DOI: 10.28919/ejma.2022.2.14

Eur. J. Math. Appl. (2022)2:14
URL: http://ejma.euap.org

K-g-FRAME IN CARTESIAN PRODUCT OF TWO HILBERT SPACES

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Abstract. The concept of K-g-frame in Cartesian product of two Hilbert spaces is being studied. We will see that the Cartesian product of two K-g-frames is also a K-g-frame. The concept of K-g-frame operator on Cartesian product of two Hilbert spaces is being presented and results of it are being established. Finally, we give a perturbation result on K-g-frame in Cartesian product of two Hilbert spaces.

1. Introduction

In 1952, Duffin and Schaeffer [3] introduced the notion of frame in Hilbert space. Later on, after some innovative work of Daubechies, Grossman, Meyer [4], the theory of frames began to be studied more widely.

The theory of frame has been generalized rapidly and various generalizations of frame in Hilbert space namely, K-frame [5], G-frame [9], fusion frame [2] etc. have been introduced in recent times. K-frame was introduced by L. Gavruta and it is a natural generalization of the frame in Hilbert space. D. L. Hua et al. [7] studied K-g-frame by combing K-frame and g-frame. Frame theory has so many application in data processing, coding theory, signal processing and so on.

In this paper, we present K-g-frame in Cartesian product of two Hilbert spaces and establish some of its properties. It is verified that the Cartesian product of two K-g-frames is a K-g-frame. An interesting topic in frame theory is a perturbation of K-g-frame. In this aspects, a perturbation result on K-g-frame in Cartesian product of two Hilbert spaces is studied.

Throughout this paper, H and X are considered to be separable Hilbert spaces with associated inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$. $\mathcal{B}(H, X)$ is a collection of all bounded linear operators from H to X. In particular $\mathcal{B}(H)$ denote the space of all bounded linear operators on H. $\{H_i\}_{i \in I}$ and $\{K_i\}_{i \in I}$ are sequences of Hilbert spaces, where I is the subset of integers $\mathbb{Z}$. Define the space

$$l^2\left(\{H_i\}_{i \in I}\right) = \left\{\{f_i\}_{i \in I} : f_i \in H_i, \sum_{i \in I} \|f_i\|_1^2 < \infty\right\}$$

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Key words and phrases. frame; g-frame; K-frame; K-g-frame; frame operator.
Received 21/04/2022.

with inner product is given by

$$\langle \{ f_i \}_{i \in I}, \{ g_i \}_{i \in I} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle_1 .$$

Clearly $l^2(\{ H_i \}_{i \in I})$ is a Hilbert space with the above inner product [2]. Similarly, we can define the space $l^2(\{ K_i \}_{i \in I})$.

2. Preliminaries

Definition 2.1. *[9] A sequence $\{ \Lambda_i \in \mathcal{B}(H, H_i) : i \in I \}$ is called a generalized frame or g-frame for H with respect to $\{ H_i \}_{i \in I}$ if there exist two positive constants A and B such that*

$$A \| f \|_1^2 \leq \sum_{i \in I} \| \Lambda_i f \|_1^2 \leq B \| f \|_1^2 \quad \forall f \in H. \tag{1}$$

A and B are called the lower and upper bounds of g-frame, respectively. If the sequence $\{ \Lambda_i \}_{i \in I}$ satisfying only right inequality of (1), it is is called a g-Bessel sequence.

Definition 2.2. *[9] Let $\{ \Lambda_i \}_{i \in I}$ be a g-frame for H. Then the synthesis operator $T_\Lambda : l^2(\{ H_i \}_{i \in I}) \to H$, is defined as*

$$T_\Lambda (\{ f_i \}_{i \in I}) = \sum_{i \in I} \Lambda_i^* f_i \quad \forall \{ f_i \}_{i \in I} \in l^2(\{ H_i \}_{i \in I})$$

and the analysis operator is given by

$$T_\Lambda^* : H \to l^2(\{ H_i \}_{i \in I}), \ T_\Lambda^* f = \{ \Lambda_i f \}_{i \in I} \quad \forall f \in H.$$

The g-frame operator $S_\Lambda : H \to H$ is defined as follows:

$$S_\Lambda f = \sum_{i \in I} \Lambda_i^* \Lambda_i f \quad \forall f \in H.$$

Definition 2.3. *[1] Let $K \in \mathcal{B}(H)$. Then a sequence $\{ \Lambda_i \}_{i \in I}$ is called a K-g-frame for H with respect to $\{ H_i \}_{i \in I}$ if there exist two positive constants A and B such that*

$$A \| K^* f \|_1^2 \leq \sum_{i \in I} \| \Lambda_i f \|_1^2 \leq B \| f \|_1^2 \quad \forall f \in H.$$

Theorem 2.4. *[1] Let $K \in \mathcal{B}(H)$. Then the following statements are equivalent:*

(I) $\{ \Lambda_i \}_{i \in I}$ is a K-g-frame for H with respect to $\{ H_i \}_{i \in I}$.

(II) $\{ \Lambda_i \}_{i \in I}$ is a g-Bessel sequence in H with respect to $\{ H_i \}_{i \in I}$ and there exists a g-Bessel sequence $\{ \Lambda_i' \}_{i \in I}$ in H with respect to $\{ H_i \}_{i \in I}$ such that

$$K f = \sum_{i \in I} \Lambda_i^* \Lambda_i' f \quad \forall f \in H.$$

Theorem 2.5. *[8] The set $\mathcal{S}(H)$ of all self-adjoint operators on H is a partially ordered set with respect to the partial order $\leq$ which is defined as for $T, S \in \mathcal{S}(H)$*

$$T \leq S \Leftrightarrow \langle T f, f \rangle_1 \leq \langle S f, f \rangle_1 \quad \forall f \in H.$$

