, β_1 is a positive real number and the function $\beta_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a positive L^∞ function.

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In the whole paper, we suppose as in [14] that the function β_2 verifying

$$\beta_1 > \int_{\tau_1}^{\tau_2} \beta_2(s)ds. \tag{2}$$

It is well known that if $\beta_2 = 0$, that is, in absence of delay, the system (1) takes the following form

$$\begin{cases} u_t(x,t) - i\Delta u(x,t) = 0 \text{ in } \Omega \times (0,+\infty) \\ u(x,t) = 0 \text{ on } \Gamma_D \times (0,+\infty) \\ \dfrac{\partial u}{\partial \nu}(x,t) - i\beta_1 u(x,t) = 0 \text{ on } \Gamma_N \times (0,+\infty) \\ u(x,0) = u_0(x) \text{ in } \Omega \end{cases} \tag{3}$$

Lasiecka and al [7] have shown that the energy of the system (3) decays exponentially using multiplier techniques and constructing energy functionals well adapted to the system.
In the presence of a delay, that is

$$\begin{cases} u_t(x,t) - i\Delta u(x,t) = 0 \text{ in } \Omega \times (0,+\infty) \\ u(x,t) = 0 \text{ on } \Gamma_D \times (0,+\infty) \\ \dfrac{\partial u}{\partial \nu}(x,t) - i\beta_1 u(x,t) + \beta_2 u(x,t-\tau) = 0 \text{ on } \Gamma_N \times (0,+\infty) \\ u(x,0) = u_0(x) \text{ in } \Omega \\ u(x,-t) = f_0(x,-\tau) \text{ on } \Gamma_N \times (0,\tau_2), \end{cases} \tag{4}$$

where β_1 and β_2 verifying the assumption that there exists a positive constant ζ verifying

$$\tau\beta_1 < \zeta < \tau(2\beta_1 - \beta_2), \tag{5}$$

it has been proved by S. Nicaise and C. Pignotti [13] that the energy of problem (4) decays exponentially by combining inequalities due to Carleman estimates and compactness-uniqueness arguments.
In [2] the authors developed an observer-predictor scheme to stabilize the 1-D Schrödinger equation with distributed input time delay. Here, we prove the fact that if the distributed delay term is small enough, then the system with delay has the same decay rate than the one without delay. The main idea is to use a duality argument already used in [1]
To our best knowledge, this idea was not used before in the context of the Schrödinger equation with distributed time-delay in boundary stabilization.

The paper is organized as follows : section 2 is devoted to the well-posedness of the problem (1) while section 3 deals with the exponential stability.

2. Well posedness result

In this section we will give the well posedness for the problem (1) using the semigroup theory. For this, we introduce the new variable $y(x,\rho,t,s) = u(x,t-\rho s)$, $\quad x \in \Gamma_N$, $\rho \in (0,1)$, $s \in$

(τ_1, τ_2), $t > 0$. Then the problem (1) is now equivalent to

$$(6) \qquad \begin{cases} u_t(x,t) - i\Delta u(x,t) = 0 \text{ in } \Omega \times (0,+\infty) \\ y_\rho(x,\rho,t,s) + s y_t(x,\rho,t,s) = 0 \text{ on } \Gamma_N \times (0,1) \times (0,+\infty) \times (\tau_1,\tau_2) \\ u(x,t) = 0 \text{ on } \Gamma_D \times (0,+\infty) \\ \dfrac{\partial u}{\partial \nu}(x,t) - i\beta_1 u(x,t) + i\displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s) y(x,1,t,s) ds = 0 \text{ on } \Gamma_N \times (0,+\infty) \\ u(x,0) = u_0(x) \text{ in } \Omega \\ y(x,0,t,s) = u(x,t) \text{ on } \Gamma_N \times (0,+\infty) \times (\tau_1,\tau_2) \\ y(x,\rho,0,s) = f_0(x,\rho,s) \text{ on } \Gamma_N \times (0,1) \times (0,\tau_2). \end{cases}$$

Let

$$V = \left\{ u \in H^1(\Omega), u = 0 \text{ on } \Gamma_D \right\}.$$

Now we define the energy space by

$$\mathcal{H} = V \times L^2\left(\Gamma_N \times (0,1) \times (\tau_1,\tau_2)\right)$$

endowed with the norm

$$\left\| (u,y)^\mathsf{T} \right\|_{\mathcal{H}}^2 = \|u\|_{L^2(\Omega)}^2 + \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 |y(\cdot,\rho,s)|^2 d\rho \right) ds d\Gamma$$

and the usual inner product

$$\left\langle \begin{pmatrix} u \\ y \end{pmatrix}, \begin{pmatrix} u^* \\ y^* \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_\Omega u\overline{u^*} dx + \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 y\overline{y^*} d\rho \right) ds d\Gamma.$$

We next define the linear operator defined by

$$\mathcal{A} \begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} i\Delta u & 0 \\ 0 & -s^{-1}\dfrac{\partial}{\partial \rho} \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{c} (u,y)^\mathsf{T} \in \left(H^{\frac{3}{2}}(\Omega) \cap V \right) \times L^2\left(\Gamma_N; H^1(\Omega)\right) \text{ such that } \Delta u \in L^2(\Omega), \\ \\ y(x,0,s) = u(x,t) \text{ on } \Gamma_N \text{ and } \dfrac{\partial u}{\partial \nu}(x) = i\beta_1 u(x) - i\displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s) y(x,1,s) ds \text{ on } \Gamma_N \end{array} \right\}.$$

Setting

$$\mathcal{U} = \left(u, y\right)^\mathsf{T}.$$

Then we have

$$\mathcal{U}_t = (u_t, y_t)^\mathsf{T} = \left(i\Delta u, -s^{-1} y_\rho\right)^\mathsf{T}.$$

