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QUALITATIVE ANALYSIS OF Ψ-HILFER FRACTIONAL IMPULSIVE DIFFERENTIAL EQUATIONS INVOLVING ALMOST SECTORIAL OPERATORS WITH NONLOCAL MULTI-POINT CONDITION

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Abstract. In this manuscript, we establish the existence results of fractional impulsive differential equations involving ψ-Hilfer fractional derivative and almost sectorial operators by using the Schauder fixed point theorem. For this purpose, we have discussed the two cases if the associated semigroup is compact and noncompact, respectively. We considered here the higher dimensional system of integral equations. We have presented here new theoretical results, structural investigations, new models and approaches, and new applications of integral equations. Finally, an example is discussed to illustrate the main result.

1. Introduction

Ordinary and partial differential equations are universally recognized as powerful tools to model and solve practical problems involving nonlinear phenomena. In particular, we mention physical processes as problems in elasticity theory, where we deal with composites made of two different materials with different hardening exponents. Therefore, the theory of differential equations has been successfully applied to establish the existence and multiplicity of solutions of initial and boundary value problems via direct methods, minimax theorems, variational methods, and topological methods. If possible, one looks to solutions in special forms by using the symmetries of the driving equation. This also leads to the study of the difference counterparts of such equations to provide exact or approximate solutions.

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We consider the following fractional impulsive differential equations involving ψ- Hilfer fractional derivative and almost sectorial operators

$$\mathfrak{D}^{\alpha,\gamma;\psi}\wp(t)+\mathscr{A}\wp(t)=\mathscr{E}(t,\wp(t))\quad t\in(0,T]=\mathcal{J} \tag{1.1}$$

$$\Delta\wp|_{t=t_k}=\mathscr{I}_k(\wp(t_k^-)),k=1,2,3,...,m \tag{1.2}$$

$$I_{0+}^{(1-\lambda)(1-\gamma);\psi}\wp(0)=\wp_0, \tag{1.3}$$

The fundamental concept and properties of integrals and derivatives of fractional order of a function with respect to the another function viz. Riemann-Liouville integral and derivative have been introduced in [2, Chapter 2]. Following the similar approach, Almeida [4] discussed Caputo fractional derivative and investigated the interesting properties of this operator and extended few preceding study concerned with the Caputo and the Caputo-Hadamard derivative operators.

On the other hand, Hilfer [11] introduced a fractional derivative $D^{\gamma,\eta}(.)$having two parameters $\eta\in(n-1,n),n\in N$ and$\gamma\in(0\leq\gamma\leq1)$ which in specific gives the Riemann– Liouville and the Caputo derivative operator. In [2] M.S. Abdo et.al, studied the existence and Ulam-stability results for ψ-Hilfer fractional integrodifferential equations. Initial value problems for nonlinear fractional differential equations with ψ- caputo derivative via monotone iterative technique was refereed in [6].In [17], Ashwini D. Mali et.al, discussed the ψ-Hilfer fractional derivative differential equations with boundary value problems. In [3], Anjali Jaiswal and Bahuguna studied the equations of Hilfer fractional derivative with almost sectorial operator in the abstract sense.

$$\mathfrak{D}^{\alpha,\gamma}(t)+\mathscr{A}u(t)=\mathscr{E}(t,u(t))\ t\in(o,T]$$
$$I_{0+}^{(1-\alpha)(1-\gamma)}u(0)=u_0.$$

In [15] Kishor D. Kucche et.al., established the On the Nonlinear ψ-Hilfer Fractional derivative Differential Equations with initial value problems of the form,

$${}^{H}\mathfrak{D}^{\alpha,\gamma;\psi}\wp(t)=\mathscr{F}(t,\wp(t))\quad 0<\alpha<1,\quad 0\leq\gamma\leq1,$$
$$I_{0+}^{(1-\zeta;\psi)}\wp(a)=\wp_a.$$

where ${}^{H}\mathfrak{D}^{\alpha,\gamma;\psi}$ is the ψ-Hilfer derivative of order α and $\mathscr{F}$ is an appropriate function. We also refer to the work in [11], where Hamdy M. Ahmed et. al. studied the existence for nonlinear Hilfer fractional derivative differential equations with control. In [30], Yong Zhoy et.al studied the factional Cauchy problems with almost sectorial operators of the form

$$\mathfrak{D}^{\alpha}\wp(t)=\mathscr{A}\wp(t)+\mathscr{F}(t,\wp(t))\ t\in(0,T],$$
$$I_{0+}^{(1-\alpha)}\wp(0)=\wp_0.$$

where $\mathfrak{D}^{\alpha}$ is Riemann–Liouville derivative of order α, $I^{(1-\alpha)}$ is Riemann–Liouville integral of order $1-\alpha$,$0<\alpha<1$. A is an almost sectorial operator on a complex Banach space, and $\mathscr{F}$ is a given function.

In [27]J. Vanterler et.al, studied the stability of the modified impulsive fractional differential equations of the form

$$\begin{aligned} {}^{H}\mathfrak{D}^{\alpha,\gamma;\psi}\wp(t) =& \mathscr{F}(t,\wp(t)) \quad 0<\alpha<1, \quad 0\le\gamma\le 1\\ \wp(t) =& g_i(t,\wp(t_i^+)) \end{aligned}$$

where ${}^{H}\mathfrak{D}^{\alpha,\gamma;\psi}$- is the ψ-Hilfer fractional derivative and $\mathscr{F}, g$-is continuous. For the fundamental properties of ψ -Hilfer fractional derivative and the basic theory of fractional differential equation involving ψ-Hilfer fractional derivative, we refer the readers to the papers of Sousa and Oliveira [25, 26].

The paper is structured as follows: we have presented some information in Section 2 about Hilfer derivative, almost sectorial operators, measure of non-compactness, mild solutions of equations $(1.1) - (1.3)$ along with some basic definitions, results and lemmas. We discuss the main results for mild solutions for the equations $(1.1) - (1.3)$ in Section 3. In Section 4 & 5, we have discussed the two cases if associated semigroup is compact and noncompact respectively. Finally, an abstract application is discussed for the main result.

The following sections describes the supporting results of the given problem and also generalizes the results in [30].

2. Preliminaries

Definition 2.1 [29] For $\alpha > 0$, the $\psi-$fractional integral of order α of a function $f(t)$ is defined by

$$I_{0+}^{\alpha;\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(r)(t-r)^{\alpha-1} dr$$

Definition 2.2 [29] For $0 < \alpha < 1$, The $\psi-$HFD (Hilfer Fractional Derivative) in Riemann-Liouville (R-L) fractional derivative with order α of a function $f(t)$ is defined by

$$\mathfrak{D}^{\alpha;\psi} f(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_0^t \frac{f(r)}{(t-r)^{\alpha}} dr$$

Definition 2.3 [29] For $0 < \alpha < 1$, the $\psi-$Caputo fractional derivative with order α of a function $f(t)$ is defined by

$${}^{c}\mathfrak{D}^{\alpha;\psi} f(t) = \mathfrak{I}^{n-v;\psi} \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n f(t)$$

Definition 2.4 [24] The $\psi-$Hilfer fractional derivative of the function h is given by

$$\mathfrak{D}^{v,\beta;\psi} f(t) = \mathfrak{I}^{\beta(1-\gamma);\psi} \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right) \mathfrak{I}^{(1-\beta)(1-\gamma)\psi} f(t)$$

Measure of Non-compactness :

Let $\mathscr{L} \subset \mathscr{Y}$ also bounded. We consider the Hausdorff measure of non-compactness as follows

$$\Theta(\mathscr{L}) = \inf\Big\{\zeta > 0 \quad \text{that is} \quad \mathscr{L} \subset \bigcup_{j=1}^{m} B_{\zeta}(x_j)$$

$$\text{where} \quad x_j \in \mathscr{Y}, m \in N\Big\}. \tag{2.1}$$

The Kuratowski measure of noncompactness Φ on a bounded set $\mathcal{B} \subset \mathscr{Y}$ is considered as follows

$$\Phi(\mathscr{L}) = \inf\Big\{\epsilon > 0 \quad \text{implies} \quad \mathscr{L} \subset \bigcup_{j=1}^{m} M_j$$

$$\text{also} \quad \text{diam}(M_j) \leq \epsilon\Big\}, \tag{2.2}$$

with the following properties

(1) $\mathscr{L}_1 \subset \mathscr{L}_2$ gives $\Theta(\mathscr{L}_1) \leq \Theta(\mathscr{L}_2)$ where $\mathscr{L}_1, \mathscr{L}_2$ are bounded subsets of $\mathscr{Y}$

(2) $\Theta(\mathscr{L}) = 0$ iff $\mathscr{L}$ is relatively compact in $\mathscr{Y}$

(3) $\Theta(\{z\} \bigcup \mathscr{L}) = \Theta(\mathscr{L})$ for all $z \in \mathscr{Y}$ $\mathscr{L} \subseteq \mathscr{Y}$

(4) $\Theta(\mathscr{L}_1 \bigcup \mathscr{L}_2) \leq \max\{\Theta(\mathscr{L}_1), \Theta(\mathscr{L}_2)\}$

(5) $\Theta(\mathscr{L}_1 + \mathscr{L}_2) \leq \Theta(\mathscr{L}_1) + \Theta(\mathscr{L}_2)$

(6) $\Theta(r\mathscr{L}) \leq |r|\Theta(\mathscr{L})$ for $r \in R$.