Let H and X be two Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$. Then the space define by $H \oplus X = \{ f \oplus g = (f, g) : f \in H, g \in X \}$ is a linear space with respect to the addition and scalar multiplication defined by

$$(f_1, g_1) + (f_2, g_2) = (f_1 + f_2, g_1 + g_2), \text{ and}$$

$$\lambda (f, g) = (\lambda f, \lambda g) \ \forall \ f, f_1, f_2 \in H, \ \ g, g_1, g_2 \in X \text{ and } \lambda \in \mathbb{K}.$$

Now, $H \oplus X$ is an inner product space with respect to the inner product given by

$$\langle (f \oplus g), (f' \oplus g') \rangle = \langle f, f' \rangle_1 + \langle g, g' \rangle_2 \ \ \forall f, f' \in H \ \text{ and } \forall g, g' \in X.$$

The norm on $H \oplus X$ is defined by

$$\| f \oplus g \| = \| f \|_1 + \| g \|_2 \ \ \forall \ f \in H, \ g \in X,$$

where $\| \cdot \|_1$ and $\| \cdot \|_2$ are norms generated by $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively.
The space $H \oplus X$ is complete with respect to the above inner product. Therefore the space $H \oplus X$ is a Hilbert space.

Note 2.6. *[6] Let $U \in \mathcal{B}(H)$, $V \in \mathcal{B}(X)$. Then for all $f \in H$, $g \in X$, define*

$$U \oplus V \in \mathcal{B}(H \oplus X) \ \text{ by } \ (U \oplus V)(f \oplus g) = (U f, V g), \text{ and}$$

$$(U \oplus V)^* (f \oplus g) = (U^* f, V^* g).$$

Furthermore, if U, V and $(U \oplus V)$ are invertible operators, then we define

$$(U \oplus V)^{-1} (f \oplus g) = \left(U^{-1} f, V^{-1} g \right).$$

Theorem 2.7. *[6] Suppose $U, U' \in \mathcal{B}(H)$ and $V, V' \in \mathcal{B}(X)$. Then*

- *(I)* $(U + U') \oplus V = U \oplus V + U' \oplus V$, $\lambda U \oplus \lambda V = \lambda (U \oplus V)$ *and* $U \oplus (V + V') = U \oplus V + U \oplus V'$.
- *(II)* $I_H \oplus I_X = I_{H \oplus X}$, *where* I_H, I_X *and* $I_{H \oplus X}$ *are identity operators on* H, X *and* $H \oplus X$, *respectively.*
- *(III)* $(U \oplus V)(U' \oplus V') = (U U' \oplus V V')$.
- *(IV)* $(U \oplus V)^* = U^* \oplus V^*$.
- *(V)* *If* U *and* V *are invertible, then* $(U \oplus V)$ *is invertible and moreover* $(U \oplus V)^{-1} = U^{-1} \oplus V^{-1}$.

3. K-g-FRAME IN $H \oplus X$

In this section, we study K-g-frame in $H \oplus X$ and establish some results.

Definition 3.1. *Let $K_1 \in \mathcal{B}(H)$ and $K_2 \in \mathcal{B}(X)$ be two operators. Then the family $\{ \Lambda_i \oplus \Gamma_i \in \mathcal{B}(H \oplus X, H_i \oplus K_i) \}_{i \in I}$ is said to be a $K_1 \oplus K_2$-g-frame for $H \oplus X$ with respect to $\{ H_i \oplus K_i \}_{i \in I}$, if there exist constants $0 < A \leq B < \infty$ such that*

$$A \, \| (K_1 \oplus K_2)^* (f \oplus g) \|^2 \leq \sum_{i \in I} \| (\Lambda_i \oplus \Gamma_i)(f \oplus g) \|^2 \leq B \, \| f \oplus g \|^2, \tag{2}$$

for all $f \oplus g \in H \oplus X$. The constants A and B are called the frame bounds. If $A = B$ then it is called a tight $K_1 \oplus K_2$-g-frame. If the family $\{ \Lambda_i \oplus \Gamma_i \}_{i \in I}$ satisfies only the right inequality of (2) then it is called a $K_1 \oplus K_2$-g-Bessel sequence in $H \oplus X$ with bound B.

Definition 3.2. *Define* $l^2\left(\{H_i \oplus K_i\}_{i\in I}\right)$

$$= \left\{ \{f_i \oplus g_i\}_{i\in I} : f_i \in H_i,\, g_i \in K_i,\ and\ \sum_i \|f_i \oplus g_i\|^2 < \infty \right\}$$

with inner product

$$\begin{aligned}
&\left\langle \{f_i \oplus g_i\}_{i\in I}, \{f_i' \oplus g_i'\}_{i\in I} \right\rangle_{l^2} \\
&= \sum_{i\in I} \langle (f_i \oplus g_i), (f_i' \oplus g_i') \rangle \\
&= \sum_{i\in I} \left[\langle f_i, f_i' \rangle_{H_i} + \langle g_i, g_i' \rangle_{K_i} \right] \\
&= \sum_{i\in I} \langle f_i, f_i' \rangle_{H_i} + \sum_{i\in I} \langle g_i, g_i' \rangle_{K_i} \\
&= \left\langle \{f_i\}_{i\in I}, \{f_i'\}_{i\in I} \right\rangle_{l^2\left(\{H_i\}_{i\in I}\right)} + \left\langle \{g_i\}_{i\in I}, \{g_i'\}_{i\in I} \right\rangle_{l^2\left(\{K_i\}_{i\in I}\right)}.
\end{aligned}$$

The space $l^2(\{H_i \oplus K_i\})$ *is complete with respect to the above inner product. Therefore the space* $l^2(\{H_i \oplus K_i\})$ *is a Hilbert space.*

In the following theorem, we show a sufficient condition for a Cartesian product of K-g-frames be also a K-g-frame.

Theorem 3.3. *If* $\{\Lambda_i\}_{i\in I}$ *is a* K_1-g-*frame for* H *and* $\{\Gamma_i\}_{i\in I}$ *is a* K_2-g-*frame for* X, *then* $\{\Lambda_i \oplus \Gamma_i\}_{i\in I}$ *is a* $K_1 \oplus K_2$-g-*frame for* $H \oplus X$.

