DOI: 10.28919/ejma.2022.2.12

Eur. J. Math. Appl. (2022)2:12
URL: http://ejma.euap.org

MIXINGALE ESTIMATION FUNCTION FOR MIXED FRACTIONAL SPDES WITH RANDOM EFFECT AND RANDOM SAMPLING

JAYA P. N. BISHWAL

Abstract. We study the mixingale estimation function estimator of the parameters in the fractional stochastic partial differential equation when the process is observed at the arrival times of a Poisson process with the presence of random effect. We use a two stage estimation procedure. We first estimate the intensity of the Poisson process. Then we plug-in this estimate in the estimation function to estimate the drift parameter. We obtain the consistency and the asymptotic normality of the mixingale estimation function estimator.

1. Introduction

Parameter estimation in fractional stochastic partial differential equations (SPDEs) is a very young and exciting area of research in view of its applications in finance, physics, biology and oceanography only to mention a few. Loges [43] initiated the study of parameter estimation in infinite dimensional stochastic differential equations. When the length of the observation time becomes large, he obtained consistency and asymptotic normality of the maximum likelihood estimator (MLE) of a real valued drift parameter in a Hilbert space valued SDE. Koski and Loges [42] extended the work of Loges [43] to minimum contrast estimators. Koski and Loges [41] applied the work to a stochastic heat flow problem. Mohapl [48] studied maximum likelihood and least squares estimators for discrete observations of an elliptic SPDE where the dependent structure of the observations is completely different and simple from the parabolic case. Cialenko *et al.* [21] studied drift estimation for discretely sampled SPDEs extending Bishwal and Bose [19]. Martingale estimation function for discretely observed diffusions was studied in Bibby and Srensen [2]. Bishwal [8] studied a new estimating function for discretely sampled diffusions by removing the stochastic integral in Girsanov likelihood. Bishwal [11] studied maximum likelihood estimation in anticipative stochastic differential equations. With random drift parameter, Bishwal [6] studied rates of convergence of the posterior distributions and the Bayes estimators in the Ornstein-Uhlenbeck process. Bishwal [9] studied likelihood asymptotics and Bayesian asymptotics for drift estimation of finite and infinite dimensional stochastic differential equations. For parameter estimation in partially observed stochastic differential equations, see Bishwal [17]. Bernstein-von Mises theorem and spectral Bayesian asymptotics for parabolic

Department of Mathematics and Statistics, University of North Carolina at Charlotte, 376 Fretwell Bldg, 9201 University City Blvd., Charlotte, NC 28223-0001, USA

E-mail address: J.Bishwal@uncc.edu.

Key words and phrases. fractional stochastic partial differential equations; space-time color noise; cylindrical fractional Brownian motion; fractional diffusion field; random effect; Poisson sampling; mixingale estimating function; quasi likelihood estimator; consistency; asymptotic normality.

Received 18/04/2022.

stochastic partial differential equations was studied in Bishwal [19]. Bernstein-von Mises theorem and small noise Bayesian asymptotics for parabolic stochastic partial differential equations was studied in Bishwal [15]. Cheng *et al.* [20] studied BVT and Bayesian estimation for a large class of prior distributions and loss functions (of at most polynomial growth) for diagonalizable bilinear SPDEs driven by a multiplicative noise. HJM type forward interest rate models viewed as an SPDE along corresponding estimation and hypothesis testing problem is studied in Bishwal [14], see also Bishwal [16]. Chong [22] established a limit theorem for integrated volatility estimation in SPDE by conducting a martingale approximation by truncation and blocking techniques to apply results by Jacod [38]. Bishwal [16] studied estimation and hypothesis testing on nonlinear SPDEs from both continuous and discrete observations.

Based on continuous observations, usually there can be two asymptotic settings in SPDE: 1) $T \to \infty$ 2) $N \to \infty$ where T is the length of the observations and N is the number of Fourier coefficients in the expansion of the solution to the SPDE. In a Bayesian approach, using the first setting, Bishwal [5] proved the Bernstein-von Mises theorem and asymptotic properties of regular Bayes estimator of the drift parameter in a Hilbert space valued SDE when the corresponding ergodic diffusion process is observed continuously over a time interval $[0, T]$. The asymptotics are studied as $T \to \infty$ under the condition of absolute continuity of measures generated by the process. Results are illustrated for the example of an SPDE. Bishwal [19](2001) proved the Bernstein-von Mises theorem and spectral asymptotics of Bayes estimators for parabolic SPDEs when the number of Fourier coefficients becomes large. In this case, the measures generated by the process for different parameters are singular.

Ditlevsen and De Gaetano [31] and Donnet and Samson [32] studied estimation of parameters of specific model of mixed effects of Brownian motion with drift. Picchini *et al.* [53] studied parameter estimation of the diffusion leaky integrate-and-fire neuronal model for a slowly fluctuating signal. Picchini *et al.* [51] studied estimation in stochastic differential mixed-effect models. Picchini and Ditlevsen [52] studied estimation of high dimensional stochastic differential mixed-effect models. Delattre *et al.* [24–26] studied maximum likelihood estimation for random effects for i.i.d. sample paths for fixed T while $M \to \infty$. Comte *et al.* [23] studied nonparametric estimation for stochastic differential equations with random effect. Maitra and Bhattacharya [45] and Maitra and Bhattacharya [46] studied maximum likelihood estimation for random effects for non-i.i.d. sample paths for fixed T while $M \to \infty$. Maitra and Bhattacharya [44] studied Bayesian asymptotics in SDEs with random effect. Nonparametric adaptive estimation of a mixed effect in the drift coefficient of an O-U process has been studied in Dion [29]. Dion and Genon-Catalot [30] studied bidimensional random effect estimation in mixed SDE model. Whitaker *et al.* [59] studied Bayesian inference for mixed effect models of SDE. Picchini and Forman [54] studied Bayesian inference for stochastic differential equation mixed effects models and applied to tumor xenography. Recently Ruse *et al.* [55] studied inference for biomedical data using diffusion models with covariates and mixed effects.