Let $\mathscr{M} \subset C(I, \mathscr{Y})$ and $\mathscr{M}(r) = \Big\{\upsilon(r) \in \mathscr{Y} | \upsilon \in \mathscr{M}\}$. We define

$$\int_0^t \mathscr{M}(r)dr = \Big\{\int_0^t \upsilon(r)dr | \upsilon \in \mathscr{M}\Big\}, \quad t \in \mathcal{J}$$

Proposition 2.2 If $\mathscr{M} \subset C(\mathcal{J}, \mathscr{Y})$ is equicontinuous and bounded, then $t \to \Theta(\mathscr{M}(t))$ is continuous on I, also

$$\Theta(\mathscr{M}) = \max \Theta(\mathscr{M}(t)), \Theta(\int_0^t \upsilon(r)dr) \leq \int_0^t \Theta(\upsilon(r))dr, \quad \text{for} \quad t \in I.$$

Proposition 2.3 Let $\{\upsilon_n : \mathcal{J} \to \mathscr{Y}, n \in N\}$ are Bochner integrable functions. This implies for $n \in N, \|\upsilon_n\| \leq m(t)$ a.e $m \in L^1(I, R^+)$. Then $\xi(t) = \Theta(\{\upsilon_n(t)\}_{n=1}^{\infty}) \in L^1(I, R^+)$ and satisfies

$$\Theta(\{\int_0^t \upsilon_n(r)dr : n \in N\}) \leq 2\int_0^t \xi(r)dr.$$

Proposition 2.4 Let $\mathscr{M}$ be a bounded set. Then for any $\zeta > 0$, $\exists$ a sequence $\{\upsilon_n\}_{n=1}^{\infty} \subset \mathscr{M}$ that is

$$\Theta(\mathscr{M}) \leq 2\Theta\{\upsilon_n\}_{n=1}^{\infty} + \zeta.$$

Almost Sectorial Operators :

Let $0 < \mu < \pi$ and $-1 < \beta < 0$. We define $S^0_\mu = \{v \in C \setminus \{0\}$ that is $|argv| < \mu\}$ and its closure by S_μ, that is $S_\mu = \{v \in C \setminus \{0\}|\text{arg}v| < \mu\} \bigcup \{0\}$.

Definition 2.6 [20] For $-1 < \beta < 0, 0 < \omega < \frac{\pi}{2}$, we define $\{\odot^\beta_\omega\}$ as a family of all closed and linear operators $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{Y} \to \mathscr{Y}$ this implies

(1) $\sigma(\mathscr{A})$ is contained in S_ω.

(2) For all $\mu \in (\omega, \pi)$, there exists M_μ implies

$$\|\mathscr{R}(z, \mathscr{A})\|_{L(X)} \leq M_\mu |z|^\beta$$

where $\mathscr{R}(z, \mathscr{A}) = (zI - \mathscr{A})^{-1}$ is the resolvent operator and $\mathscr{A} \in \odot^\beta_\omega$ is said to be an almost sectorial operator on $\mathscr{Y}$.

Proposition 2.5 [20] Let $\mathscr{A} \in \odot^\beta_\omega$ for $-1 < \beta < 0$ and $0 < \omega < \frac{\pi}{2}$. Then the below properties are completed

(1) $\Im(t)$ is analytic and $\frac{d^n}{dt^n}\Im(t) = (-\mathscr{A}^n\Im(t)(t \in S^0_{\frac{\pi}{2}})$;

(2) $\Im(t+s) = \Im(t)\Im(s) \ \forall \ t, s \in S^0_{\frac{\pi}{2}}$;

(3) $\|\Im(t)\|_{L(\mathscr{Y})} \leq C_0 t^{-\beta-1}(t > 0)$; where $C_0 = C_0(\beta) > 0$ is a constant;

(4) Let $\sum_\Im = \{x \in \mathscr{Y} \ : lim_{t\to 0_+}\Im(t)x = x\}$. Then $\mathfrak{D}(\mathscr{A}^\theta) \subset \sum_\Im$ if $\theta > 1 + \beta$;

(5) $\mathscr{R}(r, -\mathscr{A}) = \int_0^\infty e^{-rs}\Im(s)ds$, $r \in \mathbb{C}$ with $Re(r) > 0$.

We assume the following Wright-type function [29]

$$\mathfrak{M}_\alpha(\theta) = \sum_{n\in\mathbb{N}} \frac{(-\theta)^{n-1}}{\Gamma(1-\alpha n)(n-1)!}, \ \theta \in \mathbb{C}.$$

For $-1 < \sigma < \infty, \ r > 0$,

(A1) $\mathfrak{M}_\alpha(\theta) \geq 0, \ t > 0$;

(A2) $\int_0^\infty \theta^\sigma \mathfrak{M}_\alpha d\theta = \frac{\Gamma(1+\sigma)}{\Gamma(1+\alpha\sigma)}$

(A3) $\int_0^\infty \frac{\alpha}{\theta^{\alpha+1}} e^{-r\theta}\mathfrak{M}_\alpha(\frac{1}{\theta^\alpha})d\theta = e^{-r^\alpha}$.

We have define $\{\mathfrak{S}_\alpha(t)\}|_{t\in S^0_{\frac{\pi}{2}-w}}$, $\{\mathfrak{Q}_\alpha(t)\}|_{t\in S^0_{\frac{\pi}{2}-w}}$ by

$$\mathfrak{S}_\alpha(t) = \int_0^\infty \mathfrak{M}_\alpha(\zeta)\varrho(t^\alpha\zeta)d\zeta,$$

$$\mathfrak{Q}_\alpha(t) = \int_0^\infty \alpha\zeta\mathfrak{M}_\alpha(\zeta)\varrho(t^\alpha\zeta)d\zeta.$$

Theorem 2.1 (Theorem 4.6.1 [29]) For each fixed $t \in S^0_{\frac{\pi}{2}-\omega}$, $\mathfrak{S}_\alpha(t)$ and $\mathfrak{Q}_\alpha(t)$ are bounded and linear operators on $\mathscr{Y}$. Also

$$\|\mathfrak{S}_\alpha(t)\| \leq \mathfrak{C}_s t^{-\alpha(1+\beta)},\ \|\mathfrak{Q}_\alpha(t)\| \leq \mathfrak{C}_p t^{-\alpha(1+\beta)},\ t > 0,$$

where $\mathfrak{C}_s$ and $\mathfrak{C}_p$ are constants

We assume the following conditions, to prove the main results:

(H1) For $t \in \mathcal{J}'$, $\mathscr{E}(t,.) : \mathscr{Y} \to \mathscr{Y}$ is continuous function and for each $\wp \in \mathfrak{C}(\mathcal{J}', \mathscr{Y})$, $\mathscr{E}(.,\wp) : \mathcal{J}' \to \mathscr{Y}$ is strongly measurable.

(H2) $\exists\ k \in L^1(\mathcal{J}', \mathbb{R}^+)$ satisfying

$$I^{-\alpha\beta}_{0+} k \in \mathfrak{C}(\mathcal{J}', \mathbb{R}),\ \lim\ (\psi(t))^{(1+\alpha\beta)(1-\gamma)} I^{-\alpha\beta}_{0+} k(t) = 0$$

(H3)

$$\begin{aligned}&\sup_{[0,T]}(\psi(t))^{(1+\alpha\beta)(1-\gamma)}\|\psi_0^{(1-\alpha)(\gamma-1)}\| \\ &+ (\psi(t))^{(1+\alpha\beta)(1-\gamma)} \int_0^t \psi'(r)[\psi(t)-\psi(r)]^{-\alpha\beta-1} \\ &k(r)dr) \leq r,\ \text{for } r > 0.\end{aligned}$$

(H4) $\exists$ constants γ_k such that $\|\mathscr{I}_k(\wp)\| \leq \omega_k,\ k = 1, 2, \ldots, m$ for each $\wp \in \mathscr{Y}$.