Therefore problem (7) can be rewritten in an abstract framework:

$$(7) \qquad \begin{cases} \mathcal{U}_t = \mathcal{A}\mathcal{U} \\ \\ \mathcal{U}(0) = (u_0, f_0(-\cdot s))^\mathsf{T}, \end{cases}$$

Theorem 2.1. *For any initial data $U_0 \in \mathcal{H}$, there exists a unique weak solution $U \in C((0,+\infty);\mathcal{H})$ of (7). Moreover, if we assume that $U_0 \in \mathcal{D}(\mathcal{A})$, then there exists a unique strong solution $U \in C((0,+\infty);\mathcal{H})$.*

Proof. Take $U = (u, y)^{\mathsf{T}} \in \mathcal{D}(\mathcal{A})$. Then we have

$$\begin{aligned}\Re\left\langle \mathcal{A}\begin{pmatrix} u \\ y \end{pmatrix}, \begin{pmatrix} u \\ y \end{pmatrix}\right\rangle_{\mathcal{H}} &= \Re\left\langle \begin{pmatrix} i\Delta u \\ -s^{-1}y_\rho \end{pmatrix}, \begin{pmatrix} u \\ y \end{pmatrix}\right\rangle_{\mathcal{H}} \\ &= \Re\left(\int_\Omega i\Delta u\,\overline{u}\,dx\right) - \Re\left(\int_{\Gamma_N}\int_{\tau_1}^{\tau_2} \beta_2(s)\int_0^1 y_\rho \overline{y} d\rho ds d\Gamma\right)\end{aligned}$$

that is

$$\Re\left\langle \mathcal{A}\begin{pmatrix} u \\ y \end{pmatrix}, \begin{pmatrix} u \\ y \end{pmatrix}\right\rangle_{\mathcal{H}} = \underbrace{\Re\left(\int_\Omega i\Delta u\,\overline{u}\,dx\right)}_{X} - \underbrace{\int_{\Gamma_N}\int_{\tau_1}^{\tau_2} \beta_2(s)\underbrace{\Re\left(\int_0^1 y_\rho \overline{y} d\rho\right)}_{Y} ds d\Gamma}_{Z}. \tag{8}$$

Using Green formula, Cauchy Schwarz's inequality and the definition of $\mathcal{D}(\mathcal{A})$ we obtain

$$\begin{aligned} X &= \Re\left(-i\int_\Omega |\nabla u|^2 dx + \int_{\Gamma_N} i\frac{\partial u}{\partial \nu}\overline{u} d\Gamma\right) \\ &= \Re\left(\int_{\Gamma_N} i\frac{\partial u}{\partial \nu}\overline{u} d\Gamma\right) \\ &= \Re\left[\int_{\Gamma_N} i\left(i\beta_1 u - i\int_{\tau_1}^{\tau_2} \beta_2(s) y(x,1,s) ds\right)\overline{u} d\Gamma\right] \\ &= -\beta_1\int_{\Gamma_N} |u|^2 d\Gamma + \Re\left(\int_{\Gamma_N}\int_{\tau_1}^{\tau_2} \beta_2(s) y(x,1,s)\overline{u} ds d\Gamma\right) \\ &\le -\beta_1\int_{\Gamma_N} |u|^2 d\Gamma + \frac{1}{2}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2} \beta_2(s)|y(x,1,s)|^2 ds d\Gamma + \frac{1}{2}\int_{\Gamma_N}\left(\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)|u|^2 d\Gamma \end{aligned}$$

that is

$$X \le -\beta_1\int_{\Gamma_N} |u|^2 d\Gamma + \frac{1}{2}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2} \beta_2(s)|y(x,1,s)|^2 ds d\Gamma + \frac{1}{2}\int_{\Gamma_N}\left(\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)|u|^2 d\Gamma. \tag{9}$$

Integrating Y by parts we get

$$\begin{aligned} Y &= \left[|y(x,\rho,s)|^2\right]_0^1 - \Re\left(\int_0^1 y_\rho y d\rho\right) \\ &= |y(x,1,s)|^2 - |y(x,0,s)|^2 - Y \\ &= |y(x,1,s)|^2 - |u|^2 - Y. \end{aligned}$$

Then we deduce that

$$Y = \frac{1}{2}\left(|y(x,1,s)|^2 - |u|^2\right).$$

Inserting the expression of Y in Z we get

$$Z = \int_{\Gamma_N}\int_{\tau_1}^{\tau_2} \beta_2(s)\frac{1}{2}\left(|y(x,1,s)|^2 - |u|^2\right) ds d\Gamma$$

that is

$$Z = \frac{1}{2}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2} \beta_2(s)|y(x,1,s)|^2 ds d\Gamma - \frac{1}{2}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2} \beta_2(s)|u|^2 ds d\Gamma. \tag{10}$$

Now combining (8), (9) and (10) we get

$$\Re\left\langle \mathcal{A}\begin{pmatrix} u \\ y \end{pmatrix}, \begin{pmatrix} u \\ y \end{pmatrix}\right\rangle_{\mathcal{H}} \leq \left(-\beta_1 + \int_{\tau_1}^{\tau_2} \beta_2(s)ds\right)\int_{\Gamma_N} |u|^2 d\Gamma. \tag{11}$$

Thanks to (2), we deduce from the above relation (11) that $\mathcal{A}$ is dissipative .