Proof. Since $\{\Lambda_i\}_{i\in I}$ is a K_1-g-frame for H and $\{\Gamma_i\}_{i\in I}$ is a K_2-g-frame for X, there exist positive constants (A, B) and (C, D) such that

$$A\,\|K_1^* f\|_1^2 \le \sum_{i\in I} \|\Lambda_i f\|_1^2 \le B\,\|f\|_1^2 \quad \forall f \in H \tag{3}$$

$$C\,\|K_2^* g\|_2^2 \le \sum_{i\in I} \|\Gamma_i g\|_2^2 \le D\,\|g\|_2^2 \quad \forall g \in X. \tag{4}$$

Now, for each $f \oplus g \in H \oplus X$, we have

$$\begin{aligned}
\sum_{i\in I} \|(\Lambda_i \oplus \Gamma_i)(f \oplus g)\|^2 &= \sum_{i\in I} \langle (\Lambda_i \oplus \Gamma_i)(f \oplus g), (\Lambda_i \oplus \Gamma_i)(f \oplus g) \rangle \\
&= \sum_{i\in I} \langle (\Lambda_i f \oplus \Gamma_i g), (\Lambda_i f \oplus \Gamma_i g) \rangle \\
&= \sum_{i\in I} \{ \langle \Lambda_i f, \Lambda_i f \rangle_1 + \langle \Gamma_i g, \Gamma_i g \rangle_2 \} \\
&= \sum_{i\in I} \|\Lambda_i f\|_1^2 + \sum_{i\in I} \|\Gamma_i g\|_2^2 \qquad (5) \\
&\le B\,\|f\|_1^2 + D\,\|g\|_2^2 \ [\text{by (3) and (4)}] \\
&\le \max\{B, D\}\,\{\|f\|_1^2 + \|g\|_2^2\} = \max\{B, D\}\,\|f \oplus g\|^2.
\end{aligned}$$

On the other hand, from (5), we have

$$\sum_{i\in I}\|(\Lambda_i\oplus\Gamma_i)(f\oplus g)\|^2=\sum_{i\in I}\|\Lambda_i f\|_1^2+\sum_{i\in I}\|\Gamma_i g\|_2^2$$
$$\geq A\,\|K_1^* f\|_1^2+C\,\|K_2^* g\|_2^2\ [\text{ by } (3) \text{ and } (4)\ ]$$
$$\geq \min\{A,C\}\,\{\|K_1^* f\|_1^2+\|K_2^* g\|_2^2\}=\min\{A,C\}\,\|(K_1^* f\oplus K_2^* g)\|^2$$
$$=\min\{A,C\}\,\|(K_1\oplus K_2)^*(f\oplus g)\|^2.$$

Thus, $\{\Lambda_i\oplus\Gamma_i\}_{i\in I}$ is a $K_1\oplus K_2$-g-frame for $H\oplus X$ with bounds $\min\{A,C\}$ and $\max\{B,D\}$. □

Note 3.4. *Let $\{\Lambda_i\oplus\Gamma_i\}_{i\in I}$ be a $K_1\oplus K_2$-g-frame for $H\oplus X$. According to the definition (2.2), the synthesis operator $T_{\Lambda\oplus\Gamma}: l^2\left(\{H_i\oplus K_i\}_{i\in I}\right)\to H\oplus X$ is described by*

$$T_{\Lambda\oplus\Gamma}\left(\{f_i\oplus g_i\}_{i\in I}\right)=\sum_{i\in I}(\Lambda_i\oplus\Gamma_i)^*(f_i\oplus g_i)$$

for all $\{f_i\oplus g_i\}_{i\in I}\in l^2\left(\{H_i\oplus K_i\}_{i\in I}\right)$, and the corresponding frame operator $S_{\Lambda\oplus\Gamma}: H\oplus X\to H\oplus X$ is given by

$$S_{\Lambda\oplus\Gamma}(f\oplus g)=\sum_{i\in I}(\Lambda_i\oplus\Gamma_i)^*(\Lambda_i\oplus\Gamma_i)(f\oplus g)\ \forall\,(f\oplus g)\in H\oplus X.$$

Proposition 3.5. *Let $\{\Lambda_i\oplus\Gamma_i\}_{i\in I}$ be a $K_1\oplus K_2$-g-frame for $H\oplus X$ with the synthesis operator $T_{\Lambda\oplus\Gamma}$ and the corresponding frame operator $S_{\Lambda\oplus\Gamma}$. Then*

$$S_{\Lambda\oplus\Gamma}=S_\Lambda\oplus S_\Gamma,\ S_{\Lambda\oplus\Gamma}^{-1}=S_\Lambda^{-1}\oplus S_\Gamma^{-1}\ \text{ and}$$

$$T_{\Lambda\oplus\Gamma}=T_\Lambda\oplus T_\Gamma,\ T_{\Lambda\oplus\Gamma}^*=T_\Lambda^*\oplus T_\Gamma^*$$

Proof. For each $f\oplus g\in H\oplus X$, we have

$$S_{\Lambda\oplus\Gamma}(f\oplus g)=\sum_{i\in I}(\Lambda_i\oplus\Gamma_i)^*(\Lambda_i\oplus\Gamma_i)(f\oplus g)$$
$$=\sum_{i\in I}(\Lambda_i^*\Lambda_i\oplus\Gamma_i^*\Gamma_i)(f\oplus g)$$
$$=\left(\sum_{i\in I}\Lambda_i^*\Lambda_i f\right)\oplus\left(\sum_{i\in I}\Gamma_i^*\Gamma_i g\right)$$
$$=S_\Lambda f\oplus S_\Gamma g=(S_\Lambda\oplus S_\Gamma)(f\oplus g)$$

This shows that $S_{\Lambda\oplus\Gamma}=S_\Lambda\oplus S_\Gamma$. Since S_Λ and S_Γ are invertible, by (V) of Theorem 2.7, $S_{\Lambda\oplus\Gamma}$ is also invertible and $S_{\Lambda\oplus\Gamma}^{-1}=(S_\Lambda\oplus S_\Gamma)^{-1}=S_\Lambda^{-1}\oplus S_\Gamma^{-1}$.