Lemma 2.1 The fractional Cauchy problem (1.1) – (1.3) is equivalent to an integral equation given by

$$\begin{aligned}\wp(t) =&\wp_0 \frac{(\psi(t)-\psi(0))^{(1-\alpha)(\gamma-1)}}{\Gamma(v(1-\alpha)+\alpha)} t^{(1-\alpha)(\gamma-1)} \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^1 \psi'(r)\,(\psi(t)-\psi(r))^{\alpha-1}\,[-\mathscr{A}\wp(r) + \mathscr{E}(r,\wp(r))]dr \\ &+ \sum_{0<t_k<t} \psi_0^{\alpha,\gamma}(\psi(t_2)-\psi(t_k))\mathscr{I}_k(\wp(t_k^-)), \quad t \in \mathcal{J}. \end{aligned} \tag{2.3}$$

Definition 2.7 By a mild solution of the Cauchy problem (1.1) – (1.3) we mean a function $\wp \in \mathbb{C}(\mathcal{J}', X)$ that satisfies

$$\begin{aligned}\wp(t) = &(\psi_0)^{(1-\alpha)(\gamma-1)}\,\wp_0 \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(r)\mathfrak{K}_\alpha\,(\psi(t)-\psi(r))\,\mathscr{E}(r,\wp(r))dr \\ &+ \sum_{0<t_k<t} \psi_0^{\alpha,\gamma}(\psi(t_2)-\psi(t_k))\mathscr{I}_k(\wp(t_k^-)),\ t \in \mathcal{J}'.\end{aligned} \tag{2.4}$$

Where

$$\begin{aligned}\psi_0^{(1-\alpha)(\gamma-1)} =&\frac{(\psi(t)-\psi(0))^{(1-\alpha)(\gamma-1)}}{\Gamma(v(1-\alpha)+\alpha}\\ \mathfrak{K}_\alpha =&\psi(t)^{\alpha-1}\mathfrak{Q}_\alpha\psi(t)\end{aligned}$$

Now we define an operator $\mathfrak{P} : \mathfrak{B}_r(\mathcal{J}') \to \mathfrak{B}_r(\mathcal{J}')$ as

$$
\begin{aligned}
(\mathfrak{P}\wp)(t) =& \psi_0^{(1-\alpha)(\gamma-1)}\wp_0 \\
&+ \int_0^t \psi'(r)\left(\psi(t)-\psi(r)\right)^{\alpha-1}\mathfrak{Q}_\alpha(\psi(t) \\
&- \psi(r))\mathscr{E}(r,\wp(r))dr \\
&+ \sum_{0<t_k<t} \psi_0^{\alpha,\gamma}(\psi(t_2)-\psi(t_k))\mathscr{I}_k(\wp(t_k^-))
\end{aligned} \tag{2.5}
$$

Lemma 2.3 [2] $\mathfrak{K}_\alpha(t)$ and $\mathfrak{S}_{\alpha,\gamma}(t)$ are bounded linear operators on $\mathscr{Y}$, for every fixed $t \in S^0_{\frac{\pi}{2}-\omega}$. Furthermore for $t > 0$.

$$
\begin{aligned}
\|\mathfrak{K}_\alpha(t)x\| \leq& \mathfrak{C}_p t^{-1-\alpha\beta}\|x\|, \\
\|\psi_0^{(1-\alpha)(\gamma-1)}(t)x\| \leq& \frac{\Gamma(-\alpha\beta)}{\Gamma(\gamma(1-\alpha)-\alpha\beta)}\mathfrak{C}_p t^{\gamma(1-\alpha)-\alpha\beta-1}\|x\|.
\end{aligned}
$$

Proposition 2.6 [2] $\mathfrak{K}_\alpha(t)$ and $\mathfrak{S}_{\alpha,\gamma(t)}$ are strongly continuous, for $t > 0$.

3. Main results

Theorem 3.1 Let $\mathscr{A} \in \odot^\beta_\omega$ for $-1 < \beta < 0$ and $0 < \omega < \frac{\pi}{2}$. Assuming $(H1)-(H4)$ are satisfied, the operators $\{\mathfrak{F}y \ : \ y \in \mathfrak{B}_r(\mathcal{J})\}$ is equicontinuous provided $\wp_0 \in \mathcal{D}(\mathscr{A}^\theta)$ with $\theta > 1+\beta$.

Proof: For $y \in \mathfrak{B}_r(\mathcal{J})$ and $t_1 = 0 < t_2 \leq T$, we gave

$$
\begin{aligned}
&\left\|\mathfrak{F}y(t_2) - \mathfrak{F}y(0)\right\| \\
&= \Bigg\| t_2^{(1+\alpha\mu)(1-\gamma)}\Big(\left(\psi_0(t_2)\right)^{(1-\alpha)(\gamma-1)}\wp_0 \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{t_2}\psi'(r)\left(\psi(t_2)-\psi(r)\right)^{\alpha-1}\mathfrak{Q}_\alpha(\psi(t_2) \\
&- \psi(r))\mathscr{E}(r,\wp(r))dr + \sum_{0<t_k<t_2}\psi_0^{\alpha,\gamma}(\psi(t_2) \\
&- \psi(t_k))\mathscr{I}_k(\wp(t_k^-))\Big)\Bigg\| \\
&\leq \left\| t_2^{(1+\alpha\mu)(1-\gamma)}\left(\psi_0(t_2)\right)^{(1-\alpha)(\gamma-1)}\wp_0\right\| \\
&+ \frac{1}{\Gamma(\alpha)}\Bigg\| t_2^{(1+\alpha\mu)(1-\gamma)}\int_0^{t_2}\psi'(r)\left(\psi(t_2)-\psi(r)\right)^{\alpha-1} \\
&\mathfrak{Q}_\alpha(\psi(t_2)-\psi(r))\mathscr{E}(r,\wp(r))dr\Bigg\| \\
&+ \Bigg\| t_2^{(1+\alpha\mu)(1-\gamma)}\sum_{0<t_k<t_2}\psi_0^{\alpha,\gamma}(\psi(t_2) \\
&- \psi(t_k))\mathscr{I}_k(\wp(t_k^-))\Bigg\| \\
&\to 0, \text{ as } t_2 \to 0
\end{aligned}
$$

Now, let $0 < t_1 < t_2 \leq T$,

$$
\begin{aligned}
&\Big\|\mathfrak{F}y(t_2)-\mathfrak{F}y(t_1)\Big\| \\
&\leq \Big\|t_2^{(1+\alpha\mu)(1-\gamma)}((\psi_0(t_2))^{(1-\alpha)(\gamma-1)}\,\wp_0 \\
&\ -t_1^{(1+\alpha\mu)(1-\gamma)}((\psi_0(t_1))^{(1-\alpha)(\gamma-1)}\,\wp_0\Big\| \\
&+\Big\|t_2^{(1+\alpha\mu)(1-\gamma)}\int_0^{t_2}\psi'(r)\,(\psi(t_2)-\psi(r))^{\alpha-1} \\
&\mathfrak{Q}_\alpha(\psi(t_2)-\psi(r))\mathscr{E}(r,\wp(r))dr \\
&-t_1^{(1+\alpha\mu)(1-\gamma)}\int_0^{t_1}\psi'(r)\,(\psi(t_1)-\psi(r))^{\alpha-1} \\
&\mathfrak{Q}_\alpha(\psi(t_1)-\psi(r))\mathscr{E}(r,\wp(r))dr\Big\| \\
&+\Big\|t_2^{(1+\alpha\mu)(1-\gamma)}\sum_{0<t_k<t_2}\psi_0^{\alpha,\gamma}(\psi(t_2) \\
&\ -\psi(t_k))\mathscr{I}_k(\wp(t_k^-)) \\
&-t_1^{(1+\alpha\mu)(1-\gamma)}\sum_{0<t_k<t_1}\psi_0^{\alpha,\gamma}(\psi(t_2) \\
&\ -\psi(t_k))\mathscr{I}_k(\wp(t_k^-))\Big\|
\end{aligned}
$$

Here using the triangle inequality, we have

$$
\begin{aligned}
&\Big\|\mathfrak{F}y(t_2)-\mathfrak{F}y(t_1)\Big\| \\
&\leq \Big\|t_2^{(1+\alpha\mu)(1-\gamma)}((\psi_0(t_2))^{(1-\alpha)(\gamma-1)}\,\wp_0 \\
&\ -t_1^{(1+\alpha\mu)(1-\gamma)}((\psi_0(t_1))^{(1-\alpha)(\gamma-1)}\,\wp_0\Big\| \\
&+\Big\|t_2^{(1+\alpha\mu)(1-\gamma)}\int_{t_1}^{t_2}\psi'(r)\,(\psi(t_2)-\psi(r))^{\alpha-1}\,\mathfrak{Q}_\alpha(\psi(t_2) \\
&\ -\psi(r))\mathscr{E}(r,\wp(r))dr\Big\| \\
&+\Big\|t_2^{(1+\alpha\mu)(1-\gamma)}\int_0^{t_1}\psi'(r)\,(\psi(t_2)-\psi(r))^{\alpha-1}\,\mathfrak{Q}_\alpha(\psi(t_2) \\
&\ -\psi(r))\mathscr{E}(r,\wp(r))dr \\
&-t_1^{(1+\alpha\mu)(1-\gamma)}\int_0^{t_1}\psi'(r)\,(\psi(t_1)-\psi(r))^{\alpha-1}\,\mathfrak{Q}_\alpha(\psi(t_2) \\
&\ -\psi(r))\mathscr{E}(r,\wp(r))dr\Big\| \\
&+\Big\|t_1^{(1+\alpha\mu)(1-\gamma)}\int_0^{t_1}\psi'(r)\,(\psi(t_1)-\psi(r))^{\alpha-1}\,\mathfrak{Q}_\alpha(\psi(t_2) \\
&\ -\psi(r))\mathscr{E}(r,\wp(r))dr \\
&-t_1^{(1+\alpha\mu)(1-\gamma)}\int_0^{t_1}\psi'(r)\,(\psi(t_1)-\psi(r))^{\alpha-1}\,\mathfrak{Q}_\alpha(\psi(t_2) \\
&\ -\psi(r))\mathscr{E}(r,\wp(r))dr\Big\|
\end{aligned}
$$