Let us now show that for a fixed $\lambda > 0$ and given $(g,h)^{\mathsf{T}} \in \mathcal{H}$, there exists $(u,y)^{\mathsf{T}} \in \mathcal{D}(\mathcal{A})$ solution of

$$(\lambda I - \mathcal{A})\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix} \tag{12}$$

that is verifying

$$\begin{cases} \lambda u - i\Delta u = g \\ \lambda y + s^{-1} y_\rho = h. \end{cases} \tag{13}$$

Suppose that we have found u with the right regularity. Then we can determine y. Indeed, from (13) and the last lines of (6) we have

$$\begin{cases} \lambda y + s^{-1} y_\rho = h \\ y(x,0,s) = u(x). \end{cases} \tag{14}$$

The unique solution of the above initial value problem (14) is given by

$$y(x,\rho,s) = u(x)e^{-\lambda\rho s} + s e^{-\lambda\rho s}\int_0^\rho h(x,\sigma,s)e^{\lambda\sigma s}d\sigma \tag{15}$$

and in particular

$$y(x,1,s) = u(x)e^{-\lambda s} + s e^{-\lambda s}\int_0^1 h(x,\sigma,s)e^{\lambda\sigma s}d\sigma. \tag{16}$$

The first equation of (13) can be reformulated as follows

$$\int_\Omega (\lambda u - i\Delta u)\,\overline{w}dx = \int_\Omega g\overline{w}dx. \tag{17}$$

Integrating by parts the left hand side of (17) and recalling the boundary conditions and (16), we get

$$\begin{aligned}
\int_\Omega (\lambda u - i\Delta u)\,\overline{w}dx &= \int_\Omega (\lambda u\overline{w} + i\nabla u\nabla\overline{w})\,dx - i\int_{\Gamma_N}\frac{\partial u}{\partial\nu}\overline{w}d\Gamma \\
&= \int_\Omega (\lambda u\overline{w} + i\nabla u\nabla\overline{w})\,dx - i\int_{\Gamma_N}\left(i\beta_1 u - i\int_{\tau_1}^{\tau_2}\beta_2(s)y(x,1,s)ds\right)\overline{w}d\Gamma \\
&= \int_\Omega (\lambda u\overline{w} + i\nabla u\nabla\overline{w})\,dx + \int_{\Gamma_N}\beta_1 u\overline{w}d\Gamma - \int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)y(x,1,s)\overline{w}dsd\Gamma \\
&= \int_\Omega (\lambda u\overline{w} + i\nabla u\nabla\overline{w})\,dx + \int_{\Gamma_N}\beta_1 u\overline{w}d\Gamma \\
&- \int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)\left[ue^{-\lambda s} + se^{-\lambda s}\int_0^1 h(x,\sigma,s)e^{\lambda\sigma s}d\sigma\right] \\
&= \int_\Omega (\lambda u\overline{w} + i\nabla u\nabla\overline{w})\,dx + \int_{\Gamma_N}\beta_1 u\overline{w}d\Gamma - \int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)e^{-\lambda s}u\overline{w}dsd\Gamma \\
&- \int_{\Gamma_N}\int_{\tau_1}^{\tau_2}s\beta_2(s)e^{-\lambda s}\left(\int_0^1 h(x,\sigma,s)e^{\lambda\sigma s}d\sigma\right)\overline{w}d\Gamma.
\end{aligned}$$

Therefore (17) can be rewritten as follows

$$\int_\Omega (\lambda u\overline{w} + i\nabla u\nabla\overline{w})\,dx + \int_{\Gamma_N}\left(\beta_1 - \int_{\tau_1}^{\tau_2}\beta_2(s)e^{-\lambda s}ds\right)u\overline{w}d\Gamma$$

$$= \int_\Omega g\overline{w}dx + \int_{\Gamma_N}\int_{\tau_1}^{\tau_2} s\beta_2(s)e^{-\lambda s}\left(\int_0^1 h(x,\sigma,s)e^{\lambda\sigma s}d\sigma\right)\overline{w}d\Gamma \tag{18}$$

Multiplying (18) by $1-i$, we obtain

$$(1-i)\int_\Omega (\lambda u\overline{w} + i\nabla u\nabla\overline{w})\,dx + (1-i)\int_{\Gamma_N}\left(\beta_1 - \int_{\tau_1}^{\tau_2}\beta_2(s)e^{-\lambda s}ds\right)u\overline{w}d\Gamma$$

$$= (1-i)\int_\Omega g\overline{w}dx + (1-i)\int_{\Gamma_N}\int_{\tau_1}^{\tau_2} s\beta_2(s)e^{-\lambda s}\left(\int_0^1 h(x,\sigma,s)e^{\lambda\sigma s}d\sigma\right)\overline{w}d\Gamma \tag{19}$$

If we denote the left hand side of (19) by $a(u,y)$ and the right hand side by $L(y)$, we get

$$\Re\ a(u,u) = \lambda\int_\Omega |u|^2dx + \int_\Omega |\nabla u|^2dx + \int_{\Gamma_N}\left(\beta_1 - \int_{\tau_1}^{\tau_2}\beta_2(s)e^{-\lambda s}ds\right)|u|^2d\Gamma$$

which implies

$$\Re\ a(u,u) \geq C\left[\int_\Omega |u|^2dx + \int_\Omega |\nabla u|^2dx + \int_{\Gamma_N}|u|^2d\Gamma\right]$$

where $C = \min\left\{\lambda, 1, \beta_1 - \int_{\tau_1}^{\tau_2}\beta_2(s)e^{-\lambda s}ds\right\}$. We then obtain

$$\Re\ a(u,u) \geq C\|u\|^2_{H^1_{\Gamma_D}(\Omega)}.$$

Consequently a is cœrcive.

Since the right-hand side defines a continuous linear form on $H^1_{\Gamma_D}(\Omega)$, since $(g,h)\in\mathcal{H}$, the Lax-Milgram Theorem ensures the existence and uniqueness of a solution $u\in H^1_{\Gamma_D}(\Omega)$ of (17). Taking now $w\in\mathcal{D}(\Omega)$ in (18), then u solves in $\mathcal{D}'(\Omega)$

$$\lambda u - i\Delta u = g$$

and thus $\Delta u\in L^2(\Omega)$.