Delattre *et al.* [27] studied parametric inference for discrete observations of diffusion processes with mixed effects using estimators defined by estimating equation approach. The estimators of population parameters are asymptotically equivalent to the maximum likelihood estimators based on direct observations of M i.i.d. Gamma random variables. Delattre *et al.* [28] studied approximate maximum likelihood estimation for stochastic differential equations with simultaneous random effects in the drift and the diffusion coefficients by means of a joint parametric distribution. They considered M paths and n observations per path. For linear random effects and a specific joint distribution for these random effects, we have proved that the model parameters in the drift and in the diffusion can be estimated consistently and with a rate $\sqrt{M}$ under the condition $M/n \to 0$. For the parameters of the diffusion coefficient, the constraint is weaker $(M/n^2 \to 0)$.

In this paper we study the asymptotic properties of the quasi maximum likelihood estimator when we have observations of finite-dimensional projections at Poisson arrival time points. The asymptotic setting is the large number of observations at random time points which are the arrivals of a Poisson process and large number of Fourier coefficients.

The rest of the paper is organized as follows : Section 2 contains model, assumptions and preliminaries. In Section 3 we prove the main results of the paper. Section 4 demonstrates the results through several examples of fSPDE. Finally we give some concluding remarks.

2. Mixingales and Mixed Fractional SPDEs

Recall that a fractional Brownian motion (fBM) has the covariance

$$\widetilde{C}_H(s,t) = \frac{1}{2}\left[s^{2H} + t^{2H} - |s-t|^{2H}\right], \quad s,t > 0.$$

For $H > 0.5$ the process has long range dependence or long memory and the process is self-similar. For $H \neq 0.5$, the process is neither a Markov process nor a semimartingale. For $H = 0.5$, the process reduces to standard Brownian motion.

Let us fix θ_0, the unknown true value of the parameter θ. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $W(t,x)$ be a process on this space with values in the Schwarz space of distributions $D'(G)$ where $x \in G \subset \mathbb{R}^d$ such that for $\phi, \psi \in C_0^\infty(G)$, $\|\phi\|^{-1}_{L^2(G)}\, \langle W(t,\cdot), \phi(\cdot)\rangle$ is a one dimensional Wiener process and

$$E(\langle W(s,\cdot), \phi(\cdot)\rangle\langle W(t,\cdot), \psi(\cdot)\rangle) = (s \wedge t)(\phi, \psi)_{L^2(G)}. \tag{2.1}$$

This process is usually referred to as the cylindrical Brownian motion (CBM).

Consider the stochastic evolution equation

$$du(t,x) + (\mathcal{A}_0 + \theta\mathcal{A}_1)u(t,x)dt = dW^H(t,x), \quad t \in [0,T], x \in G, \quad u(0,x) = 0 \tag{2.2}$$

where G is a smooth bounded domain in $\mathbb{R}^d$, $\mathcal{A}_0$ and $\mathcal{A}_1$ are linear operators on a smooth bounded domain G in $\mathbb{R}^d$ with orders m_0 and m_1 respectively with $m_1 \geq m - d/2$ where $2m = \max(m_0, m_1)$, $\{W_t(x)\}$ is a cylindrical fractional Brownian motion based on the observations of the solution $u(t,x),\ t \in [0,T],\ x \in G$ and $\theta \in \Theta$ is a random variable independent of W^H whose distribution will be specified later. Let $A^\theta := \mathcal{A}_0 + \theta\mathcal{A}_1$.

We assume that there exists a complete orthonormal system $\{h_j\}_{j=1}^\infty$ in $L_2(G))$ such that for every $j = 1, 2, \ldots,$ the system $h_j \in W_0^{m,2}(G) \cap C^\infty(\overline{G})$ and

$$\Lambda_\theta h_j = \alpha_j(\theta)h_j, \text{ and } \mathcal{L}_\theta h_j = \mu_j(\theta)h_j \text{ for all } \theta \in \Theta \tag{2.3}$$

where $\mathcal{L}_\theta$ is a closed self adjoint extension of A^θ, $\Lambda_\theta := (k(\theta)I - \mathcal{L}_\theta)^{1/2m}, k(\theta)$ is a constant and the spectrum of the operator Λ_θ consists of eigenvalues $\{\alpha_j(\theta)\}_{j=1}^\infty$ of finite multiplicities and $\mu_j(\theta) = -\alpha_j^{2m} + k(\theta)$.

The fractional cylindrical Brownian motion $W(t,x)$ can be expanded in the series

$$W^H(t,x) = \sum_{j=1}^\infty W_j^H(t) h_j \tag{2.4}$$

where $\{W_j(t), t \geq 0\}_{j=1}^\infty$ are independent one dimensional fractional Brownian motions. The latter series converges P-a.s. in H^{-a} for $a > d/2$. Indeed

$$\|W^H(t,x)\|_{-a}^2 = \sum_{j=1}^\infty W_j^2(t)\|h_j\|_{-a}^2 = \sum_{j=1}^\infty (W_j^H)^2(t)\alpha_j^{-2a}. \tag{2.5}$$

and the later series converges P-a.s.

Let

$$\psi_N := \sum_{j=1}^N \frac{\alpha_j^2}{\mu_j}. \tag{2.6}$$

Here $\theta \in \Theta \subseteq \mathbb{R}$ is the unknown parameter to be estimated on the basis of the observations of the random field $u^\theta(t,x), t \geq 0,\ x \in [0,1]$.

Consider the Fourier expansion of the process

$$u(t,x) = \sum_{j=1}^\infty u_j(t)\phi_j(x) \tag{2.7}$$

corresponding to some orthogonal basis $\{\phi_j(x)\}_{j=1}^\infty$. Bagchi and Kumar (2001) used this representation for infinite factor model. Note that $u_j^\theta(t), j \geq 1$ are independent one dimensional Ornstein-Uhlenbeck processes

$$du_j^\theta(t) = \mu_j(\theta_j)u_j^\theta(t)dt + \alpha_j^{-a}dW_j^H(t) \tag{2.8}$$

$$u_j^\theta(0) = u_{0j}^\theta,$$

Recall that $\mu_j(\theta_j) = k(\theta_j) - \alpha_j^{2m}$. Thus the j^{th} Fourier coefficient satisfies the linear SDE of the Ornstein-Uhlenbeck type

$$du_j^\theta(t) = (k(\theta_j) - \alpha_j^{2m})u_j^\theta(t)dt + \alpha_j^{-a}dW_j^H(t) \tag{2.9}$$

Note that for a fixed t, the processes are $\{u_1(t), u_2(t), u_3(t), \ldots\}$ independent. This is like a continuous version of cross section time series, i.e, a joint regression auto-regression model of order 1. The random field $u(t,x)$ is observed at discrete times t and discrete positions x. Equivalently, the Fourier coefficients $u_j^\theta(t)$ are observed at discrete time points.