$$
\begin{aligned}
&+\left\| t_2^{(1+\alpha\mu)(1-\gamma)} \sum_{0<t_k<t_2} \psi_0^{\alpha,\gamma}(\psi(t_2)-\psi(t_k))\mathscr{I}_k(\wp(t_k^-)) \right. \\
&\left. - t_1^{(1+\alpha\mu)(1-\gamma)} \sum_{0<t_k<t_1} \psi_0^{\alpha,\gamma}(\psi(t_2)-\psi(t_k))\mathscr{I}_k(\wp(t_k^-)) \right\| \\
=&\mathfrak{I}_1+\mathfrak{I}_2+\mathfrak{I}_3+\mathfrak{I}_4+\mathfrak{I}_5
\end{aligned}
$$

to use $\mathfrak{S}_{\alpha,\gamma}(t)$, we get $\mathfrak{I}_1 \to 0$ as $t_2 \to t_1$. Also

$$
\begin{aligned}
\mathfrak{I}_2 \leq &\mathfrak{C}_p \psi_0 t_2^{(1+\alpha\beta)(1-\gamma)} \int_{t_1}^{t_2} \psi'(r)\left(\psi(t_1)-\psi(r)\right)^{-\alpha\beta-1}\kappa(r)dr \\
\leq &\mathfrak{C}_p \left| \psi_0 t_2^{(1+\alpha\beta)(1-\gamma)} \int_0^{t_2} \psi'(r)\left(\psi(t_2)-\psi(r)\right)^{-\alpha\beta-1}\kappa(r)dr \right. \\
&\left. - t_2^{(1+\alpha\beta)(1-\gamma)} \int_0^{t_1} \psi'(r)\left(\psi(t_1)-\psi(r)\right)^{-\alpha\beta-1}\kappa(r)dr \right| \\
\leq &\mathfrak{C}_p \int_0^{t_1} \left| \psi_0 t_1^{(1+\alpha\beta)(1-\gamma)} \psi'(r)\left(\psi(t_1)-\psi(r)\right)^{-\alpha\beta-1} \right. \\
&\left. - t_2^{(1+\alpha\beta)(1-\gamma)} \psi'(r)\left(\psi(t_2)-\psi(r)\right)^{-\alpha\beta-1} \right| \kappa(r)dr.
\end{aligned}
$$

Then $\mathfrak{I}_2 \to 0$ as $t_2 \to t_1$, by using $(H2)$ and the dominated convergence theorem. Since

$$
\begin{aligned}
\mathfrak{I}_3 \leq &\mathfrak{C}_p \int_0^{t_1} \psi'(r)\left(\psi(t_2)-\psi(r)\right)^{-\alpha-\alpha\beta} \\
&\left| t_2^{(1+\alpha\beta)(1-\gamma)} \psi'(r)\left(\psi(t_2)-\psi(r)\right)^{\alpha-1} \right. \\
&\left. - t_1^{(1+\alpha\beta)(1-\gamma)} \psi'(r)\left(\psi(t_1)-\psi(r)\right)^{\alpha-1} \right| \kappa(r)dr
\end{aligned}
$$

and

$$
\begin{aligned}
&\psi'(r)\left(\psi(t_2)-\psi(r)\right)^{-\alpha-\alpha\beta} \left| t_2^{(1+\alpha\beta)(1-\gamma)} \psi'(r) \right. \\
&\left(\psi(t_2)-\psi(r)\right)^{\alpha-1} - t_1^{(1+\alpha\beta)(1-\gamma)} \psi'(r) \\
&\left. \left(\psi(t_1)-\psi(r)\right)^{\alpha-1} \right| \kappa(r) \\
&\leq t_2^{(1+\alpha\beta)(1-\gamma)} \psi'(r)\left(\psi(t_2)-\psi(r)\right)^{\alpha-1}\kappa(r) \\
&\quad + t_1^{(1+\alpha\beta)(1-\gamma)} \psi'(r)\left(\psi(t_1)-\psi(r)\right)^{\alpha-1}\kappa(r) \\
&\leq 2t_1^{(1+\alpha\beta)(1-\gamma)} \psi'(r)\left(\psi(t_1)-\psi(r)\right)^{\alpha-1}\kappa(r).
\end{aligned}
$$

and $\int_0^{t_1} 2t_1^{(1+\alpha\beta)(1-\gamma)} \psi'(r)\left(\psi(t_1)-\psi(r)\right)^{\alpha-1}\kappa(r)$ exists, i.e, $I_3 \to 0$ as $t_2 \to t_1$.

For $\epsilon > 0$, we have

$$
\begin{aligned}
\mathfrak{I}_4 =& \Big\| \int_0^{t_1} t_1^{(1+\alpha\beta)(1-\gamma)} [\mathfrak{Q}_\alpha(\psi(t_2)-\psi(r)) - \mathfrak{Q}_\alpha(\psi(t_1) \\
& - \psi(r))]\psi'(r)(\psi(t_1)-\psi(r))^{\alpha-1}\mathscr{E}(r,\wp(r))dr \Big\| \\
& \leq \int_0^{t_1-\epsilon} t_1^{(1+\alpha\beta)(1-\gamma)} \Big\| \mathfrak{Q}_\alpha(\psi(t_2)-\psi(r)) \\
& - \mathfrak{Q}_\alpha(\psi(t_1)-\psi(r)) \Big\|_{L(X)} \psi'(r) \\
& (\psi(t_1)-\psi(r))^{\alpha-1}\kappa(r) \\
& + \int_{t_1}^{t_1} t_1^{(1+\alpha\beta)(1-\gamma)} \Big\| \mathfrak{Q}_\alpha(\psi(t_2)-\psi(r)) \\
& - \mathfrak{Q}_\alpha(\psi(t_2)-\psi(r)) \Big\|_{L(X)} \psi'(r) \\
& (\psi(t_1)-\psi(r))^{\alpha-1}\kappa(r) \\
\leq & t_1^{(1+\alpha\beta)(1-\gamma)} \int_0^{t_1} \psi'(r)(\psi(t_1)-\psi(r))^{\alpha-1}\kappa(r)dr \\
& \sup_{s\in[0,t_1-\epsilon]} \Big\| \mathfrak{Q}_\alpha(\psi(t_2)-\psi(r)) \\
& - \mathfrak{Q}_\alpha(\psi(t_1)-\psi(r)) \Big\|_{L(X)} \\
& + \mathfrak{C}_p \int_{t_1}^{t_1} t_1^{(1+\alpha\beta)(1-\gamma)} (\psi'(r)(\psi(t_2)-\psi(r))^{-\alpha-\alpha\beta} \\
& + \psi'(r)(\psi(t_1)-\psi(r))^{-\alpha-\alpha\beta})\psi'(r) \\
& (\psi(t_1)-\psi(r))^{\alpha-1}\kappa(r)dr \\
\leq & \ t_1^{(1+\alpha\beta)(1-\gamma)+\alpha(1+\beta)} \int_0^{t_1} \psi'(r)(\psi(t_1)-\psi(r))^{-\alpha\beta-1} \\
& \kappa(r)dr \ \sup_{s\in[0,t_1-\epsilon]} \\
& \Big\| \mathfrak{Q}_\alpha(\psi(t_2)-\psi(r)) - \mathfrak{Q}_\alpha(\psi(t_1)-\psi(r)) \Big\|_{L(X)} \\
& + 2\mathfrak{C}_p \int_{t_1-\epsilon}^{t_1} t_1^{(1+\alpha\beta)(1-\gamma)} \psi'(r)(\psi(t_1)-\psi(r))^{-\alpha\beta-1}\kappa(r)dr.
\end{aligned}
$$

Since $\mathfrak{Q}_\alpha(t)$ is uniformly continuous and $\lim_{t_2\to t_1}\mathfrak{I}_2 = 0$, then $\mathfrak{I}_4 \to 0$ as $t_2 \to t_1$ i.e. independent of $y \in \mathfrak{B}_r(\mathcal{J})$.