On the other hand, for all $\{f_i \oplus g_i\}_{i \in I} \in l^2\left(\{H_i \oplus K_i\}_{i \in I}\right)$, we have

$$\begin{aligned} T_{\Lambda \oplus \Gamma}\left(\{f_i \oplus g_i\}_{i \in I}\right) &= \sum_{i \in I} (\Lambda_i \oplus \Gamma_i)^* (f_i \oplus g_i) \\ &= \sum_{i \in I} (\Lambda_i^* \oplus \Gamma_i^*) (f_i \oplus g_i) \\ &= \left(\sum_{i \in I} \Lambda_i^* f_i\right) \oplus \left(\sum_{i \in I} \Gamma_i^* g_i\right) \\ &= (T_\Lambda \oplus T_\Gamma)\left(\{f_i \oplus g_i\}_{i \in I}\right) \end{aligned}$$

Hence, $T_{\Lambda \oplus \Gamma} = T_\Lambda \oplus T_\Gamma$ and therefore by (IV) of Theorem 2.7, $T^*_{\Lambda \oplus \Gamma} = T^*_\Lambda \oplus T^*_\Gamma$. □

Theorem 3.6. *Let $\{\Lambda_i\}_{i \in I}$ be a K_1-g-frame for H with bounds A, B and $\{\Gamma_i\}_{i \in I}$ be a K_2-g-frame for X with bounds C, D having their associated frame operators S_Λ and S_Γ, respectively. Then*

$$\min(A, C)\,(K_1 \oplus K_2)\,(K_1 \oplus K_2)^* \leq S_{\Lambda \oplus \Gamma} \leq \max(B, D)\, I_{H \oplus X}.$$

Proof. Since S_Λ and S_Γ are the frame operators for $\{\Lambda_i\}_{i \in I}$ and $\{\Gamma_i\}_{i \in I}$, respectively,

$$A\,\|K_1^* f\|_1^2 \leq \langle S_\Lambda f, f\rangle_1 \leq B\,\|f\|_1^2 \;\; \forall f \in H, \text{ and}$$

$$C\,\|K_2^* g\|_2^2 \leq \langle S_\Gamma g, g\rangle_2 \leq D\,\|g\|_2^2 \;\; \forall g \in X.$$

Adding above two inequalities, we get

$$\begin{aligned} & A\,\|K_1^* f\|_1^2 + C\,\|K_2^* g\|_2^2 \leq \langle S_\Lambda f, f\rangle_1 + \langle S_\Gamma g, g\rangle_2 \leq B\,\|f\|_1^2 + D\,\|g\|_2^2 \\ & \Rightarrow E\left\{\|K_1^* f\|_1^2 + \|K_2^* g\|_2^2\right\} \leq \langle S_\Lambda f, f\rangle_1 + \langle S_\Gamma g, g\rangle_2 \leq F\left\{\|f\|_1^2 + \|g\|_2^2\right\} \\ & \Rightarrow E\,\|(K_1 \oplus K_2)^* (f \oplus g)\|^2 \leq \langle S_\Lambda f, f\rangle_1 + \langle S_\Gamma g, g\rangle_2 \leq F\,\|f \oplus g\|^2 \end{aligned}$$

where $E = \min(A, C)$ and $F = \max(B, D)$. Thus, for each $f \oplus g \in H \oplus X$,

$$\begin{aligned} & E\,\langle (K_1 \oplus K_2)\,(K_1 \oplus K_2)^* (f \oplus g), (f \oplus g)\rangle \leq \langle (S_\Lambda f \oplus S_\Gamma g), (f \oplus g)\rangle \\ & \qquad \leq F\,\langle (f \oplus g), (f \oplus g)\rangle \\ & \Rightarrow E\,\langle (K_1 \oplus K_2)\,(K_1 \oplus K_2)^* (f \oplus g), (f \oplus g)\rangle \\ & \qquad \leq \langle (S_\Lambda \oplus S_\Gamma)\,(f \oplus g), (f \oplus g)\rangle \\ & \qquad \leq F\,\langle (I_H \oplus I_X)\,(f \oplus g), (f \oplus g)\rangle \end{aligned}$$

Now, according to the Theorem (2.5), we can write

$$\min(A, C)\,(K_1 \oplus K_2)\,(K_1 \oplus K_2)^* \leq S_{\Lambda \oplus \Gamma} \leq \max(B, D)\, I_{H \oplus X}.$$

□

Theorem 3.7. *Let $\{\Lambda_i\}_{i \in I}$ be a g-frame for H with bounds A, B and $\{\Gamma_i\}_{i \in I}$ be a g-frame for X with bounds C, D. Suppose $K_1 \in \mathcal{B}(H)$ and $K_2 \in \mathcal{B}(X)$. Then $\{\Lambda_i \oplus \Gamma_i\}_{i \in I}$ is a $K_1 \oplus K_2$-g-frame for $H \oplus X$.*

Proof. By theorem 3.3, $\{\Lambda_i \oplus \Gamma_i\}_{i\in I}$ is a g-frame for $H \oplus X$ with bounds $\min\{A, C\}$ and $\max\{B, D\}$. Now, for each $f \oplus g \in H \oplus X$, we have

$$\begin{aligned}
&\frac{\min\{A, C\}}{max\{\|K_1\|^2, \|K_2\|^2\}} \|(K_1 \oplus K_2)^* (f \oplus g)\|^2 \\
&= \frac{\min\{A, C\}}{max\{\|K_1\|^2, \|K_2\|^2\}} \{\|K_1^* f\|_1^2 + \|K_2^* g\|_2^2\} \\
&\le \frac{\min\{A, C\}}{max\{\|K_1\|^2, \|K_2\|^2\}} \{\|K_1\|^2 \|f\|_1^2 + \|K_2\|^2 \|g\|_2^2\} \\
&\le \min\{A, C\} \{\|f\|_1^2 + \|g\|_2^2\} = \min\{A, C\} \|f \oplus g\|^2 \\
&\le \sum_{i\in I} \|(\Lambda_i \oplus \Gamma_i)(f \oplus g)\|^2 \le \max\{B, D\} \|f \oplus g\|^2.
\end{aligned}$$