Now we focus on the fundamental semimartingale behind the fSPDE model. Define

$$\begin{aligned} \kappa_H &:= 2H\Gamma(3/2-H)\Gamma(H+1/2), \quad k_H(t,s) := \kappa_H^{-1}(s(t-s))^{\frac{1}{2}-H}, \\ \lambda_H &:= \frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(3/2-H)}, \quad v_t \equiv v_t^H := \lambda_H^{-1}t^{2-2H}, \quad \mathcal{M}_t^H := \int_0^t k_H(t,s)dW_s^H. \end{aligned}$$

From Norros *et al.* [49] it is well known that $\mathcal{M}_{k,t}^H$ is a Gaussian martingale, called the fundamental martingale whose variance function $\langle \mathcal{M}_k^H \rangle_t$ is v_t^H. Moreover, the natural filtration of the martingale $\mathcal{M}^H$ coincides with the natural filtration of the fBm W^H since

$$W^H_{k,t} := \int_0^t K(t,s)d\mathcal{M}^H_{k,s}$$

holds for $H \in (1/2, 1)$ where

$$K_H(t,s) := H(2H-1)\int_s^t r^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}}dr, \quad 0 \le s \le t$$

and for $H = 1/2$, the convention $K_{1/2} \equiv 1$ is used.

Define

$$Q_{k,t} := \frac{d}{dv_t}\int_0^t k_H(t,s)u_{k,s}ds.$$

Define the process $Z_k = (Z_{k,t}, t \in [0,T], k \ge 1)$ by

$$Z_{k,t} := \int_0^t k_H(t,s)du_{k,s}.$$

It is easy to see that

$$Q_{k,t} = \frac{\lambda_H}{2(2-2H)}\left\{t^{2H-1}Z_{k,t} + \int_0^t r^{2H-1}dZ_{k,s}\right\}.$$

The following facts are known from Kleptsyna and Le Breton [40]:

(i) Z_k is the fundamental semimartingale associated with the process u_k.
(ii) Z_k is a $(\mathcal{F}_t)$ -semimartingale with the decomposition

$$Z_{k,t} = \theta_k \int_0^t Q_{k,s}dv_s + \mathcal{M}^H_{k,t}.$$

(iii) u_k admits the representation

$$u_{k,t} = \int_0^t K_H(t,s)dZ_{k,s}.$$

(iv) The natural filtration $(\mathcal{Z}_t)$ of Z_k and $(\mathcal{U}_t)$ of u_k coincide.

We have

$$\begin{aligned}
Q_{k,t} &= \frac{d}{dv_t}\int_0^t k_H(t,s)u_{k,s}ds \\
&= \kappa_H^{-1}\frac{d}{dv_t}\int_0^t s^{1/2-H}(t-s)^{1/2-H}u_{k,s}ds \\
&= \kappa_H^{-1}\lambda_H t^{2H-1}\frac{d}{dt}\int_0^t s^{1/2-H}(t-s)^{1/2-H}u_{k,s}ds \\
&= \kappa_H^{-1}\lambda_H t^{2H-1}\int_0^t \frac{d}{dt}s^{1/2-H}(t-s)^{1/2-H}u_{k,s}ds \\
&= \kappa_H^{-1}\lambda_H t^{2H-1}\int_0^t s^{1/2-H}(t-s)^{-1/2-H}u_{k,s}ds.
\end{aligned}$$

The discretized version of

$$Z_{k,t} = \theta_k \int_0^t Q_{k,s}dv_s + \mathcal{M}^H_{k,t}$$

is given by

$$\Delta Z_{k,t_i} = \theta_k Q_{k,t_i} \Delta v_{t_i} + \Delta \mathcal{M}^H_{k,t_i}, k \geq 1, t \geq 0$$

which is a regression model of Z on Q.

The process Q_k depends continuously on u_k and therefore, the discrete observations of u_k does not allow one to obtain the discrete observations of Q. The process Q_k can be approximated by

$$\widetilde{Q}_{k,n} = \kappa_H^{-1} \lambda_H n^{2H-1} \sum_{j=0}^{n-1} j^{1/2-H}(n-j)^{-1/2-H} u_{k,j}. \tag{2.10}$$

It is easy to show that $\widetilde{Q}_n \to Q_t$ almost surely as $n \to \infty$, see Tudor and Viens [58].

Define a new partition $0 \leq r_1 < r_2 < r_3 < \cdots < r_{m_k} = t_k, \quad k = 1, 2, \cdots, n$.

Define

$$\widetilde{Q}_i(t_k) = \kappa_H^{-1} \eta_H t_k^{2H-1} \sum_{j=1}^{m_k} r_j^{1/2-H}(r_{m_k} - r_j)^{-1/2-H} u_i(r_j)(r_j - r_{j-1}), \tag{2.11}$$

$k = 1, 2, \cdots, n$. It is easy to show that $\widetilde{Q}_i(t_k) \to Q_i(t)$ almost surely as $m_k \to \infty$ for each $k = 1, 2, \cdots, n$ and $i \geq 1$, see Tudor and Viens [58].

We use this approximate observation in the calculation of our estimators. Thus our observations are

$$u_i(t) \approx \int_0^t K_H(t,s) d\widetilde{Z}_i(s) \quad \text{where} \quad \widetilde{Z}_i(t) = \theta \int_0^t \widetilde{Q}_i(s) dv_s + \mathcal{M}^H_{i,t} \tag{2.12}$$

observed at $t_1, t_2, \ldots, t_n$.

Note that for equally spaced data

$$\Delta v_{t_i} := v_{t_i} - v_{t_{i-1}} = \lambda_H^{-1} \left(\frac{T}{n}\right)^{2-2H} [i^{2-2H} - (i-1)^{2-2H}]. \tag{2.13}$$

For $H = 0.5$,

$$v_{t_i} - v_{t_{i-1}} = \lambda_H^{-1} \left(\frac{T}{n}\right)^{2-2H} [i^{2-2H} - (i-1)^{2-2H}] = \frac{T}{n}, \quad i = 1, 2, \ldots, n \tag{2.14}$$

the standard equispaced partition. In this paper we do not need to assume $T/n \to 0$ unlike the finite dimensional diffusion models as we take advantage of the increasing spatial dimension $K \to \infty$ in this paper.