Clearly, since the strongly continuous of of $\mathfrak{S}_{\alpha,\gamma}(t)$, we get

$$
\begin{aligned}
\mathfrak{I}_5 =& \Big\| t_2^{(1+\alpha\mu)1-\gamma} \sum_{0<t_k<t_2} \psi_0^{\alpha,\gamma}(\psi(t_2)-\psi(t_k))\mathscr{I}_k(\wp(t_k^-)) \\
& - t_1^{(1+\alpha\mu)(1-\gamma)} \sum_{0<t_k<t_1} \psi_0^{\alpha,\gamma}(\psi(t_2)-\psi(t_k))\mathscr{I}_k(\wp(t_k^-)) \Big\| \to 0 \\
& \text{as } t_2 \to t_1.
\end{aligned}
$$

Hence $\left\|\mathfrak{F}y(t_2) - \mathfrak{F}y(t_1)\right\| \to 0$ independently of $y \in \mathfrak{B}_r(\mathcal{J})$ as $t_2 \to t_1$ therefore $\{\mathfrak{F}y : y \in \mathfrak{B}_r(\mathcal{J})\}$ is equicontinuous.

Theorem 3.2 Let $-1 < \beta < 0$ and $0 < \omega < \frac{\pi}{2}$ and $\mathscr{A} \in \odot^{\beta}_{\omega}$. Then under consideration $(H1)-(H3)$ the operator $\{\mathfrak{F}y : y \in \mathfrak{B}_r(\mathcal{J})\}$ is continuous and bounded. Also $\wp_0 \in \mathcal{D}(\mathscr{A}^{\theta})$ where $\theta > 1+\beta$.

Proof We verify that $\mathfrak{F}$ maps $\mathfrak{B}_r(\mathcal{J})$. Taking $y \in \mathfrak{B}_r(\mathcal{J})$ and define $\wp(t) = t^{-(1+\alpha\beta)(1-\gamma)}y(t)$, we have $\wp \in \mathfrak{B}^{\mathfrak{Y}}_r(\mathcal{J}')$.

$$\begin{aligned}\|\mathfrak{F}\| \leq & \|t^{(1+\alpha\beta)(1-\gamma)}\psi_0^{\alpha,\gamma;\psi}(t)\wp_0\| + t^{(1+\alpha\beta)(1-\gamma)} \\ & \left\| \int_0^t \psi^{'}(r)(\psi(t)-\psi(r))^{\alpha-1}\mathfrak{Q}_\alpha(\psi(t)-\psi(r))\mathscr{E}(r,\wp(r))dr \right\|\end{aligned}$$

From $(H2)-(H3)$, we get

$$\begin{aligned}\|\mathfrak{F}y(t)\| &\leq t^{(1+\alpha\beta)(1-\gamma)}\|\mathfrak{S}_{\alpha,\gamma}(t)u_0\| \\ &+ t^{(1+\alpha\beta)(1-\gamma)}\int_0^t \psi^{'}(r)(\psi(t)-\psi(r))^{-\alpha\beta-1}\kappa(r)dr \\ &\leq \ \sup_{[0,T]} t^{(1+\alpha\beta)(1-\gamma)}\int_0^t (t-r)^{-\alpha\beta-1}\kappa(r)dr \\ &\leq r\end{aligned}$$

Hence $\|\mathfrak{F}y\| \leq r$, for any $y \in \mathcal{B}_r(I)$.

Now, to verify $\mathfrak{F}$ is continuous in $\mathcal{B}_r(I)$, let $y_n, y \in \mathcal{B}(I), n = 1, 2, \cdots$, with $\lim_{n\to\infty} y_n = y$. That is $\lim_{n\to\infty} y_n(t) = y(t)$ and $\lim_{n\to\infty} t^{-(1+\alpha\beta)1-\gamma}\ y_n(t) = t^{-(1+\alpha\beta)1-\gamma}\ y(t)$ and $\lim_{n\to\infty} t^{-(1+\alpha\beta)(1-\gamma)}y_n(t) = t^{-(1+\alpha\beta)(1-\gamma)}y(t)$, on $\mathcal{J}'$
$(H1)$ implies

$$\begin{aligned}\mathscr{E}(t,\wp_n(t)) &= \mathscr{E}(t, t^{-(1+\alpha\beta)(1-\gamma)}y_n(t)) \\ &\to \mathscr{E}(t, t^{-(1+\alpha\beta)(1-\gamma)}y(t)),\end{aligned}$$

as $n \to \infty$.

From $(H2)$ to obtain the inequality $\psi^{'}(r)(\psi(t) - \psi(r))^{-\alpha\beta-1}|\mathscr{E}(r,\wp_n(r))| \leq 2\psi^{'}(r)(\psi(t)-\psi(r))^{-(\alpha\beta)(1-\gamma)}\kappa(r)$

i.e

$$\begin{aligned}&\int_0^t \psi^{'}(r)(\psi(t)-\psi(r))^{-\alpha\beta-1}\|\mathscr{E}(r,\wp_n(r)) \\ &- \mathscr{E}(r,\wp(r))\|dr \ \to \ 0, \ \text{when } n \to \ \infty.\end{aligned}$$

Now

$$\begin{aligned}
&\|\mathfrak{F}y_n(t)-\mathfrak{F}y(t)\| \\
&\leq t^{(1+\alpha\beta)(1-\gamma)}\bigg\|\int_0^t \psi'(r)(\psi(t)-\psi(r))^{\alpha-1} \\
&\mathfrak{Q}_\alpha(\psi(t)-\psi(r))(\mathscr{E}(r,\wp_n(r))-\mathscr{E}(r,\wp(r))dr\bigg\|
\end{aligned}$$

Applying Theorem (2.1), we have

$$\|\mathfrak{F}y_n(t)-\mathfrak{F}y(t)\| \leq \mathfrak{C}_p t^{(1+\alpha\beta)(1-\gamma)}\int_0^t \psi'(r)(\psi(t)-\psi(r))^{-\alpha\beta-1}\|\mathscr{E}(r,\wp_n(r))-\mathscr{E}(r,\wp(r))\|dr$$

$$\to 0 \text{ as } n\to\infty.$$

i.e, $\mathfrak{F}y_n \to \mathfrak{F}y$ pointwise on $\mathcal{J}$. Also Theorem(3.1) implies that $\mathfrak{F}y_n \to \mathfrak{F}y$ uniformly on $\mathcal{J}$ as $n\to\infty$. That is $\mathfrak{F}$ is continuous.

4. $\Im(t)$ is Compact

Theorem 4.1 Let $-1<\beta<0, 0<\omega<\frac{\pi}{2}$ and $\mathscr{A}\in\Theta^{\beta}_{\omega}$. If $\mathfrak{T}(t)(t>0)$ is compact. Also $(H1)-(H4)$ hold then $\exists$ a mild solution of $(1.1)-(1.3)$ in $\mathcal{B}^{\mathcal{Y}}_r(I')$.

Proof Since we have assumed $\Im(t)$ is compact, then the equicontinuity of $\Im(t)(t>0)$. Moreover, by Theorem 3.1 and 3.2, $\mathfrak{F}:\mathfrak{B}^{\mathcal{Y}}_r(\mathcal{J}')\to\mathfrak{B}^{Y}_r(\mathcal{J}')$ is continuous and bounded and $\varepsilon:\mathfrak{B}_r(\mathcal{J})\to\mathfrak{B}_r(\mathcal{J})$ is bounded, continuous and $\{\varepsilon y: y\in\mathfrak{B}_r(\mathcal{J})\}$ equicontinuous. We can write $\epsilon:\mathfrak{B}_r(\mathcal{J})\to\mathfrak{B}_r(\mathcal{J})$ by

$$(\epsilon y)(t)=(\epsilon^1 y)(t)+(\epsilon^2 y)(t)$$

where

$$\begin{aligned}
(\epsilon^1 y)(t) &= (\psi(t))^{(1+\alpha\beta)(1-\gamma)}\wp_0 \\
&= (\psi(t))^{1+\alpha\beta)(1-\gamma)}\mathcal{I}^{\gamma(1-\alpha)}_{0+}t^{\alpha-1}\mathfrak{Q}_\alpha(\psi(t))\wp_0 \\
&= \frac{(\psi(t))^{(1+\alpha\beta)(1-\gamma)}}{\Gamma(\gamma(1-\alpha))}\int_0^t \psi'(\psi(t)-\psi(r))^{\gamma(1-\alpha)-1}r^{\alpha-1} \\
&\int_0^\infty \alpha\theta M_\alpha(\theta)\Im(r^\alpha\theta)\wp_0 d\theta dr \\
&= \frac{\alpha(\psi(t))^{(1+\alpha\beta)(1-\gamma)}}{\Gamma(\gamma(1-\alpha))}\int_0^t\int_0^\infty(\psi(t)-\psi(r))^{\gamma(1-\alpha)-1} \\
&r^{\alpha-1}\theta M_\alpha(\theta)\Im(r^\alpha\theta)\wp_0 d\theta dr.
\end{aligned}$$

and

$$\begin{aligned}
(\varepsilon^2 y)(t) = (\psi(t))^{(1+\alpha\beta)(1-\gamma)}&\int_0^t \psi'(\psi(t)-\psi(r))^{\alpha-1} \\
&\mathfrak{Q}_\alpha(\psi(t)-\psi(r))\mathscr{E}(r,\wp(r),)dr \\
&+\sum_{0<t_k<t}\psi_0^{\alpha,\gamma}(\psi(t)-\psi(t_k))\mathscr{I}_k(\wp(t_k^-))
\end{aligned}$$