Thus, the family $\{\Lambda_i \oplus \Gamma_i\}_{i\in I}$ is a $K_1 \oplus K_2$-g-frame for $H \oplus X$ with bounds $\min\{A, C\} / max\{\|K_1\|^2, \|K_2\|^2\}$ and $\max\{B, D\}$. □

Theorem 3.8. *Let $\{\Lambda_i\}_{i\in I}$ be a K_1-g-frame for H with bounds A, B and $\{\Gamma_i\}_{i\in I}$ be a K_2-g-frame for X with bounds C, D having their associated frame operators S_Λ and S_Γ, respectively. Suppose $T_1 \in \mathcal{B}(H)$ and $T_2 \in \mathcal{B}(X)$. Then $\{(\Lambda_i \oplus \Gamma_i)(T_1 \oplus T_2)^*\}_{i\in I}$ is a $(T_1 \oplus T_2)(K_1 \oplus K_2)$-$g$-frame for $H \oplus X$.*

Proof. By theorem 3.3, $\{\Lambda_i \oplus \Gamma_i\}_{i\in I}$ is a $K_1 \oplus K_2$-g-frame for $H \oplus X$ with bounds $\min\{A, C\}$ and $\max\{B, D\}$. Now, for each $f \oplus g \in H \oplus X$, we have

$$\begin{aligned}
&\min\{A, C\} \|[(T_1 \oplus T_2)(K_1 \oplus K_2)]^* (f \oplus g)\|^2 \\
&= \min\{A, C\} \|(K_1 \oplus K_2)^* (T_1 \oplus T_2)^* (f \oplus g)\|^2 \\
&= \min\{A, C\} \|K_1^* T_1^* f \oplus K_2^* T_2^* g\|^2 \\
&= \min\{A, C\} \{\|K_1^* T_1^* f\|_1^2 + \|K_2^* T_2^* g\|_2^2\} \\
&\le A \|K_1^* T_1^* f\|_1^2 + C \|K_2^* T_2^* g\|_2^2 \\
&\le \sum_{i\in I} \|\Lambda_i T_1^* f\|_1^2 + \sum_{i\in I} \|\Gamma_i T_2^* g\|_2^2 = \sum_{i\in I} \|\Lambda_i T_1^* f \oplus \Gamma_i T_2^* g\|^2 \\
&= \sum_{i\in I} \|(\Lambda_i \oplus \Gamma_i)(T_1 \oplus T_2)^* (f \oplus g)\|^2.
\end{aligned}$$

On the other hand, for each $f \oplus g \in H \oplus X$, we have

$$\begin{aligned}
&\sum_{i\in I} \|(\Lambda_i \oplus \Gamma_i)(T_1 \oplus T_2)^* (f \oplus g)\|^2 \\
&= \sum_{i\in I} \|\Lambda_i T_1^* f\|_1^2 + \sum_{i\in I} \|\Gamma_i T_2^* g\|_2^2 \\
&\le B \|T_1^* f\|_1^2 + D \|T_2^* g\|_2^2 \\
&\le \max\{B \|T_1\|^2, D \|T_2\|^2\} \{\|f\|_1^2 + \|g\|_2^2\} \\
&= \max\{B \|T_1\|^2, D \|T_2\|^2\} \|f \oplus g\|^2.
\end{aligned}$$

Thus, $\{(\Lambda_i \oplus \Gamma_i)(T_1 \oplus T_2)^*\}_{i\in I}$ is a $(T_1 \oplus T_2)(K_1 \oplus K_2)$-$g$-frame for $H \oplus X$ with bounds $\min\{A, C\}$ and $\max\{B \|T_1\|^2, D \|T_2\|^2\}$. □

Remark 3.9. *Let By Theorem 2.7, for each $f \oplus g \in H \oplus X$, we have*

$$\begin{aligned}
&\sum_{i \in I} [(\Lambda_i \oplus \Gamma_i)(T_1 \oplus T_2)^*]^* (\Lambda_i \oplus \Gamma_i)(T_1 \oplus T_2)^* (f \oplus g) \\
&= \sum_{i \in I} (T_1 \oplus T_2)(\Lambda_i^* \oplus \Gamma_i^*)(\Lambda_i \oplus \Gamma_i)(T_1^* \oplus T_2^*)(f \oplus g) \\
&= \sum_{i \in I} (T_1 \Lambda_i^* \Lambda_i T_1^* \oplus T_2 \Gamma_i^* \Gamma_i T_2^*)(f \oplus g) \\
&= \left(\sum_{i \in I} T_1 \Lambda_i^* \Lambda_i T_1^* f \right) \oplus \left(\sum_{i \in I} T_2 \Gamma_i^* \Gamma_i T_2^* g \right) \\
&= T_1 S_\Lambda T_1^* f \oplus T_2 S_\Gamma T_2^* g \\
&= (T_1 \oplus T_2)(S_\Lambda \oplus S_\Gamma)(T_1^* \oplus T_2^*)(f \oplus g) \\
&= (T_1 \oplus T_2) S_{\Lambda \oplus \Gamma} (T_1 \oplus T_2)^* (f \oplus g)
\end{aligned}$$

This shows that $(T_1 \oplus T_2) S_{\Lambda \oplus \Gamma} (T_1 \oplus T_2)^*$ is the corresponding frame operator for $\{(\Lambda_i \oplus \Gamma_i)(T_1 \oplus T_2)^*\}_{i \in I}$.