Using Green's formula in (18), we get

$$\int_\Omega (\lambda u - i\Delta u)\,\overline{w}dx + i\int_{\Gamma_N}\frac{\partial u}{\partial\nu}\overline{w}d\Gamma + \int_{\Gamma_N}\left(\beta_1 - \int_{\tau_1}^{\tau_2}\beta_2(s)e^{-\lambda s}ds\right)u\overline{w}d\Gamma$$

$$= \int_\Omega g\overline{w}dx + \int_{\Gamma_N}\int_{\tau_1}^{\tau_2} s\beta_2(s)e^{-\lambda s}\left(\int_0^1 h(x,\sigma,s)e^{\lambda\sigma s}d\sigma\right)\overline{w}d\Gamma$$

which implies

$$\int_\Omega g\overline{w}dx + i\int_{\Gamma_N}\frac{\partial u}{\partial\nu}\overline{w}d\Gamma + \int_{\Gamma_N}\left(\beta_1 - \int_{\tau_1}^{\tau_2}\beta_2(s)e^{-\lambda s}ds\right)u\overline{w}d\Gamma$$

$$= \int_\Omega g\overline{w}dx + \int_{\Gamma_N}\int_{\tau_1}^{\tau_2} s\beta_2(s)e^{-\lambda s}\left(\int_0^1 h(x,\sigma,s)e^{\lambda\sigma s}d\sigma\right)\overline{w}d\Gamma$$

that is

$$i\int_{\Gamma_N}\frac{\partial u}{\partial\nu}\overline{w}d\Gamma + \int_{\Gamma_N}\left(\beta_1 - \int_{\tau_1}^{\tau_2}\beta_2(s)e^{-\lambda s}ds\right)u\overline{w}d\Gamma = \int_{\Gamma_N}\int_{\tau_1}^{\tau_2} s\beta_2(s)e^{-\lambda s}\left(\int_0^1 h(x,\sigma,s)e^{\lambda\sigma s}d\sigma\right)\overline{w}d\Gamma.$$

We deduce that

$$i\frac{\partial u}{\partial \nu} + \left(\beta_1 - \int_{\tau_1}^{\tau_2} \beta_2(s)e^{-\lambda s}ds\right) u = \int_{\tau_1}^{\tau_2} s\beta_2(s)e^{-\lambda s}\left(\int_0^1 h(x,\sigma,s)e^{\lambda\sigma s}d\sigma\right) \quad \text{on} \quad \Gamma_N$$

that is

$$\frac{\partial u}{\partial \nu} = i\left(\beta_1 - \int_{\tau_1}^{\tau_2} \beta_2(s)e^{-\lambda s}ds\right) u - i\int_{\tau_1}^{\tau_2} s\beta_2(s)e^{-\lambda s}\left(\int_0^1 h(x,\sigma,s)e^{\lambda\sigma s}d\sigma\right) \quad \text{on} \quad \Gamma_N$$

from which follows

$$\frac{\partial u}{\partial \nu} = i\beta_1 u - i\int_{\tau_1}^{\tau_2} \beta_2(s)\left(ue^{-\lambda s}ds - se^{-\lambda s}\int_0^1 h(x,\sigma,s)e^{\lambda\sigma s}d\sigma\right) \quad \text{on} \quad \Gamma_N$$

and then recalling (16) we retrieve the boundary condition

$$\frac{\partial u}{\partial \nu} = i\beta_1 u - i\int_{\tau_1}^{\tau_2} \beta_2(s)y(\cdot,1,s) \quad \text{on} \quad \Gamma_N. \tag{20}$$

As the right hand side of (20) belongs to $L^2(\Gamma_N)$, we deduce that $\frac{\partial u}{\partial \nu} \in L^2(\Gamma_N)$ and by the theorem 2.7.4 of [8], we deduce that $u \in H^{\frac{3}{2}}(\Omega)$. Finally we have found $(u,y) \in \mathcal{D}(\mathcal{A})$ satisfying (13). Thanks to Lumer-Phillips' theorem we conclude that the operator $\mathcal{A}$ generates a $\mathcal{C}_0$ semigroup of contractions on $\mathcal{H}$, and thus problem (1) is well posed. The rest of the proof directly follows from the Hille-Yosida theorem. □

3. Exponential stability

In this section, we will show that the system (1) is exponentially stable. Our future computations are based on frequency domain approach for exponential stability (see Huang [6] and Pruss [16]), more precisely on the below result.

Lemma 3.1.
A C_0-semigroup $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ of contractions on a Hilbert space $\mathcal{H}$ is exponentially stable, namely satisfies

$$\left\|e^{t\mathcal{A}}U_0\right\|_{\mathcal{H}} \leq Ce^{-\omega t}\|U_0\|_{\mathcal{H}} \quad \forall\, U_0 \in \mathcal{H},\ \forall\, t \geq 0, \tag{21}$$

for some positive constants C and ω if and only if

$$\rho(\mathcal{A}) \supset \{i\beta,\ \beta \in \mathbb{R}\} \equiv i\mathbb{R} \tag{22}$$

and

$$\sup_{\beta\in\mathbb{R}} \left\|(i\beta - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty \tag{23}$$

where $\rho(\mathcal{A})$ denotes the resolvent set of the operator $\mathcal{A}$.

Now, we are ready to state and prove the main result of this section.

Theorem 3.2.
The system (1) is exponentially stable in the energy space $\mathcal{H}$.

Proof.