For $\sigma > 0$ and $\zeta \in (0, t)$, we define an operator $\varepsilon^1_{\zeta,\sigma}$ on $\mathfrak{B}_r(\mathcal{J})$ by

$$(\varepsilon^1_{\zeta,\sigma}y)(t) = \frac{(\psi(t))^{(1+\alpha\beta)(1-\gamma)}}{\Gamma(\gamma(1-\alpha))}\int_\zeta^t\int_\sigma^\infty \psi^{'}(\psi(t)-\psi(r))^{(1-\alpha)\gamma-1}$$
$$r^{\alpha-1}\theta M_\alpha(\theta)\Im(r^\alpha\theta)\wp_0 d\theta dr$$
$$= \frac{\alpha(\psi(t))^{(1+\alpha\beta)(1-\gamma)}}{\Gamma(\gamma(1-\alpha))}\mathfrak{T}(\zeta^\alpha\sigma)$$
$$\int_\zeta^t\int_\sigma^\infty \psi^{'}(\psi(t)-\psi(r))^{(1-\alpha)\gamma-1}r^{\alpha-1}\theta$$
$$M_\alpha(\theta)\Im(r^\alpha\theta-\zeta^\alpha\sigma)\wp_0 d\theta dr.$$

Since $\mathfrak{T}(\epsilon^\alpha\delta)$ is compact $\mathcal{V}^1_{\zeta,\sigma}(t) = \{\varepsilon^1_{\zeta,\sigma}y)(t), y \in \mathfrak{B}_r(\mathcal{J})\}$ is precompact in $\mathscr{Y}$. Moreover,

$$\|(\varepsilon^1 y)(t) - (\varepsilon^1_{\zeta,\sigma}y)(t)\|$$
$$\leq \mathscr{K}(\alpha,\gamma)\Big\|(\psi(t))^{(1+\alpha\beta)(1-\gamma)}$$
$$\int_0^t\int_0^\sigma \psi^{'}(\psi(t)-\psi(r))^{\gamma(1-\alpha)-1}r^{\alpha-1}\theta M_\alpha(\theta)$$
$$\Im(r^\alpha\theta)\wp_0 d\theta dr\Big\|$$
$$+ \mathscr{K}(\alpha,\gamma)\Big\|(\psi(t))^{(1+\alpha\beta)(1-\gamma)}$$
$$\int_0^\zeta\int_\sigma^\infty \psi^{'}(\psi(t)-\psi(r))^{\gamma(1-\alpha)-1}r^{\alpha-1}\theta M_\alpha(\theta)$$
$$\Im(r^\alpha\theta)\wp_0 d\theta dr\Big\|$$
$$\leq \mathscr{K}(\alpha,\gamma)(\psi(t))^{(1+\alpha\beta)(1-\gamma)}$$
$$\int_0^t\int_0^\sigma \psi^{'}(\psi(t)-\psi(r))^{\gamma(1-\alpha)-1}r^{\alpha-1}$$
$$\theta M_\alpha(\theta)r^{-\alpha\gamma-\alpha}\|\wp_0\|\theta^{-\beta-1}d\theta dr$$
$$+ \mathscr{K}(\alpha,\gamma)(\psi(t))^{(1+\alpha\beta)(1-\gamma)}$$
$$\int_0^\zeta\int_\sigma^\infty \psi^{'}(\psi(t)-\psi(r))^{\gamma(1-\alpha)-1}r^{\alpha-1}$$
$$\theta M_\alpha(\theta)r^{-\alpha\beta-\alpha}\theta^{-\beta-1}\|\wp_0\|d\theta dr$$
$$= \mathscr{K}(\alpha,\gamma)(\psi(t))^{(1+\alpha\beta)(1-\gamma)}$$
$$\int_0^t \psi^{'}(\psi(t)-\psi(r))^{\gamma(1-\alpha)-1}r^{-\alpha\beta-1}\|\wp_0\|dr$$
$$\int_0^\sigma \theta^{-\beta}M_\alpha(\theta)d\theta$$

$$+\mathscr{K}(\alpha,\gamma)(\psi(t))^{(1+\alpha\beta)(1-\gamma)}$$
$$\int_0^{\zeta}\psi^{'}(\psi(t)-\psi(r))^{\gamma(1-\alpha)-1}r^{-\alpha\beta-1}\|\wp_0\|dr$$
$$\int_{\eta}^{\infty}\theta^{-\beta}M_{\alpha}(\theta)d\theta$$
$$\leq\mathscr{K}(\psi(t))^{-\alpha\gamma(1+\beta)}\|\wp_0\|\int_0^{\eta}\theta^{-\beta}M_{\alpha}(\theta)d\theta$$
$$+\mathscr{K}(\psi(t))^{-\alpha\gamma(1+\beta)}\|\wp_0\|$$
$$\int_0^{\zeta}(1-s)^{\gamma(1-\alpha)-1}r^{-\alpha\beta-1}dr$$
$$\int_{\eta}^{\infty}\theta^{-\beta}M_{\alpha}(\theta)d\theta,$$
$$\to 0,\ \text{as}\ \zeta\to 0,\ \sigma\to 0,$$

where, $\mathscr{K}(\alpha,\gamma)=\frac{\alpha}{\Gamma(\gamma(1-\alpha))}$.
Therefore, $\mathcal{V}^1_{\zeta,\sigma}(t)=\{\varepsilon^1_{\zeta,\sigma}y)(t),y\in\mathfrak{B}_r(\mathcal{J})\}$ are arbitrarily close to $\mathcal{V}^1(t)=\{\varepsilon^1 y)(t),y\in\mathfrak{B}_r(\mathcal{J})\}$, for $t>0$. Hence $\mathcal{V}^1(t)$, for $t>0$, is precompact in $\mathscr{Y}$. Now we can present $\varepsilon^2_{\zeta,\sigma}$ on $\mathfrak{B}_r(\mathcal{J})$ by

$$(\varepsilon^2_{\zeta,\sigma}y)(t)$$
$$=\alpha(\psi(t))^{(1+\alpha\beta)(1-\gamma)}\int_0^{t-\zeta}\int_{\sigma}^{\infty}\theta M_{\alpha}(\theta)\psi^{'}(\psi(t)-\psi(r))^{\alpha-1}$$
$$\Im(\psi^{'}(\psi(t)-\psi(r))^{\alpha}\theta)\mathscr{E}(r,\wp(r))d\theta dr$$
$$+\sum_{0<t_k<t}\psi_0^{\alpha,\gamma}(\psi(t)-\psi(t_k))\mathscr{I}_k(\wp(t_k^-))$$
$$=\alpha(\psi(t))^{(1+\alpha\beta)(1-\gamma)}\mathfrak{T}(\zeta^{\alpha}\sigma)\int_0^{t-\zeta}\int_{\sigma}^{\infty}\theta M_{\alpha}(\theta)\psi^{'}$$
$$(\psi(t)-\psi(r))^{\alpha-1}\Im(\psi^{'}(\psi(t)$$
$$-\psi(r))^{\alpha}\theta-\zeta^{\alpha}\sigma)\mathscr{E}(r,\wp(r))d\theta dr$$
$$+\sum_{0<t_k<t}\psi_0^{\alpha,\gamma}(\psi(t)-\psi(t_k))\mathscr{I}_k(\wp(t_k^-)).$$