Theorem 3.10. *Suppose $K_1, K_2 \in \mathcal{B}(H)$ and $T_1, T_2 \in \mathcal{B}(X)$. Let $\{\Lambda_i\}_{i \in I}$ be a K_1-g-frame and K_2-g-frame for H and $\{\Gamma_i\}_{i \in I}$ be a T_1-g-frame and T_2-g-frame for X. Then $\{\Lambda_i \oplus \Gamma_i\}_{i \in I}$ is a $\alpha(K_1 \oplus T_1) + \beta(K_2 \oplus T_2)$-g-frame and $(K_1 \oplus T_1)(K_2 \oplus T_2)$-g-frame for $H \oplus X$.*

Proof. Since $\{\Lambda_i\}_{i \in I}$ is a K_1-g-frame and K_2-g-frame for H and $\{\Gamma_i\}_{i \in I}$ is a T_1-g-frame and T_2-g-frame for X, there exist positive constants A_m, B_m, $m = 1, 2$ and C_n, D_n, $n = 1, 2$ such that

$$A_m \| K_m^* f \|_1^2 \leq \sum_{i \in I} \| \Lambda_i f \|_1^2 \leq B_m \| f \|_1^2 \quad \forall f \in H \tag{6}$$

$$C_n \| T_n^* g \|_2^2 \leq \sum_{i \in I} \| \Gamma_i g \|_2^2 \leq D_n \| g \|_2^2 \quad \forall g \in X. \tag{7}$$

Now, for each $f \oplus g \in H \oplus X$, we have

$$\begin{aligned}
&\| [\alpha (K_1 \oplus T_1)^* + \beta (K_2 \oplus T_2)^*](f \oplus g) \|^2 \\
&= \| [\alpha (K_1^* \oplus T_1^*) + \beta (K_2^* \oplus T_2^*)](f \oplus g) \|^2 \\
&\leq \| \alpha (K_1^* \oplus T_1^*)(f \oplus g) \|^2 + \| \beta (K_2^* \oplus T_2^*)(f \oplus g) \|^2 \\
&\leq |\alpha|^2 \{ \| K_1^* f \|_1^2 + \| T_1^* g \|_2^2 \} + |\beta|^2 \{ \| K_2^* f \|_1^2 + \| T_2^* g \|_2^2 \} \\
&\leq \max\{|\alpha|^2, |\beta|^2\} \{ \| K_1^* f \|_1^2 + \| K_2^* f \|_1^2 + \| T_1^* g \|_2^2 + \| T_2^* g \|_2^2 \} \\
&\leq \max\{|\alpha|^2, |\beta|^2\} \left\{ \left(\frac{1}{A_1} + \frac{1}{A_2} \right) \sum_{i \in I} \| \Lambda_i f \|_1^2 + \left(\frac{1}{C_1} + \frac{1}{C_2} \right) \sum_{i \in I} \| \Gamma_i g \|_2^2 \right\} \\
&\leq \max\{|\alpha|^2, |\beta|^2\} \max\left\{ \left(\frac{1}{A_1} + \frac{1}{A_2} \right), \left(\frac{1}{C_1} + \frac{1}{C_2} \right) \right\} \times \\
&\qquad \sum_{i \in I} \| (\Lambda_i \oplus \Gamma_i)(f \oplus g) \|^2 .
\end{aligned}$$

On the other hand, from (6) and (7), for each $f \oplus g \in H \oplus X$, we have

$$\sum_{i \in I} \| \Lambda_i f \|_1^2 + \sum_{i \in I} \| \Gamma_i g \|_2^2 \leq (B_1 + B_2) \| f \|_1^2 + (D_1 + D_2) \| g \|_2^2$$
$$\Rightarrow \sum_{i \in I} \| (\Lambda_i \oplus \Gamma_i)(f \oplus g) \|^2 \leq \max\{(B_1 + B_2), (D_1 + D_2)\} \| f \oplus g \|^2.$$

Thus, $\{\Lambda_i \oplus \Gamma_i\}_{i \in I}$ is a $\alpha(K_1 \oplus T_1) + \beta(K_2 \oplus T_2)$-$g$-frame for $H \oplus X$.

Now, for each $f \oplus g \in H \oplus X$, we have

$$\begin{aligned} & \| [(K_1 \oplus T_1)(K_2 \oplus T_2)]^* (f \oplus g) \|^2 \\ & = \| (K_2^* K_1^* \oplus T_2^* T_1^*)(f \oplus g) \|^2 \\ & = \| K_2^* K_1^* f \|_1^2 + \| T_2^* T_1^* g \|_2^2 \\ & \leq \| K_2 \|^2 \| K_1^* f \|_1^2 + \| T_2 \|^2 \| T_1^* g \|_2^2 \\ & \leq \max\{ \| K_2 \|^2, \| T_2 \|^2 \} \left\{ \frac{1}{A_1} \sum_{i \in I} \| \Lambda_i f \|_1^2 + \frac{1}{C_1} \sum_{i \in I} \| \Gamma_i g \|_2^2 \right\} \\ & \leq \max\{ \| K_2 \|^2, \| T_2 \|^2 \} \max\left\{ \frac{1}{A_1}, \frac{1}{C_1} \right\} \sum_{i \in I} \| (\Lambda_i \oplus \Gamma_i)(f \oplus g) \|^2. \end{aligned}$$

Hence, $\{\Lambda_i \oplus \Gamma_i\}_{i \in I}$ is a $(K_1 \oplus T_1)(K_2 \oplus T_2)$-$g$-frame for $H \oplus X$. This completes the proof. □

Theorem 3.11. *Let $\{\Lambda_i\}_{i \in I}$ be a tight K_1-g-frame for H with bound A and $\{\Gamma_i\}_{i \in I}$ be a tight K_2-g-frame for X with bound B. Then there exists a g-Bessel sequence $\{\Lambda_i' \oplus \Gamma_i'\}_{i \in I}$ in $H \oplus X$ with bound F such that*

$$(K_1 \oplus K_2)(f \oplus g) = \sum_{i \in I} (\Lambda_i \oplus \Gamma_i)^* (\Lambda_i' \oplus \Gamma_i')(f \oplus g),$$

for all $f \oplus g \in H \oplus X$ and $F \max(A, B) \geq 1$.