Lemma 3.3.
There is no eigenvalue of $\mathcal{A}$ on the imaginary axis, that is

$$i\mathbb{R} \subset \rho(\mathcal{A}).$$

Proof of lemma 3.3. Let $i\lambda$ be an eigenvalue of $\mathcal{A}$ and $U = (u, y)^\intercal \in \mathcal{D}(\mathcal{A})$ the associated eigenvector. Then we have

$$\mathcal{A}U = i\lambda U. \tag{24}$$

So using (24) and the dissipativity of $\mathcal{A}$, we get

$$0 = \Re\left(i\lambda\|U\|_{\mathcal{H}}^2\right) = \Re\left(\mathcal{A}U, U\right)_{\mathcal{H}} \leq \left(-\beta_1 + \int_{\tau_1}^{\tau_2} \beta_2(s)ds\right)\|u\|_{L^2(\Gamma_N)}^2 \leq 0.$$

We deduce that

$$\|u\|_{L^2(\Gamma_N)}^2 = 0$$

that is

$$u = 0 \quad \text{on} \quad \Gamma_N. \tag{25}$$

The relation (24) can be rewritten as

$$\begin{cases} i\Delta u = i\lambda u \text{ in } \ \Omega \\ s^{-1} y_\rho = i\lambda y \text{ in } \ (0,1) \end{cases} \tag{26}$$

with boundary conditions

$$\begin{cases} u = 0 \ \text{ on } \ \Gamma_D \\ \dfrac{\partial u}{\partial \nu} = i\beta_1 u - i\displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s) y(x,1,s)ds \ \text{ on } \ \Gamma_N. \end{cases} \tag{27}$$

Recalling the definition of $\mathcal{D}(\mathcal{A})$ and using (25) it follows that $y(x,0,s) = 0$ on Γ_N. Then from the last equation of (26) we have the system

$$\begin{cases} y_\rho - i\lambda s y = 0 \\ y(x,0,s) = 0 \end{cases} \tag{28}$$

which admits a unique solution $y = 0$. Consequently the boundary conditions become

$$\begin{cases} u = 0 \ \text{ on } \ \Gamma \\ \dfrac{\partial u}{\partial \nu} = 0 \ \text{ on } \ \Gamma_N. \end{cases} \tag{29}$$

Then from (26) and (29) we get

$$\begin{cases} \Delta u - \lambda u = 0 \text{ in } \ \Omega \\ u = 0 \ \text{ on } \ \Gamma \\ \dfrac{\partial u}{\partial \nu} = 0 \ \text{ on } \ \Gamma_N. \end{cases} \tag{30}$$

It is obvious that (30) admits a unique solution $u = 0$. In short we get $U = 0$ which contradicts the fact that U is an eigenvector.
Finally we have found $(u, y)^\top = (0, 0)^\top$ that is $U = 0$ which contradicts the fact that U is an eigenvector. The proof is thus completed.

Proof of theorem 3.2. As the condition (22) is guaranteed by Lemma 3.3, it suffices now to check the condition (23) in other words, the boundedness of the resolvent on the imaginary axis. For that, we will establish that for any $\lambda \in \mathbb{R}$ and $F = (g,h)^{\mathsf{T}} \in \mathcal{H}$, the solution $U = (u,y)^{\mathsf{T}} \in \mathcal{D}(\mathcal{A})$ of

$$(i\lambda I - \mathcal{A})\, U = F \tag{31}$$

satisfies

$$\|U\|_{\mathcal{H}} \leq C\|F\|_{\mathcal{H}}; \tag{32}$$

where C is positive constant (independent of λ and F).
Problem (1) without delay (corresponding to $\beta_2 = 0$) is the following one

$$\begin{cases} u_t(x,t) - i\Delta u(x,t) = 0 \text{ in } \ \Omega \times (0,+\infty) \\ u(x,t) = 0 \ \text{ on } \ \Gamma_D \times (0,+\infty) \\ \dfrac{\partial u}{\partial \nu}(x,t) = i\beta_1 u(x,t) \ \text{ on } \ \Gamma_N \times (0,+\infty) \\ u(x,0) = u_0(x) \text{ in } \ \Omega. \end{cases} \tag{33}$$

It has been studied by Lasiecka et al. in [7]. This problem is well-posed in

$$\mathcal{H}_0 = L^2(\Omega) \tag{34}$$

endowed with the norm

$$\|u\|^2_{\mathcal{H}_0} = \|u\|^2_{L^2(\Omega)} \tag{35}$$

The generator of its semigroup is $\mathcal{A}_0$ defined by

$$\mathcal{A}_0 u = i\Delta u \tag{36}$$

with domain

$$\mathcal{D}(\mathcal{A}_0) = \left\{ \ u \in H^2(\Omega) \cap V : \frac{\partial u}{\partial \nu} = i\beta_1 u \text{ on } \Gamma_N \ \right\} \tag{37}$$

In [7], it has been proved that $\mathcal{A}_0$ generates an exponentially stable semigroup, then we have that $i\mathbb{R} \subset \rho(\mathcal{A}_0)$ and there exist a constant $C_0 > 0$ such that

$$\left\|(i\xi - \mathcal{A}_0)^{-1}\right\|_{\mathcal{L}(\mathcal{H}_0)} \leq C_0, \quad \forall\ \xi \in \mathbb{R}. \tag{38}$$