The discrete time points could be deterministic (equally spaced/homoscedastic or unequally spaced

/heteroscadastic) or random. We consider random time points. For $x \in [0,1]$, for fixed j, we observe the process $\{u_j(t), t \geq 0\}$ at times $\{t_0, t_1, t_2, ...\}$. We assume that the sampling instants $\{t_i, i = 0, 1, 2...\}$ are generated by a Poisson process on $[0, \infty)$, i.e., $t_0 = 0$, $t_i = t_{i-1} + \tau_i, i = 1, 2, ...$ where τ_i are i.i.d. positive random variables with a common exponential distribution $F(x) = 1 - \exp(-\lambda x)$. Note that intensity parameter $\lambda > 0$ is the average sampling rate which is needs to be estimated. It is also assumed that the sampling process $t_i, i = 0, 1, 2, ...$ is independent of the observation process $\{u_j(t), t \geq 0, j \geq 1\}$. We note that the probability density function of $t_{k+i} - t_k$ is independent of k and is given by the gamma density

$$f_i(t) = \lambda(\lambda t)^{i-1} \exp(-\lambda t) I_t/(i-1)!, \; i = 0, 1, 2, ... \tag{2.15}$$

where $I_t = 1$ if $t \geq 0$ and $I_t = 0$ if $t < 0$.

For a fixed $1 \leq j \leq N$, we denote $u_j(t_i)$ by $u_{j,t_i}, i = 1, 2, \ldots, n$. We observe the first N Fourier coefficient at the random time points $t_i, i = 1, 2, \ldots, n$. Thus the data set is given by $u_{j,t_i}, j = 1, 2, \ldots, N, i = 1, 2, \ldots, n$. Thus the total number of observations of the random field u on the grid is nN. For instance, in the case of an application to term structure data, this is quite natural, since we can expect many observations over a time of a number of different maturities. Our asymptotic set up is $n \to \infty$ and $N \to \infty$.

Define

$$\rho := \rho(\lambda, \theta) = \frac{\lambda}{\lambda - \kappa(\theta) + \alpha_j^{2m}}, j = 1, 2, \ldots, N. \tag{2.16}$$

Now we consider mixed fSPDE where the parameters are independent random variables.

FSPDE with mixed effect are useful for modeling neuronal data. Consider the parabolic mixed FSPDE

$$du^\theta(t,x) = \theta_j u^\theta(t,x) + \frac{\partial^2}{\partial x^2} u^\theta(t,x)dt + dW^H(t,x),\ t \geq 0,\ x \in [0,1] \tag{2.17}$$

$$u(0,x) = u_0(x) \in L_2([0,1]) \tag{2.18}$$

$$u^\theta(t,0) = u^\theta(t,1),\ t \in\ [0,T], \tag{2.19}$$

Consider the Fourier expansion of the process

$$u(t,x) = \sum_{i=1}^{\infty} u_i(t)\phi_i(x) \tag{2.20}$$

corresponding to some orthogonal basis $\{\phi_i(x)\}_{i=1}^{\infty}$. Note that $u_i^\theta(t), i \geq 1$ are independent one dimensional fractional Ornstein-Uhlenbeck processes

$$du_i^j(t) = \mu_i(\theta_j)u_i^j(t)dt + \alpha_i^{-a} dW_i^{j,H}(t) \tag{2.21}$$

$$u_i^j(0) = u_{0i}^\theta,$$

where $\mu_i(\theta_j) := k(\theta_j) - \alpha_i^{2m}$. Thus for $0 \leq t \leq T$, we have

$$du_i^j(t) = (k(\theta_j) - \alpha_i^{2m})u_i^j(t)dt + \alpha_i^{-a} dW_i^{j,H}(t), \quad j = 1, 2, \ldots, M, \quad i = 1, 2, \ldots, N. \tag{2.22}$$

Consider the more general set up where the particles could possible be interacting. We study asymptotics as both N and M tend to infinity while T is fixed. The random field $u(t,x)$ is observed at discrete times t and discrete positions x. Equivalently, the Fourier coefficients $u_i^j(t)$ are observed at discrete time points $t_k = k\frac{T}{n}, k = 1, 2, \ldots, n$ and we study asymptotics as $n \to \infty$. Thus we study triple asymptotics $n \to \infty$, $N \to \infty$ and $M \to \infty$. Euler scheme is given by

$$u_i^j(t_k) - u_i^j(t_{k-1}) = (k(\theta_j) - \alpha_i^{2m})u_i^j(t_{k-1})(t_k - t_{k-1}) + \alpha_i^{-a} W_i^{j,H}(t_k - t_{k-1}),$$
$$j = 1, 2, \ldots, M, \quad i = 1, 2, \ldots, N, \quad k = 1, 2, \ldots, n. \tag{2.23}$$

Here θ_j has the mixture normal distribution with density $g(x, \theta) = \sum_{r=1}^{R} \pi_r \mathcal{N}'(x, \varpi_r)$ with $\varpi_l = (\mu_l, \sigma_l)$ where R is the number of components in the mixture, π_r is the proportions of mixtures with $\sum_{l=1}^{R} \pi_r = 1$. We want to estimate the components of mixtures as well as the parameters and proportions. The convergence rate of estimators differ when deterministic components are present in the random effect.

Genon-Catalot and Laredo [36] studied estimation of the parameters of the distribution of the random effect from M i.i.d. diffusion processes $\{X_j(t), 0 \leq t \leq T, j = 1, 2. \ldots, M\}$ when M and $T(M)$ tend to infinity. Thus each sample path X_j is associated with a random effect

θ_j. They consider a two-stage procedure. For each j, an estimator of θ_j from the trajectories of $X_j(t), 0 \le t \le T, \;\; j = 1, 2, \dots, M$. Then use a plug-in technique to estimate the parameters of the distribution of θ_j (the population parameters). Thus (θ, X) is a two dimensional strong Markov process and hence a diffusion process. This is a state-space model.