Proof. Since $\{\Lambda_i\}_{i \in I}$ is a tight K_1-g-frame for H and $\{\Gamma_i\}_{i \in I}$ is a tight K_2-g-frame for X, by Theorem 2.4, there exist g-Bessel sequences $\{\Lambda_i'\}_{i \in I}$ and $\{\Gamma_i'\}_{i \in I}$ in H and X with bounds C and D, respectively, such that

$$K_1 f = \sum_{i \in I} \Lambda_i^* \Lambda_i' f \;\; \forall f \in H, \text{ and } K_2 g = \sum_{i \in I} \Gamma_i^* \Gamma_i' g \;\; \forall g \in X.$$

Then for each $f \oplus g \in H \oplus X$, we have

$$\begin{aligned} (K_1 \oplus K_2)(f \oplus g) &= K_1 f \oplus K_2 g \\ &= \left(\sum_{i \in I} \Lambda_i^* \Lambda_i' f \right) \oplus \left(\sum_{i \in I} \Gamma_i^* \Gamma_i' g \right) \\ &= \sum_{i \in I} (\Lambda_i^* \Lambda_i' f \oplus \Gamma_i^* \Gamma_i' g) \\ &= \sum_{i \in I} (\Lambda_i \oplus \Gamma_i)^* (\Lambda_i' \oplus \Gamma_i')(f \oplus g). \end{aligned}$$

By Theorem 3.3, $\{\Lambda_i' \oplus \Gamma_i'\}_{i \in I}$ is a g-Bessel sequence in $H \oplus X$ with bound F, where $F = \max(C, D)$. Now, for each $f \oplus g \in H \oplus X$, we have

$$\sum_{i \in I} \|(\Lambda_i \oplus \Gamma_i)(f \oplus g)\|^2 = \sum_{i \in I} \|\Lambda_i f\|_1^2 + \sum_{i \in I} \|\Gamma_i g\|_2^2$$
$$= A \|K_1^* f\|_1^2 + B \|K_2^* g\|_2^2$$
$$\leq \max(A, B) \left\{ \sup_{f_1 \in H, \|f_1\| = 1} \left\| \sum_{i \in I} \left\langle (\Lambda_i')^* \Lambda_i f, f_1 \right\rangle_1 \right\|^2 \right\} +$$
$$\max(A, B) \left\{ \sup_{g_1 \in X, \|g_1\| = 1} \left\| \sum_{i \in I} \left\langle (\Gamma_i')^* \Gamma_i g, g_1 \right\rangle_2 \right\|^2 \right\}$$
$$= \max(A, B) \left\{ \sup_{f_1 \in H, \|f_1\| = 1} \left\| \sum_{i \in I} \langle \Lambda_i f, \Lambda_i' f_1 \rangle_1 \right\|^2 \right\} +$$
$$\max(A, B) \left\{ \sup_{g_1 \in X, \|g_1\| = 1} \left\| \sum_{i \in I} \langle \Gamma_i g, \Gamma_i' g_1 \rangle_2 \right\|^2 \right\}$$
$$\leq \max(A, B) \left\{ \sup_{f_1 \in H, \|f_1\| = 1} \sum_{i \in I} \|\Lambda_i f\|_1^2 \sum_{i \in I} \|\Lambda_i' f_1\|_1^2 \right\} +$$
$$\max(A, B) \left\{ \sup_{g_1 \in X, \|g_1\| = 1} \sum_{i \in I} \|\Gamma_i g\|_2^2 \sum_{i \in I} \|\Gamma_i' g_1\|_2^2 \right\}$$
$$\leq \max(A, B) \left\{ C \sum_{i \in I} \|\Lambda_i f\|_1^2 + D \sum_{i \in I} \|\Gamma_i g\|_2^2 \right\}$$
$$\leq \max(A, B) \max(C, D) \sum_{i \in I} \|(\Lambda_i \oplus \Gamma_i)(f \oplus g)\|^2.$$
$$\Rightarrow \max(A, B) \max(C, D) \geq 1.$$

Furthermore, for each $f \oplus g \in H \oplus X$, we have

$$\langle (K_1 \oplus K_2)^* (f \oplus g), (f_1 \oplus g_1) \rangle = \langle K_1^* f \oplus K_2^* g, (f_1 \oplus g_1) \rangle$$
$$= \langle K_1^* f, f_1 \rangle_1 + \langle K_2^* g, g_1 \rangle_2 = \langle f, K_1 f_1 \rangle_1 + \langle g, K_2 g_1 \rangle_2$$
$$= \left\langle f, \sum_{i \in I} \Lambda_i^* \Lambda_i' f_1 \right\rangle_1 + \left\langle g, \sum_{i \in I} \Gamma_i^* \Gamma_i' g_1 \right\rangle_2$$
$$= \left\langle \sum_{i \in I} (\Lambda_i')^* \Lambda_i f, f_1 \right\rangle_1 + \left\langle \sum_{i \in I} (\Gamma_i')^* \Gamma_i g, g_1 \right\rangle_2$$
$$= \left\langle \sum_{i \in I} (\Lambda_i' \oplus \Gamma_i')^* (\Lambda_i \oplus \Gamma_i)(f \oplus g), (f_1 \oplus g_1) \right\rangle.$$

Thus, for each $f \oplus g \in H \oplus X$, we have

$$(K_1 \oplus K_2)^* (f \oplus g) = \sum_{i \in I} (\Lambda_i' \oplus \Gamma_i')^* (\Lambda_i \oplus \Gamma_i)(f \oplus g).$$

□

4. Perturbation of K-g-frame in $H \oplus X$

In frame theory, an important concept is stability of frame. In this section, we study the stability of $K_1 \oplus K_2$-g-frame for $H \oplus X$ under some perturbations.