Hence $\mathcal{V}^2_{\zeta,\sigma}(t)=\{\varepsilon^2_{\zeta,\sigma}y)(t),y\in\mathfrak{B}_r(\mathcal{J})\}$ is precompact in $\mathscr{Y}$. For every $y\in\mathfrak{B}_r(\mathcal{J})$, we get

$$\|(\varepsilon^2 y)(t)-(\varepsilon^2_{\zeta,\sigma}y)(t)\|\leq\Bigg\|\alpha(\psi(t))^{(1+\alpha\beta)(1-\gamma)}$$
$$\Big(\int_0^{t}\int_0^{\sigma}\theta M\alpha(\theta)\psi^{'}(\psi(t)-\psi(r))^{\alpha-1}$$
$$\Im((\psi(t)-\psi(r))^{\alpha}\theta)\mathscr{E}(r,\wp(r))d\theta dr$$
$$+\sum_{0<t_k<t}\psi_0^{\alpha,\gamma}(\psi(t)-\psi(t_k))\mathscr{I}_k(\wp(t_k^-))\Big)\Bigg\|$$

$$+\Bigg\|\alpha(\psi(t))^{(1+\alpha\beta)(1-\gamma)}$$

$$\Big(\int_{t-\zeta}^{t}\int_{\sigma}^{\infty}\psi^{'}(\psi(t)-\psi(r))^{\alpha-1}\theta M_{\alpha}(\theta)$$

$$\Im((\psi(t)-\psi(r))^{\alpha}\theta)\mathscr{E}(r,\wp(r))d\theta dr$$

$$+\sum_{0<t_k<t}\psi_0^{\alpha,\gamma}(\psi(t)-\psi(t_k))\mathscr{I}_k(\wp(t_k^-))\Big)\Bigg\|$$

$$\leq \alpha\mathfrak{C}_0(\psi(t))^{(1+\alpha\beta)(1-\gamma)}$$

$$\Big(\int_{0}^{t}\psi^{'}(\psi(t)-\psi(r))^{-\alpha\beta-1}k(r)dr$$

$$\int_0^{\sigma}\theta^{-\beta}M_{\alpha}(\theta)d\theta+\sum_{0<t_k<\sigma}\omega_k\Big)$$

$$+\alpha\mathfrak{C}_0(\psi(t))^{(1+\alpha\beta)(1-\gamma)}$$

$$\Big(\int_{t-\zeta}^{t}\psi^{'}(\psi(t)-\psi(r))^{-\alpha\beta-1}k(r)dr$$

$$\int_0^{\infty}\theta^{-\beta}M_{\alpha}(\theta)d\theta+\sum_{0<t_k<\zeta}\omega_k\Big)$$

$$\leq \alpha\mathfrak{C}_0(\psi(t))^{(1+\alpha\beta)(1-\gamma)}\Big(\int_{0}^{t}\psi^{'}(\psi(t)-\psi(r))^{-\alpha\beta-1}$$

$$k(r)dr\int_0^{\sigma}\theta^{-\beta}M_{\alpha}(\theta)d\theta+\sum_{0<t_k<\eta}\omega_k\Big)$$

$$+\frac{\alpha\mathfrak{C}_0\Gamma(1-\beta)}{\Gamma(1-\alpha\beta)}(\psi(t))^{(1+\alpha\beta)(1-\gamma)}$$

$$\Big(\int_{t-\zeta}^{t}\psi^{'}(\psi(t)-\psi(r))^{-\alpha\beta-1}k(r)dr+\sum_{0<t_k<\zeta}\omega_k\Big)$$

$$\to 0 \text{ as } \sigma\to 0,\sigma\to 0.$$

Therefore, $\mathcal{V}^2_{\zeta,\sigma}(t)=\{\varepsilon^2_{\zeta,\sigma}y)(t),y\in\mathfrak{B}_r(\mathcal{J})\}$ are arbitrarily close to $\mathcal{V}^2(t)=\{\varepsilon^2 y)(t),y\in\mathfrak{B}_r(\mathcal{J})\},t>0$. That is, $\{\varepsilon y,y\in\mathfrak{B}_r(\mathcal{J})\}$ is relatively compact by using the Arzela-Ascoli Theorem. Then $\wp^*$ is a mild solution of (1.1)-(1.3).

5. $\Im(t)$is Noncompact

We consider as follows,

$(H5)$ $\exists$ a constant $k>0$ satisfies the following

$$\Theta(\mathscr{E}(t,\mathfrak{E}_1,\mathfrak{E}_2))\leq k\Theta(\mathfrak{E}_1,\mathfrak{E}_2) \text{ for a.e } t\in[0,T].$$

and $\mathfrak{E}_1,\mathfrak{E}_2\subset\mathscr{Y}$,

Theorem 5.1 Let $-1<\beta<0,0<\omega<\frac{\pi}{2}$ and $\mathscr{A}\in\Theta^{\beta}_{\omega}$. Suppose $(H1)-(H5)$ hold. Then $(1.1)-(1.3)$ has a mild solution in $\mathfrak{B}^{\mathcal{Y}}_r(J')$ for every $u_0\in D(\mathscr{E}^{\theta})$ with $\theta>1+\beta$.

Proof By Theorem 3.1 and 3.2, we get $\varepsilon : \mathfrak{B}_r(\mathcal{J}) \to \mathfrak{B}_r(\mathcal{J})$ is continuous, bounded and $\{\varepsilon y : y \in \mathfrak{B}_r(\mathcal{J})\}$ is equicontinuous. Also, we prove ε is campact in $\mathfrak{B}_r(\mathcal{J})$.

For bounded set $\mathbb{P}_0 \subset \mathfrak{B}_r(\mathcal{J})$, set

$$\varepsilon^{(1)}(\mathbb{P}_0) = \varepsilon(\mathbb{P}_0), \varepsilon^{(n)}(\mathbb{P}_0) = \varepsilon(\bar{co}(\varepsilon^{(n-1)}(\mathbb{P}_0))), \quad n = 2, 3, \cdots.$$

For $\epsilon > 0$, we obtain from Propositions $((2.2)-(2.4))$, a subsequence $\{y_n^{(1)}\}_{n=1}^{\infty} \subset \mathbb{P}_0$ satisfying

$$\begin{aligned}\Theta(\varepsilon^{(1)}(\mathbb{P}_0(t))) \leq 2\Theta\Big(&(\psi(t))^{(1+\alpha\beta)(1-\gamma)} \int_0^t \psi'(\psi(t)-\psi(r))^{\alpha-1}\mathfrak{Q}_\alpha(\psi(t) \\ &- \psi(r))\mathscr{E}(r, \{r^{-(1+\alpha\beta)(1-\gamma)}(y_n^{(1)}(r))\}_{n=1}^{\infty})dr + \sum_{0<t_k<t}\omega_k\Big) \\ \leq &4\mathfrak{C}_p\psi(t)^{(1+\alpha\beta)(1-\gamma)} \Big(\int_0^t \psi'(\psi(t)-\psi(r))^{-\alpha\beta-1} \\ &\Theta(\mathscr{E}(r, \{r^{-(1+\alpha\beta)(1-\gamma)}(y_n^{(1)}(r))\}_{n=1}^{\infty}))dr + \sum_{0<t_k<t}\omega_k\Big) \\ \leq &4\mathfrak{C}_p k(\psi(t))^{(1+\alpha\beta)(1-\gamma)}\Theta(\mathfrak{P}_0) \Big(\int_0^t \psi'(\psi(t)-\psi(r))^{-\alpha\beta-1}r^{-(1+\alpha\beta)(1-\gamma)}dr + \sum_{0<t_k<t}\omega_k\Big) \\ = &4\mathfrak{C}_p k(\psi(t))^{-\alpha\beta}\Theta(\mathfrak{P}_0) \Big(\frac{\Gamma(-\alpha\beta)\Gamma((-\alpha\beta+\gamma(1+\alpha\beta))}{\Gamma(-2\alpha\beta+\gamma(1+\alpha\beta))} + \sum_{0<t_k<t}\omega_k\Big).\end{aligned}$$

From ϵ is arbitrary,

$$\Theta(\varepsilon^{(1)}(\mathbb{P}_0(t))) \leq 4\mathfrak{C}_p k(\psi(t))^{-\alpha\beta}\Theta(\mathbb{P}_0) \Big(\frac{\Gamma(-\alpha\beta)\Gamma((-\alpha\beta+\gamma(1+\alpha\beta))}{\Gamma(-2\alpha\beta+\gamma(1+\alpha\beta))} + \sum_{0<t_k<t}\omega_k\Big).$$

Again for any $\epsilon > 0$, we can get from Propositions $((2.2)-(2.4))$ a subsequence $\{y_n^{(2)}\}_{n=1}^{\infty} \subset \bar{co}(\varepsilon^{(1)}(\mathbb{P}_0))$ implies that