For the finite dimensional fractional Ornstein-Uhlenbeck process, Berry-Esseen inequalities of minimum contrast estimators based on continuous and discrete observations was studied in Bishwal [13]. Large deviations in hypothesis testing for fractional Ornstein-Uhlenbeck models was studied in Bishwal [10].

In our case for the f-O-U process with one multiplicative random effect

$$du_j(t) = -\theta_j u_j(t)dt + dW_j^H(t), \;\; u_j(0) = \eta_j, \; j = 1, 2, \dots, M \tag{2.24}$$

where $\theta_j \in (0, \infty)$. We assume that θ_j has Gamma distribution with parameters $\beta = (\mu, \delta) \in (0, \;\infty)^2$ with the density

$$f(\beta, x) = \frac{\delta^\mu}{\Gamma(\mu)} x^{\mu-1} e^{-\delta x} 1_{x>0}. \tag{2.25}$$

We estimate the distribution of the random effect from M i.i.d. trajectories.

Consider the log-likelihood function associated with the observations of $(\theta_1, \theta_2, \dots, \theta_M)$

$$l_M(\beta) = M\mu \log \delta - M \log \Gamma(\mu) + (\mu - 1) \sum_{j=1}^{M} \log \theta_j - \delta \sum_{j=1}^{M} \theta_j \tag{2.26}$$

and define

$$\beta_M = \operatorname{Arg}\max_{\theta} \;\; l_M(\beta) \tag{2.27}$$

which is the the MLE based on direct observations $(\theta_1, \theta_2, \dots, \theta_M)$.

As $M \to \infty$, we have β_M is consistent and

$$\sqrt{M}(\beta_M - \beta) \to^{\mathcal{D}} N_2(0, I^{-1}(\beta)) \tag{2.28}$$

where

$$I(\beta) = \begin{pmatrix} \psi'(\mu) & -\delta^{-1} \\ -\delta^{-1} & \mu\delta^{-2} \end{pmatrix}$$

where

$$\psi(\mu) := \frac{\Gamma'(\mu)}{\Gamma(\mu)} = -\gamma + \int_0^1 \frac{1 - t^{\mu-1}}{1-t} dt \tag{2.29}$$

is the di-gamma function where $\gamma = -\Gamma'(1)$ is the Euler constant.

As the random variables θ_j's are not directly observed, a natural strategy consists in plugging $l_M(\beta)$ the estimators of θ_j's. For the plug-in we must define estimators of θ_j. One can use the MLE which is given by

$$\hat{\theta}_j = -\frac{\int_0^T Q_j(t) dZ_j(t)}{\int_0^T Q_j^2(t) dv_t} \tag{2.30}$$

and the MCE is which is given by

$$\tilde{\theta}_j = -\frac{T/2}{\int_0^T Q_j^2(t) dv_t}. \tag{2.31}$$

Truncation based estimator is given by

$$\hat{\theta}_j^{(k)} = \hat{\theta}_j 1_{\{V_{j,T}/T > k/\sqrt{T}\}} \tag{2.32}$$

where

$$V_{j,T} = \int_0^T Q_j^2(t)dv_t \tag{2.33}$$

where k is a numerical constant calibrated in practice. This estimator may be negative and null. Therefore we define a specific estimator of $L_j = \log\theta_j$ and set, for k a constant

$$\hat{L}_j^{(k)} = (\log\hat{\theta}_j)I_{\{\hat{\theta}_j\ge k/\sqrt{T},\ \ V_{j,T}/T>k/\sqrt{T}\}} = (\log\hat{\theta}_j^{(k)})I_{\{\hat{\theta}_j\ge k/\sqrt{T},\ \ V_{j,T}/T>k/\sqrt{T}\}}.$$

The MLE and the truncated estimator are asymptotically equivalent:

$$\sqrt{T}(\hat{\theta}_j - \hat{\theta}_j^{(k)}) = o_P(1). \tag{2.34}$$

Based on these estimators one can proceed to the estimation of density. Set

$$V_M(\beta) = M\mu\log\delta - M\log\Gamma(\mu) + (\mu-1)\sum_{j=1}^M \log\hat{\theta}_j^{(k)} 1_{\{\hat{\theta}_j\ge k/\sqrt{T},\ \ V_{j,T}/T>k/\sqrt{T}\}} - \delta\sum_{j=1}^M \hat{\theta}_j^{(k)} \tag{2.35}$$

and define

$$\tilde{\beta}_M = \operatorname{Arg}\max_{\beta}\ V_M(\beta). \tag{2.36}$$

Theorem 2.8 *Assume that $\mu > 8$. a) The estimator $\tilde{\beta}_M$ is consistent as $M\to\infty$ and $T\to\infty$. b) $\sqrt{M}(\tilde{\beta}_M - \beta_M) = o_P(1)$ when $M\to\infty$ and $T\to\infty$ such that $\frac{M}{T}\to 0$ where β_M is the MLE of β based on the observations $(\theta_1,\theta_2,\ldots,\theta_M)$.*

This shows the asymptotic equivalence of the plug-in estimator and the MLE.

3. Random Effect and Random Sampling

For the finite dimensional SDE with fBM noise, Maitra and Bhattacharya [47] considered discretization if the likelihood. They obtained strong consistency and asymptotic normality of the resulting estimator under increasing domain infill asymptotics, i.e., when the time domain in increases, the number of discrete time points in the domain is increased faster than T, attempting to fill up the domain. $(T\to\infty, n\to\infty, n/T\to\infty, n/T^2\to\infty.)$ They also obtained posterior asymptotic normality using regularilty conditions of Schervish [56]. We use random temporal discretization. Bishwal [12] studied conditional least squares estimation in finite dimensional diffusion models based on Poisson sampling. Usually optimal discretization is achieved by random time interval, see Hofmann *et al.* [37]. We observe the process at the arrival times of a Poisson process. We study the parameter estimation in two steps: The rate λ of the Poisson process can be estimated given the jump times t_i, therefore it is done at a first step. Since we observe total number of jumps n of the Poisson process over the T intervals of length one, the MLE of λ is given by

$$\hat{\lambda}_n := \frac{n}{T}.$$

Theorem 3.1 *We have*

$$\text{a)}\quad \hat{\lambda}_n \to \lambda \text{ a.s. as } \ n\to\infty.$$

$$\text{b)}\quad \sqrt{n}(\hat{\lambda}_n - \lambda) \to^{\mathcal{D}} \mathcal{N}(0,\ e^{\lambda}(1-e^{-\lambda})) \text{ as } n\to\infty.$$