Theorem 4.1. *Let $\{\Lambda_i\}_{i\in I}$ be a K_1-g-frame for H with bounds A, B and $\{\Gamma_i\}_{i\in I}$ be a K_2-g-frame for X with bounds C, D. Consider the family of operators $\Lambda_i' \oplus \Gamma_i' \in \mathcal{B}(H \oplus X, H_i \oplus K_i)$. If there exists $0 < R < \min\{A, C\}$ such that*

$$\sum_{i\in I} \|(\Lambda_i \oplus \Gamma_i)(f \oplus g) - (\Lambda_i' \oplus \Gamma_i')(f \oplus g)\|^2 \leq R\,\|(K_1 \oplus K_2)^*(f \oplus g)\|^2 \tag{8}$$

for all $f \oplus g \in H \oplus X$. Then the family $\{\Lambda_i' \oplus \Gamma_i'\}_{i\in I}$ is a $K_1 \oplus K_2$-g-frame for $H \oplus X$.

Proof. For each $f \oplus g \in H \oplus X$, from (8), we have

$$\sum_{i\in I} \|(\Lambda_i f \oplus \Gamma_i g) - (\Lambda_i' f \oplus \Gamma_i' g)\|^2 \leq R\,\|(K_1 \oplus K_2)^*(f \oplus g)\|^2$$

$$\Rightarrow \sum_{i\in I} \|(\Lambda_i f - \Lambda_i' f) \oplus (\Gamma_i g - \Gamma_i' g)\|^2 \leq R\,\|(K_1^* f \oplus K_2^* g)\|^2$$

$$\Rightarrow \sum_{i\in I} \|\Lambda_i f - \Lambda_i' f\|_1^2 + \sum_{i\in I} \|\Gamma_i g - \Gamma_i' g\|_2^2 \leq R\left(\|K_1^* f\|_1^2 + \|K_2^* g\|_2^2\right). \tag{9}$$

Now, by the triangle inequality, we have

$$\begin{aligned}\sum_{i\in I} \|\Lambda_i' f\|_1^2 &\geq \sum_{i\in I} \|\Lambda_i f\|_1^2 - \sum_{i\in I} \|\Lambda_i f - \Lambda_i' f\|_1^2 \\ &\geq A\,\|K_1^* f\|_1^2 - \sum_{i\in I} \|\Lambda_i f - \Lambda_i' f\|_1^2, \text{ and}\end{aligned}$$

$$\sum_{i\in I} \|\Gamma_i' g\|_2^2 \geq C\,\|K_2^* g\|_2^2 - \sum_{i\in I} \|\Gamma_i g - \Gamma_i' g\|_2^2.$$

Adding these above two inequalities, we get

$$\begin{aligned}&\sum_{i\in I} \left\{\|\Lambda_i' f\|_1^2 + \|\Gamma_i' g\|_2^2\right\} \\ &\geq A\,\|K_1^* f\|_1^2 + C\,\|K_2^* g\|_2^2 - \left\{\sum_{i\in I} \|\Lambda_i f - \Lambda_i' f\|_1^2 + \sum_{i\in I} \|\Gamma_i g - \Gamma_i' g\|_2^2\right\} \\ &\geq A\,\|K_1^* f\|_1^2 + C\,\|K_2^* g\|_2^2 - R\left(\|K_1^* f\|_1^2 + \|K_2^* g\|_2^2\right) \quad [\text{ by (9)} \\ &\geq \min(A, C)\left(\|K_1^* f\|_1^2 + \|K_2^* g\|_2^2\right) - R\left(\|K_1^* f\|_1^2 + \|K_2^* g\|_2^2\right) \\ &= \{\min(A, C) - R\}\left(\|K_1^* f\|_1^2 + \|K_2^* g\|_2^2\right) \\ &= \{\min(A, C) - R\}\,\|(K_1 \oplus K_2)^*(f \oplus g)\|^2. \\ &\Rightarrow \{\min(A, C) - R\}\,\|(K_1 \oplus K_2)^*(f \oplus g)\|^2 \\ &\qquad \leq \sum_{i\in I} \|(\Lambda_i' \oplus \Gamma_i')(f \oplus g)\|^2,\end{aligned}$$

for all $f \oplus g \in H \oplus X$. On the other hand,

$$\sum_{i \in I} \| \Lambda_i' f \|_1^2 \leq \sum_{i \in I} \| \Lambda_i f \|_1^2 + \sum_{i \in I} \| \Lambda_i f - \Lambda_i' f \|_1^2$$
$$\leq B \| f \|_1^2 - \sum_{i \in I} \| \Lambda_i f + \Lambda_i' f \|_1^2, \text{ and}$$

$$\sum_{i \in I} \| \Gamma_i' g \|_2^2 \leq D \| g \|_2^2 + \sum_{i \in I} \| \Gamma_i g - \Gamma_i' g \|_2^2 .$$

Again adding the above two inequalities, and using (9), we get

$$\begin{aligned}
&\sum_{i \in I} \| (\Lambda_i' \oplus \Gamma_i') (f \oplus g) \|^2 \\
&\leq B \| f \|_1^2 + D \| g \|_2^2 + R \left(\| K_1^* f \|_1^2 + \| K_2^* g \|_2^2 \right) \\
&\leq B \| f \|_1^2 + D \| g \|_2^2 + R \left(\| K_1 \|^2 \| f \|_1^2 + \| K_2 \|^2 \| g \|_2^2 \right) \\
&\leq \left\{ \max(B, D) + R \max \left(\| K_1 \|^2 , \| K_2 \|^2 \right) \right\} \left(\| f \|_1^2 + \| g \|_2^2 \right) \\
&= \left\{ \max(B, D) + R \max \left(\| K_1 \|^2 , \| K_2 \|^2 \right) \right\} \| f \oplus g \|^2,
\end{aligned}$$

for all $f \oplus g \in H \oplus X$. Thus, $\{ \Lambda_i' \oplus \Gamma_i' \}_{i \in I}$ is a $K_1 \oplus K_2$-g-frame for $H \oplus X$ with bounds $\{ \min(A, C) - R \}$ and $\left\{ \max(B, D) + R \max \left(\| K_1 \|^2 , \| K_2 \|^2 \right) \right\}$. This completes the proof. □

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