The relation (38) implies that the solution $u^* \in \mathcal{D}\left(\mathcal{A}_0\right)$ of

$$i\lambda u^* - i\Delta u^* = u \tag{39}$$

verifies

$$\|u^*\|_{\mathcal{H}_0} \leq C_0 \left\|u\right\|_{\mathcal{H}_0}. \tag{40}$$

We have

$$
\begin{aligned}
& \left\langle (i\lambda I - \mathcal{A}) \begin{pmatrix} u \\ y \end{pmatrix}, \begin{pmatrix} u^* \\ \alpha y \end{pmatrix} \right\rangle_{\mathcal{H}} \\
= & \left\langle \begin{pmatrix} i\lambda u - i\Delta u \\ i\lambda y + s^{-1} y_\rho \end{pmatrix}, \begin{pmatrix} u^* \\ \alpha y \end{pmatrix} \right\rangle_{\mathcal{H}} \\
= & \int_\Omega (i\lambda u - i\Delta u)\, \overline{u^*} dx + \alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 \left(i\lambda y + s^{-1} y_\rho\right) \overline{y} d\rho \right) ds d\Gamma \\
= & \; i \int_\Omega \lambda u \overline{u^*} dx - i \int_\Omega \Delta u \; \overline{u^*} dx + i\lambda\alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 |y|^2 d\rho \right) ds d\Gamma \\
+ & \; \alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(\beta_2(s) \int_0^1 y_\rho \overline{y} d\rho \right) ds d\Gamma \\
= & \; i\lambda \int_\Omega u \overline{u^*} dx - i \int_\Omega u \Delta \overline{u^*} dx + i \int_{\Gamma_N} u \frac{\partial \overline{u^*}}{\partial \nu} d\Gamma - i \int_{\Gamma_N} \frac{\partial u}{\partial \nu} \overline{u^*} d\Gamma \\
+ & \; i\lambda\alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 |y|^2 d\rho \right) ds d\Gamma \\
+ & \; \alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(\beta_2(s) \int_0^1 y_\rho \overline{y} d\rho \right) ds d\Gamma \\
= & \int_\Omega u \left(i\lambda \overline{u^*} - i\Delta \overline{u^*} \right) dx - \beta_1 \int_{\Gamma_N} u \overline{u^*} d\Gamma \\
- & \; i \int_{\Gamma_N} \left(i\beta_1 u - i \int_{\tau_1}^{\tau_2} \beta_2(s) y(\cdot, 1, s) ds \right) \overline{u^*} d\Gamma \\
+ & \; i\lambda\alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 |y|^2 d\rho \right) ds d\Gamma \\
+ & \; \alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(\beta_2(s) \int_0^1 y_\rho \overline{y} d\rho \right) ds d\Gamma \\
= & \int_\Omega u \left(\overline{-i\lambda u^* + i\Delta u^*} \right) dx - \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \beta_2(s) y(\cdot, 1, s) \overline{u^*} ds d\Gamma \\
+ & \; i\lambda\alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 |y|^2 d\rho \right) ds d\Gamma \\
+ & \; \alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(\beta_2(s) \int_0^1 y_\rho \overline{y} d\rho \right) ds d\Gamma \\
= & - \int_\Omega |u|^2 dx - \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \beta_2(s) y(\cdot, 1, s) \overline{u^*} ds d\Gamma \\
+ & \; i\lambda\alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 |y|^2 d\rho \right) ds d\Gamma \\
+ & \; \alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(\beta_2(s) \int_0^1 y_\rho \overline{y} d\rho \right) ds d\Gamma
\end{aligned}
$$

Then using (35) we get

$$\left\langle (i\lambda I - \mathcal{A}) \begin{pmatrix} u \\ y \end{pmatrix}, \begin{pmatrix} u^* \\ \alpha y \end{pmatrix} \right\rangle_{\mathcal{H}} = -\|u\|^2_{\mathcal{H}_0} + \int_{\Gamma_N} \left(\int_{\tau_1}^{\tau_2} \beta_2(s) y(\cdot, 1, s) ds \right) \overline{u^*} d\Gamma + i\lambda\alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 |y|^2 d\rho \right) ds d\Gamma + \alpha \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(\beta_2(s) \int_0^1 y_\rho \overline{y} d\rho \right) ds d\Gamma \tag{41}$$

At this point, we set $\alpha = -\frac{1}{\varepsilon}$, and taking the real part in (41), we obtain

$$\|u\|^2_{\mathcal{H}_0} = -\Re \left\langle F, \begin{pmatrix} u^* \\ -\frac{1}{\varepsilon} y \end{pmatrix} \right\rangle_{\mathcal{H}} + \Re \left(\int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \beta_2(s) y(\cdot, 1, s) \overline{u^*} ds d\Gamma \right) - \Re \left(\frac{1}{\varepsilon} \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(\beta_2(s) \int_0^1 y_\rho \overline{y} d\rho \right) ds d\Gamma \right) \tag{42}$$

Using (40) and the Cauchy-Schwarz inequality we have

$$\begin{aligned} -\Re \left\langle F, \begin{pmatrix} u^* \\ -\frac{1}{\varepsilon} y \end{pmatrix} \right\rangle_{\mathcal{H}} &\leq \|F\|_{\mathcal{H}} \|u^*\|_{\mathcal{H}_0} + \frac{1}{\varepsilon} \|F\|_{\mathcal{H}} \left\| (0, y)^\mathsf{T} \right\|_{\mathcal{H}} \\ &\leq \|F\|_{\mathcal{H}} \|u^*\|_{\mathcal{H}_0} + \frac{1}{\varepsilon} \|F\|_{\mathcal{H}} \left\| (u, y)^\mathsf{T} \right\|_{\mathcal{H}} \\ &\leq C_0 \|F\|_{\mathcal{H}} \|u\|_{\mathcal{H}_0} + \frac{1}{\varepsilon} \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} \end{aligned} \tag{43}$$

From the dissipativity of $\mathcal{A}$, we deduce using (31) and the Cauchy-Schwarz inequality that

$$\left(\beta_1 - \int_{\tau_1}^{\tau_2} \beta_2(s) ds \right) \int_{\Gamma_N} |u|^2 d\Gamma \leq \langle (i\lambda I - \mathcal{A}) U, U \rangle_{\mathcal{H}} \leq \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} \tag{44}$$