Proof. Let J_i be the number of jumps in the interval $(i-1, i]$. Then J_i, $i = 1, 2, \ldots, n$ are i.i.d. Poisson distributed with parameter λ. Since Φ is continuous, we have $I_{\{0\}}(J_i) = I_{\{0\}}(Q_{t_i})$, a.s. $i = 1, 2, \ldots, n$. Note that

$$\frac{1}{nN}\sum_{k=1}^{N}\sum_{i=1}^{n} I_{\{0\}}(Q_{k,t_i}) \to^{a.s.} E(I_{\{0\}}J_1) = P(J_1 = 0) = e^{-\lambda} \text{ as } n \to \infty.$$

LLN and CLT and delta method applied to the sequence $I_{\{0\}}(Q_{t_i})$, $i = 1, 2, \ldots, n$ give the results. □

The CLT result above allows us to construct confidence interval for the jump rate λ. A $100(1-\alpha)\%$ confidence interval for λ is given by

$$\left[\frac{n}{T} - \varepsilon_{1-\frac{\alpha}{2}}\sqrt{\frac{1}{n} - \frac{1}{T}},\ \ \frac{n}{T} + \varepsilon_{1-\frac{\alpha}{2}}\sqrt{\frac{1}{n} - \frac{1}{T}}\right]$$

where $\varepsilon_{1-\frac{\alpha}{2}}$ is the $(1-\frac{\alpha}{2})$-quantile of the standard normal distribution.

We have a time series representation of the model. It is well known that the discretized version of the O-U process is an first order autoregressive process (AR(1)). Hence we have

$$Q_{k,t_{i+1}} = e^{-\mu_k(\theta_k)\Delta t_i} Q_{k,t_i} + \epsilon_{k,i}$$

where

$$\epsilon_{k,i} \sim \mathcal{MN}\left(0, \frac{1 - e^{-2\mu(\theta_k)\Delta t_i}}{\nu(\theta)}\sigma_i^2\right),\ i \geq 1, k \geq 1$$

conditionally on θ_k and $\mathcal{MN}$ denotes mixed normal distribution.

First we are interested in the estimation of the random variable θ. Note that it is not a statistical parameter in the usual sense since θ is not a parameter but a random variable. Several work these days deal with estimation of random variables. One of the most popular example is the estimation of stochastic integrated volatility in semimartingale models based on high frequency observations of the sample path, see Bishwal [17]. We approach the problem through estimation function method, whose ideas resembles GMM method. See also Bishwal [11] where estimation was based on N i.i.d. trajectories of $[0, T]$.

Define $\rho_k := \frac{\lambda}{\lambda + \theta_k}$. Mixingale estimation function (MEF) estimator, which is also the quasi maximum likelihood estimator (QMLE) is the solution of the estimating equation: $G^*_{n,N}(\theta) = 0$ where

$$G^*_{n,N}(\theta) = \sum_{k=1}^{K}\sum_{i=1}^{n} \frac{\alpha_k^{2a}\lambda(\rho(\lambda,\theta_k))^2}{\rho(\lambda, 2\theta_k)} Q_{k,t_{i-1}} \left[(Q_{k,t_{i-1}}\theta_k\rho(\lambda,\theta_k))^2 + \lambda\right]^{-1} [Q_{k,t_i} - \rho(\lambda,\theta_k)Q_{k,t_{i-1}}].$$

We call the solution of the estimating equation the quasi maximum likelihood estimator (QMLE). There is no explicit solution for this equation.

The optimal estimating function for estimation of the unknown parameter θ is given by

$$G_{n,N}(\theta) = \sum_{k=1}^{N}\sum_{i=1}^{n} \alpha_k^{2a} Q_{k,t_{i-1}}[Q_{k,t_i} - \rho_k(\lambda,\theta_k)Q_{k,t_{i-1}}].$$

The mixingale estimation function (MEF) estimator of ρ is the solution of $G_{n,N}(\theta) = 0$ and is given by

$$\hat{\rho}_{N,n} := \frac{\sum_{k=1}^{N}\sum_{i=1}^{n} Q_{k,t_{i-1}} Q_{k,t_i}}{\sum_{k=1}^{N}\sum_{i=1}^{n} Q_{k,t_{i-1}}^2}. \tag{3.4}$$

We obtain the strong consistency and asymptotic normality of the estimator.

Theorem 3.2

$$\text{a)} \quad \hat{\rho}_{N,n} \to^P \rho \text{ as } \; n \to, \infty \; \text{ and } N \to \infty,$$

such that $\frac{N}{n} \to 0$.

$$\text{b)} \quad \sqrt{n\Psi_N}(\hat{\rho}_{N,n} - \rho) \to^{\mathcal{D}} \mathcal{MN}(0,\ \lambda^{-i}(1 - e^{-\rho})) \;\; \text{as } n \to \infty \text{ and } N \to \infty$$

such that $\frac{N}{\sqrt{n}} \to 0$ where $\mathcal{MN}$ denotes mixed normal distribution.

Proof: By using the fact that every stationary mixing process is ergodic, it is easy to show that if $Q_k(t)$ is a stationary ergodic O-U process and t_i is a process with nonnegative i.i.d. increments which is independent of $Q_k(t)$, then $\{Q_{k,t_i}, i \geq 1, k \geq 1\}$ is a stationary ergodic Markov process. Hence $\{Q_{j,t_i}, i \geq 1\}$ is a stationary ergodic process. Thus the extra randomness of the sampling instants preserves the stationarity and ergodicity of the process in order for the law of large numbers to be applicable.