$$\Theta(\varepsilon^{(2)}(\mathbb{P}_0(t))) = \Theta(\varepsilon(\bar{co}(\varepsilon^{(1)}(\mathbb{P}_0(t)))))$$
$$\leq 2\Theta\Big(t^{(1+\alpha\beta)(1-\gamma)}\int_0^t \psi'(\psi(t)-\psi(r))^{\alpha-1}\mathcal{Q}_\alpha$$
$$(\psi(t)-\psi(r))$$
$$\mathscr{E}(r,\{r^{-(1+\alpha\beta)(1-\gamma)}(y_n^{(2)}(r))\}_{n=1}^{\infty})dr + \sum_{0<t_k<t}\omega_k\Big)$$
$$\leq 4\mathfrak{C}_p(\psi(t))^{(1+\alpha\beta)(1-\gamma)}\Big(\int_0^t \psi'(\psi(t)-\psi(r))^{-\alpha\beta-1}$$
$$\Theta(\mathscr{E}(r,\{r^{-(1+\alpha\beta)(1-\gamma)}(y_n^{(2)}(r))\}_{n=1}^{\infty})dr + \sum_{0<t_k<t}\omega_k\Big)$$
$$\leq 4\mathfrak{C}_p k(\psi(t))^{(1+\alpha\beta)(1-\gamma)}\Big(\int_0^t \psi'(\psi(t)-\psi(r))^{-\alpha\beta-1}$$
$$\Theta(r^{-(1+\alpha\beta)(1-\gamma)}(\{y_n^{(2)}(r)\}_{n=1}^{\infty}))dr + \sum_{0<t_k<t}\omega_k\Big)$$
$$\leq 4\mathfrak{C}_p k(\psi(t))^{(1+\alpha\beta)(1-\gamma)}\Big(\int_0^t \psi'(\psi(t)-\psi(r))^{-\alpha\beta-1}$$
$$r^{-(1+\alpha\beta)(1-\gamma)}\Theta(\{y_n^{(2)}(r)\}_{n=1}^{\infty})dr + \sum_{0<t_k<t}\omega_k\Big)$$
$$\leq \frac{(4\mathfrak{C}_p k)^2(\psi(t))^{(1+\alpha\beta)(1-\gamma)}\Gamma(-\alpha\beta)\Gamma(-\alpha\beta+\gamma(1+\alpha\beta))}{\Gamma(-2\alpha\beta+\gamma(1+\alpha\beta))}\Theta(\mathbb{P}_0)$$
$$\times\Big(\int_0^t \psi'(\psi(t)-\psi(r))^{-\alpha\beta-1}r^{-(1+\alpha\beta)(1-\gamma)-\alpha\beta}dr$$
$$+\sum_{0<t_k<t}\omega_k\Big)$$
$$=\Big(\frac{(4\mathfrak{C}_p k)^2(\psi(t))^{-2\alpha\beta}\Gamma^2(-\alpha\beta)\Gamma(-\alpha\beta+\gamma(1+\alpha\beta))}{\Gamma(-3\alpha\beta+\gamma(1+\alpha\beta))}$$
$$+\sum_{0<t_k<t}\omega_k\Big)\Theta(\mathbb{P}_0).$$

Now,

$$\Theta(\varepsilon^{(n)}(\mathbb{P}_0(t)))$$
$$\leq \frac{(4\mathfrak{C}_p k)^n(\psi(t))^{-n\alpha\beta}\Gamma^n(-\alpha\beta)\Gamma(-\alpha\beta+\gamma(1+\alpha\beta))}{\Gamma(-(n+1)\alpha\beta+\gamma(1+\alpha\beta))}$$
$$\Theta(\mathbb{P}_0), n\in\mathbb{N}.$$

Let $M = 4\mathfrak{C}_p kT^{-\alpha\beta}\Gamma(-\alpha\beta)$. We can put $m,k\in\mathbb{N}$ big enough to $\frac{1}{k}<\alpha\beta<\frac{1}{k-1}$ and $\frac{n+1}{k}>2$ for $n>m$. $\Gamma(-(n+1)\alpha\beta+\gamma(1+\alpha\beta))>\Gamma(\frac{n+1}{k})$. That is

$$\frac{(4\mathfrak{C}_p k)^n T^{-n\alpha\beta}\Gamma^n(-\alpha\beta)\Gamma(-\alpha\beta+\gamma(1+\alpha\beta))}{\Gamma(-(n+1)\alpha\beta+\gamma(1+\alpha\beta))} <$$
$$\frac{(4\mathfrak{C}_p k)^n T^{-n\alpha\beta}\Gamma^n(-\alpha\beta)\Gamma(-\alpha\beta+\gamma(1+\alpha\beta))}{\Gamma\left(\frac{n+1}{k}\right)}.$$

Change $(n+1)$ by $(j+1)k$. We obtain

$$\frac{M^{(j+1)k-1}\Gamma(-\alpha\beta+\gamma(1+\alpha\beta))}{\Gamma(j+1)} =$$
$$\frac{(M^k)^j M^{k-1}\Gamma(-\alpha\beta+\gamma(1+\alpha\beta))}{j!} \to 0 \text{ as } j \to \infty.$$

Therefore, $\exists$ a constant $n_0 \in \mathbb{N}$ that is

$$\frac{(4\mathfrak{C}_p k^n t^{-n\alpha\beta}\Gamma^n(-\alpha\beta)\Gamma(-\alpha\beta+\gamma(1+\alpha\beta)}{\Gamma(-(n+1)\alpha\beta+\gamma(1+\alpha\beta))} \leq$$
$$\frac{(4\mathfrak{C}_p k)^{n_0} T^{-n_0\alpha\beta}\Gamma^{n_0}(-\alpha\beta)\Gamma(-\alpha\beta+\gamma(1+\alpha\beta))}{\Gamma(-(n_0+1)\alpha\beta+\gamma(1+\alpha\beta))} = p < 1.$$

Now

$$\Theta(\varepsilon^{(n_0)}(\mathbb{P}_0(t))) \leq p\Theta(\mathbb{P}_0).$$

From $\varepsilon^{(n_0)}(\mathbb{P}_0(t))$ is bounded and equicontinuous, applying Proposition (2.2), we get

$$\Theta(\varepsilon^{(n_0)}(\mathbb{P}_0)) = \max_{t\in[0,T]} \Theta(\varepsilon^{n_0}(\mathbb{P}_0(t))).$$

Hence,

$$\Theta(\varepsilon^{n_0}(\mathbb{P}_0)) \leq p\Theta(\mathbb{P}_0),$$

where $p < 1$. By applying the Schauder fixed point theorem, we obtain a fixed point y^* in $\mathfrak{B}_r(\mathcal{J})$ of ε. Let $\wp^*(t) = (\psi(t))^{(1+\alpha\beta)(\gamma-1)}y^*(t)$. Then $\wp^*(t)$ is a mild solution of $(1.1)-(1.3)$.

6. Illustrate an abstract application

We consider an abstract application via Hilfer fractional derivative system:

$$\begin{aligned} \mathfrak{D}^{\alpha,\gamma;\psi}_{0+}\wp(t,\mathfrak{x}) - \partial^2_{\mathfrak{x}}\wp(t,\mathfrak{x}) =& \mathscr{E}(t,\wp(t,\mathfrak{x}))\ t\in[0,T],\ \mathfrak{x}\in[0,\iota] \\ \wp(t,0) = \wp(t,\iota) =& 0 \text{ on } t\in[0,T] \\ \mathcal{I}^{(1-\alpha)(1-\gamma);\psi}_{0+}\wp(0,\mathfrak{x}) =& \wp_0(\mathfrak{x}),\ \mathfrak{x}\in[0,a] \\ \Delta\wp|_{t=1/2} =& \mathscr{I}_1\big(\wp(\frac{1}{2}^-)\big), \end{aligned} \tag{6.1}$$

in Banach space $\mathscr{Y} = C^{\alpha}([0,a])(0<\alpha<1)$, where $\alpha = \frac{1}{4},\ \gamma = \frac{1}{2},\ \mathscr{E}(t,\wp) = t^{-\frac{1}{5}}\cos^2\wp$. Here, we can convert the above problem $(1.1-1.3)$ in abstract form as

$$\begin{aligned} \mathfrak{D}^{\alpha,\gamma;\psi}\wp(t) + \mathscr{A}\wp(t) =& \mathscr{E}(t,\wp(t)) \quad t\in(0,T]=\mathcal{J} \\ \Delta\wp|_{t=t_k} =& \mathscr{I}_k(\wp(t_k^-)), k=1,2,3,...m \\ I^{(1-\alpha)(1-\gamma);\psi}_{0+}\wp(0) =& \wp_0. \end{aligned} \tag{6.2}$$

Here $\mathscr{A} = -\partial_x^2$ with $\mathfrak{D}(\mathscr{A}) = \{\wp \in C^{2+\alpha}([0,\iota])$ therefore $\wp(t,0) = 0 = \wp(t,\iota)\}$. Since [20] $\exists$ constants $\delta, \epsilon > 0$, implies $\mathscr{A} + \delta \in \odot_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}-1}(\mathscr{Y})$. To verify the compactness of semigroup $\Im(t)$, it is enough to prove that $\mathscr{R}(\alpha, -(\mathscr{A}+\delta)$ is compact. We take $l(t) = t^{-\frac{1}{5}}$.

$$r = \sup_{[0,T]}(\psi_0(t))^{(1+\alpha\beta)(1-\gamma)}\wp_0\|) + \frac{T^{\frac{17}{20}}\Gamma(-\frac{\beta}{4})\Gamma(\frac{4}{5})}{\Gamma(\frac{4}{5}-\frac{\beta}{4})}.$$

Then the hypotheses $(H1)-(H5)$ are satisfied. According to Theorem 4.1, the problem (6.1) has mild solution in $\mathfrak{B}_r^{\mathfrak{Y}}((0,T])$.

7. Conclusion

In this manuscript, we deal with the mild solutions for ψ-Hilfer fractional derivative differential equations involving jump conditions and almost sectorial operator when associated semigroup is compact and noncompact using Schauder fixed point theorem. Finally, an example is discussed to illustrate the main result. Our theorems guarantee the effectiveness of existence results It is the results of the equations concerned.

8. Data Availability

The data used to support the findings of this study are included in the article.

9. Conflicts of Interest

The authors declare that they have no conflicts of interest.

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