Note further that (40) and the dissipativeness of $\mathcal{A}_0$ directly yield

$$\beta_1 \int_{\Gamma_N} |u^*|^2 d\Gamma \leq \Re \langle (i\lambda I - \mathcal{A}_0) u^*, u^* \rangle_{\mathcal{H}_0} \leq \|u\|_{\mathcal{H}_0} \|u^*\|_{\mathcal{H}_0} \leq C_0 \|u\|^2_{\mathcal{H}_0}. \tag{45}$$

Thanks to the Young's inequality, we get

$$\Re \left(\int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \beta_2(s) \overline{u^*} y(\cdot, 1, s) ds d\Gamma \right) \leq \frac{1}{2\varepsilon} \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \beta_2(s) |y(\cdot, 1, s)|^2 ds d\Gamma + \varepsilon \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \beta_2(s) |u^*|^2 ds d\Gamma$$

that is using (45)

$$\Re \left(\int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \beta_2(s) \overline{u^*} y(\cdot, 1, s) ds d\Gamma \right) \leq \frac{1}{2\varepsilon} \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \beta_2(s) |y(\cdot, 1, s)|^2 ds d\Gamma + \frac{\varepsilon C_0 \int_{\tau_1}^{\tau_2} \beta_2(s) ds}{\beta_1} \|u\|^2_{\mathcal{H}_0} \tag{46}$$

Furthermore, we have

$$\begin{aligned}
-\Re\left(\frac{1}{\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\left(\beta_2(s)\int_0^1 y_\rho\overline{y}d\rho\right)dsd\Gamma\right) &= -\frac{1}{2\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)\left[|y|^2\right]_0^1 dsd\Gamma \\
&= -\frac{1}{2\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)\left|y(\cdot,1,s)\right|^2 dsd\Gamma \\
&+ \frac{1}{2\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)\left|y(\cdot,0,s)\right|^2 dsd\Gamma \\
&= -\frac{1}{2\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)\left|y(\cdot,1,s)\right|^2 dsd\Gamma \\
&+ \frac{1}{2\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)\left|u\right|^2 dsd\Gamma \\
&= -\frac{1}{2\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)\left|y(\cdot,1,s)\right|^2 dsd\Gamma \\
&+ \frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)ds\int_{\Gamma_N}\left|u\right|^2 d\Gamma
\end{aligned}$$

that is, using (44)

$$\begin{aligned}
-\Re\left(\frac{1}{\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\left(\beta_2(s)\int_0^1 y_\rho\overline{y}d\rho\right)dsd\Gamma\right) &\leq -\frac{1}{2\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)\left|y(\cdot,1,s)\right|^2 dsd\Gamma \\
&+ \frac{\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1-\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\left\|F\right\|_{\mathcal{H}}\left\|U\right\|_{\mathcal{H}}.
\end{aligned}$$

(47)

Now adding (47), (46) and (43) one gets

$$\begin{aligned}
\left\|u\right\|_{\mathcal{H}_0}^2 &\leq C_0\left\|F\right\|_{\mathcal{H}}\left\|u\right\|_{\mathcal{H}_0}+\frac{1}{\varepsilon}\left\|F\right\|_{\mathcal{H}}\left\|U\right\|_{\mathcal{H}}+\frac{1}{2\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)\left|y(\cdot,1,s)\right|^2 dsd\Gamma \\
&+ \frac{\varepsilon C_0\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{\beta_1}\left\|u\right\|_{\mathcal{H}_0}^2-\frac{1}{2\varepsilon}\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\beta_2(s)\left|y(\cdot,1,s)\right|^2 dsd\Gamma \\
&+ \frac{\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1-\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\left\|F\right\|_{\mathcal{H}}\left\|U\right\|_{\mathcal{H}}
\end{aligned}$$

that is

$$\begin{aligned}
\left\|u\right\|_{\mathcal{H}_0}^2 &\leq C_0\left\|F\right\|_{\mathcal{H}}\left\|u\right\|_{\mathcal{H}_0}+\frac{1}{\varepsilon}\left\|F\right\|_{\mathcal{H}}\left\|U\right\|_{\mathcal{H}}+\frac{\varepsilon C_0\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{\beta_1}\left\|u\right\|_{\mathcal{H}_0}^2 \\
&+ \frac{\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1-\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\left\|F\right\|_{\mathcal{H}}\left\|U\right\|_{\mathcal{H}}
\end{aligned}$$

At this level we chose ε sufficiently small such that $\varepsilon \ll \dfrac{\beta_1}{C_0 \int_{\tau_1}^{\tau_2} \beta_2(s) ds}$ and we notice that $\|u\|_{\mathcal{H}_0}^2 \leq \|U\|_{\mathcal{H}}$. So the above relation becomes

$$\|u\|_{\mathcal{H}_0}^2 \leq \left(C_0 + \frac{1}{\varepsilon} + \frac{\int_{\tau_1}^{\tau_2} \beta_2(s) ds}{2\varepsilon \left(\beta_1 - \int_{\tau_1}^{\tau_2} \beta_2(s) ds \right)} \right) \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} \tag{48}$$

from which follows

$$\|U\|_{\mathcal{H}}^2 \leq \left(C_0 + \frac{1}{\varepsilon} + \frac{\int_{\tau_1}^{\tau_2} \beta_2(s) ds}{2\varepsilon \left(\beta_1 - \int_{\tau_1}^{\tau_2} \beta_2(s) ds \right)} \right) \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 |y(\cdot,\rho,s)|^2 d\rho \right) ds d\Gamma. \tag{49}$$

Now we need a best estimation for $\int_{\Gamma_N} \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 |y(\cdot,\rho,s)|^2 d\rho \right) ds d\Gamma$.