Observe that $Q_j^\theta(t) := v_j$ is a stationary ergodic sequence and $v_j \sim \mathcal{N}(0, \sigma^2)$ where σ^2 is the variance of Q_{1,t_0}. Thus by SLLN for zero mean square integrable mixingales (Theorem 2.5 in Bishwal [18], Peligrad and Utev ([50], Theorem B) and arguments in Bibinger and Trabs ([3], Proposition 7.6), we have

$$\sum_{k=1}^{N}\sum_{i=1}^{n} Q_{k,t_{i-1}} Q_{k,t_i} \to^{a.s.} E(Q_{1,t_1} Q_{k,t_0}) = \rho E(Q_{1,t_0}^2) \tag{3.5}$$

and

$$\frac{1}{n\Psi_N}\sum_{k=1}^{N}\sum_{i=1}^{n} Q_{k,t_{i-1}}^2 \to^{a.s.} E(Q_{1,t_0}^2). \tag{3.6}$$

Further $Q_{k,i}(t) := S_i$ is a stationary ergodic Markov chain and $S_i \sim \mathcal{N}(0, \sigma^2)$ where σ^2 is the variance of $Q_{k,0}$. SLLN for martingales proves the result.

Thus

$$\frac{\sum_{i=1}^{n} Q_{k,t_{i-1}} Q_{k,t_i}}{\sum_{i=1}^{n} Q_{k,t_{i-1}}^2} \to^P \rho. \tag{3.7}$$

Further,

$$\sqrt{n\Psi_N}(\hat{\rho}_n - \rho) = \frac{(n\Psi_N)^{-1/2}\sum_{k=1}^{N}\sum_{i=1}^{n} Q_{k,t_{i-1}}(Q_{k,t_i} - \theta Q_{k,t_{i-1}})}{(n\Psi_N)^{-1}\sum_{k=1}^{N}\sum_{i=1}^{n} Q_{k,t_{i-1}}^2}. \tag{3.8}$$

Since

$$E(Q_{k,t_2} Q_{k,t_1} | Q_{k,t_1}) = \theta Q_{k,t_1}^2 \tag{3.9}$$

it follows by Theorem 2.7 and Theorem 2.2 in Bishwal [18] which is a generalization of Peligrad and Utev ([50], Theorem B), along with the arguments in Bibinger and Trabs [4], that

$$(n\Psi_N)^{-1/2}\sum_{k=1}^{N}\sum_{i=1}^{n} Q_{k,t_{i-1}}(Q_{k,t_i} - \theta Q_{k,t_{i-1}})$$

converges in distribution to normal distribution with mean zero and variance equal to

$$E[(Q_{k,t_1}Q_{k,t_2}) - E(Q_{k,t_1}Q_{k,t_2}|Q_{k,t_1})]^2 = (1 - e^{2(\theta-\lambda_i\delta)})\{2(\lambda_i - \theta)(\lambda_i + 1)\}^{-1}. \tag{3.10}$$

Applying delta method the result follows. □

In the next step, we use the estimator of λ to estimate θ.

Note that

$$\frac{1}{\hat{\rho}_{n,N}} = \frac{\sum_{k=1}^{N}\sum_{i=1}^{n} Q_{j,t_{i-1}}^2}{\sum_{k=1}^{N}\sum_{i=1}^{n} Q_{j,t_{i-1}}Q_{j,t_i}}. \tag{3.11}$$

Thus

$$1 + \frac{\alpha_1^{2m} - \kappa(\theta)}{\lambda} = \frac{\sum_{k=1}^{N}\sum_{i=1}^{n} Q_{k,t_{i-1}}^2}{\sum_{k=1}^{N}\sum_{i=1}^{n} Q_{k,t_{i-1}}Q_{k,t_i}}. \tag{3.12}$$

Hence

$$\begin{aligned}\frac{\alpha_1^{2m} - \kappa(\theta)}{\lambda} &= \frac{\sum_{k=1}^{N}\sum_{i=1}^{n} Q_{k,t_{i-1}}^2}{\sum_{k=1}^{N}\sum_{i=1}^{n} Q_{k,t_{i-1}}Q_{k,t_i}} - 1 \\ &= -\frac{\sum_{k=1}^{N}\sum_{i=1}^{n} Q_{k,t_{i-1}}[Q_{k,t_i} - Q_{k,t_{i-1}}]}{\sum_{k=1}^{N}\sum_{i=1}^{n} Q_{k,t_{i-1}}Q_{j,t_i}}\end{aligned} \tag{3.13}$$

Now replace λ by its estimator MLE $\hat{\lambda}_n = \frac{n}{T}$.

$$\alpha_1^{2m} - \kappa(\theta) = -\frac{\sum_{k=1}^{N}\sum_{i=1}^{n} Q_{j,t_{i-1}}[Q_{k,t_i} - Q_{j,t_{i-1}}]}{\frac{T}{n}\sum_{k=1}^{N}\sum_{i=1}^{n} Q_{k,t_{i-1}}Q_{k,t_i}} \tag{3.14}$$

Thus

$$\hat{\theta}_{N,n} = \kappa^{-1}\left(\alpha_1^{2m} + \frac{\sum_{k=1}^{N}\sum_{i=1}^{n} Q_{k,t_{i-1}}[Q_{k,t_i} - Q_{k,t_{i-1}}]}{\frac{T}{n}\sum_{k=1}^{N}\sum_{i=1}^{n} Q_{k,t_{i-1}}Q_{k,t_i}}\right). \tag{3.15}$$

Since the function $\kappa^{-1}(\cdot)$ is a continuous function, by application of delta method, the following result is a consequence of Theorem 3.2.

Since the function $\kappa^{-1}(\cdot)$ is a continuous function, applying delta method the following result follows.

Theorem 3.3

$$\text{a)}\quad \hat{\theta}_{N,n} \to^{P} \theta \quad \text{as} \quad n \to \infty \text{ and } N \to \infty$$

such that $\frac{N}{n} \to 0$.

$$\text{b)}\quad \sqrt{n\Psi_N}(\hat{\theta}_{N,n} - \theta) \to^{\mathcal{D}} \mathcal{MN}(0,\ (\kappa'(\theta))^{-2}\lambda^2(1 - e^{-2\lambda^{-1}(\kappa(\theta)-\alpha_1^{2m})}))$$

as $n \to \infty$ and $N \to \infty$ such that $\frac{N}{\sqrt{n}} \to 0$ *and* $\mathcal{MN}$ *denotes mixed normal distribution.*

In the second stage, we substitute λ *by its estimator* $\hat{\lambda}_n$.