Following (31) and solving the next Cauchy problem (50)

$$\begin{cases} s^{-1} y_\rho + i\lambda y = h \\ y(\cdot, 0, s) = u \end{cases} \tag{50}$$

we obtain

$$y(\cdot,\rho,s) = u e^{-i\lambda s\rho} + s \int_0^\rho e^{-i\lambda s(\rho-\sigma)} h(\cdot,\sigma,s)\, d\sigma, \quad \forall\, \rho \in (0,1)\,. \tag{51}$$

Using the triangular inequality, it follows from (51) that

$$|y(\cdot,\rho,s)| \leq |u| + s \int_0^\rho |h(\cdot,\sigma,s)|\, d\sigma, \quad \forall\, \rho \in (0,1)\,,$$

that is

$$|y(\cdot,\rho,s)|^2 \leq |u|^2 + s^2 \left(\int_0^\rho |h(\cdot,\sigma,s)|\, d\sigma \right)^2 + 2|u|\, s \left(\int_0^\rho |h(\cdot,\sigma,s)|\, d\sigma \right), \quad \forall\, \rho \in (0,1)\,. \tag{52}$$

On the one hand, by Cauchy-Schwarz's inequality we obtain

$$\begin{aligned} \left(\int_0^\rho |h(\cdot,\sigma,s)|\, d\sigma \right)^2 &\leq \left(\int_0^\rho |h(\cdot,\sigma,s)|^2 d\sigma \right) \left(\int_0^\rho d\sigma \right) \\ &\leq \rho \int_0^\rho |h(\cdot,\sigma,s)|^2 d\sigma \\ &\leq \int_0^\rho |h(\cdot,\sigma,s)|^2 d\sigma; \end{aligned}$$

that is

$$\left(\int_0^\rho |h(\cdot,\sigma,s)|\, d\sigma \right)^2 \leq \int_0^\rho |h(\cdot,\sigma,s)|^2 d\sigma. \tag{53}$$

On the other hand Young's inequality guarantees that

$$2|u|\, s \left(\int_0^\rho |h(\cdot,\sigma,s)|\, d\sigma \right) \leq |u|^2 + s^2 \left(\int_0^\rho |h(\cdot,\sigma,s)|\, d\sigma \right)^2. \tag{54}$$

A combination of (54), (53) and (52) gives

$$|y(\cdot,\rho,s)|^2 \leq 2\,|u|^2 + 2s^2 \int_0^{\rho} |h(\cdot,\sigma,s)|^2\, d\sigma. \tag{55}$$

Integrating (55) on $\Gamma_N \times (\tau_1,\tau_2) \times (0,1)$ yields

$$\begin{aligned}
\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1 |y(\cdot,\rho,s)|^2 d\rho\right) ds d\Gamma \ &\leq\ 2\int_{\tau_1}^{\tau_2} s\beta_2(s)ds \int_{\Gamma_N} |u|^2\, d\Gamma \\
&+\ 2\int_{\Gamma_N}\int_{\tau_1}^{\tau_2} s^3\beta_2(s)\int_0^1 |h(\cdot,\rho,s)|^2\, ds d\rho d\Gamma \\
&\leq\ 2\tau_2 \int_{\tau_1}^{\tau_2} \beta_2(s)ds \int_{\Gamma_N} |u|^2\, d\Gamma \\
&+\ 2\tau_2^2 \int_{\Gamma_N}\int_{\tau_1}^{\tau_2} s\beta_2(s)\int_0^1 |h(\cdot,\rho,s)|^2\, ds d\rho d\Gamma.
\end{aligned}$$

Then using (44) and the $\mathcal{H}$-norm definition, the above relation can be rewritten as

$$\int_{\Gamma_N}\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1 |y(\cdot,\rho,s)|^2 d\rho\right) ds d\Gamma \leq \left(\frac{2\tau_2\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{\beta_1 - \displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}\right)\|F\|_{\mathcal{H}}\,\|U\|_{\mathcal{H}} + 2\tau_2^2\,\|F\|_{\mathcal{H}}^2\,. \tag{56}$$

Inserting (56) in (49) leads to

$$\|U\|_{\mathcal{H}}^2 \leq \left(C_0 + \frac{1}{\varepsilon} + \frac{4\varepsilon\tau_2\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1 - \displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\right)\|F\|_{\mathcal{H}}\,\|U\|_{\mathcal{H}} + 2\tau_2^2\,\|F\|_{\mathcal{H}}^2$$

that is

$$\|U\|_{\mathcal{H}}^2 \leq C_\varepsilon\,\|F\|_{\mathcal{H}}\,\|U\|_{\mathcal{H}} + 2\tau_2^2\,\|F\|_{\mathcal{H}}^2 \tag{57}$$

where C_ε independent on λ is set as

$$C_\varepsilon = C_0 + \frac{1}{\varepsilon} + \frac{4\varepsilon\tau_2\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1 - \displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}. \tag{58}$$

Applying Young's inequality to (57) it follows that

$$\|U\|_{\mathcal{H}}^2 \ \leq\ \frac{C_\varepsilon}{2\varepsilon'}\,\|F\|_{\mathcal{H}}^2 + \frac{\varepsilon' C_\varepsilon}{2}\,\|U\|_{\mathcal{H}}^2 + 2\tau_2^2\,\|F\|_{\mathcal{H}}^2\,, \quad \text{with } \varepsilon' > 0. \tag{59}$$

Choosing ε' sufficiently small such that $\varepsilon' \ll \dfrac{2}{C_\varepsilon}$, (59) becomes

$$\|U\|_{\mathcal{H}}^2 \leq \left(\frac{C_\varepsilon}{2\varepsilon'} + 2\tau_2^2\right)\|F\|_{\mathcal{H}}^2\,. \tag{60}$$

Finally (60) directly leads to (32) with

$$C = \sqrt{\frac{C_\varepsilon}{2\varepsilon'} + 2\tau_2^2}. \tag{61}$$

That means the resolvent of $\mathcal{A}$ is uniformly bounded on the imaginary axis. The proof of theorem 3.2 is thus completed. □

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