As a consequence of (2.8) and Theorem 2.8, in the case of mixed FSPDE, based on the observations of the N Fourier coefficients and the M MEF estimators of the random parameters, we have the following results :

In the case of continuous sampling with fixed T, we have

Theorem 3.4 *As $M \to \infty$ and $N \to \infty$ such that $\frac{M}{N} \to 0$, we have*

$$\hat{\beta}_{M,N} \to^P \beta$$

and

$$\sqrt{M\Psi_N}(\hat{\beta}_{M,N} - \beta) \to^{\mathcal{D}} N_2(0, I^{-1}(\beta))$$

as $M \to \infty$ and $N \to \infty$ such that $\frac{M}{\sqrt{N}} \to 0$.

In the case of discrete sampling, we have

Theorem 3.5 *As $M \to \infty$, $N \to \infty$ and $n \to \infty$ such that $\frac{M}{N} \to 0$, $\frac{N}{n} \to 0$, we have*

$$\hat{\beta}_{M,N,n} \to^P \beta$$

and

$$\sqrt{M\Psi_N n}(\hat{\beta}_{M,N,n} - \beta) \to^{\mathcal{D}} N_2(0, I^{-1}(\beta))$$

as $M \to \infty$, $N \to \infty$ and $n \to \infty$ such that $\frac{M}{\sqrt{N}} \to 0$ and $\frac{N}{\sqrt{n}} \to 0$.

4. Examples

1) Consider the stochastic heat equation

$$du^\theta(t,x) = \theta \frac{\partial^2}{\partial x^2} u^\theta(t,x)dt + dW^H(t,x) \tag{4.1}$$

for $0 \le t \le T$ and $x \in (0,1)$ and $\theta > 0$ with periodic boundary conditions.

Here $2m = m_1 = 2$ and $\mu_j = -\theta\pi^2 j^2$, $\gamma > 1/2$ and $\psi_N = N^3$. Hence

$$\sqrt{nN^3}(\hat{\theta}_{n,N} - \theta) \to^{\mathcal{D}} \mathcal{MN}(0,\ (\kappa'(\theta))^{-2}\lambda^2(1 - e^{-2\lambda^{-1}(\kappa(\theta)-\alpha_1^{2m})})) \text{ as } n \to \infty \text{ and } N \to \infty.$$

For the fixed n and fixed parameter θ case, Es-Sebaiy *et al.* [33] obtained Berry-Esseen bound of the order $O(N^{-3/2})$ using the Stein-Malliavin theory for the MLE there by improving the bound $O(N^{-1})$ of Kim and Park [39].

2) As another example of the evolution equation consider the linear parabolic equation

$$du^\theta(t,x) = \theta u^\theta(t,x) + \frac{\partial^2}{\partial x^2} u^\theta(t,x)dt + dW^H(t,x),\ t \ge 0,\ x \in [0,1] \tag{4.2}$$

$$u(0,x) = u_0(x) \in L_2([0,1]) \tag{4.3}$$

$$u^\theta(t,0) = u^\theta(t,1),\ t \in\ [0,T], \tag{4.4}$$

If $d = 2$, then we have

$$\sqrt{n \log N}(\hat{\theta}_{n,N} - \theta) \to^{\mathcal{D}} \mathcal{MN}(0,\ (\kappa'(\theta))^{-2}\lambda^2(1 - e^{-2\lambda^{-1}(\kappa(\theta)-\alpha_1^{2m})})) \text{ as } n \to \infty \text{ and } N \to \infty.$$

If $d > 2$, then we have

$$\sqrt{nN^{(d-2)/d}}(\hat{\theta}_{n,N} - \theta) \to^{\mathcal{D}} \mathcal{MN}(0,\ (\kappa'(\theta))^{-2}\lambda^2(1 - e^{-2\lambda^{-1}(\kappa(\theta)-\alpha_1^{2m})})) \text{ as } n \to \infty \text{ and } N \to \infty.$$

Concluding Remarks: This problem has some connection to diffusion in a random environment, see e.g., Follmer [34] and Follmer and Schweizer [35] and estimation in stochastic volatility models, see Bishwal [17]. As is very well known, randomizing the environment slows down the random walk, see Solomon [57].

References

[1] A. Bagchi, K.S. Kumar, An infinite factor model for the interest rate derivatives, in: M. Kohlmann, S. Tang (Eds.), Mathematical Finance, Birkhäuser Basel, Basel, 2001: pp. 59–68. https://doi.org/10.1007/978-3-0348-8291-0_5.

[2] B.M. Bibby, M. Sørensen, M. Sorensen, Martingale estimation functions for discretely observed diffusion processes, Bernoulli. 1 (1995) 17-39. https://doi.org/10.2307/3318679.

[3] M. Bibinger, M. Trabs, Volatility estimation for stochastic PDEs using high-frequency observations, Stoch. Proc. Appl. 130 (2020) 3005–3052. https://doi.org/10.1016/j.spa.2019.09.002.

[4] M. Bibinger, M. Trabs, On central limit theorems for power variations of the solution to the stochastic heat equation, in: A. Steland, E. Rafajłowicz, O. Okhrin (Eds.), Stochastic Models, Statistics and Their Applications, Springer International Publishing, Cham, 2019: pp. 69–84. https://doi.org/10.1007/978-3-030-28665-1_5.

[5] J.P.N. Bishwal, Bayes and sequential estimation in Hilbert space valued stochastic differential equations, J. Korean Stat. Soc. 28 (1999) 93-106.

[6] J.P.N. Bishwal, Rates of convergence of the posterior distributions and the Bayes estimations in the Ornstein-Uhlenbeck process, Rand. Oper. Stoch. Equations. 8 (2000) 51-70. https://doi.org/10.1515/rose.2000.8.1.51.

[7] J.P.N. Bishwal, The Bernstein-von Mises theorem and spectral asymptotics of Bayes estimators for parabolic SPDEs, J. Aust. Math. Soc. 72 (2002) 287–298. https://doi.org/10.1017/S1446788700003906.

[8] J.P.N. Bishwal, A new estimating function for discretely sampled diffusions, Rand. Oper. Stoch. Equ. 15 (2007) 65-88. https://doi.org/10.1515/rose.2007.005.

[9] J.P.N. Bishwal, Parameter estimation in stochastic differential equations, Lecture Notes in Mathematics, 1923, Springer-Verlag, (2008).

[10] J.P.N. Bishwal, Large deviations in testing fractional Ornstein–Uhlenbeck models, Stat. Probab. Lett. 78 (2008) 953–962. https://doi.org/10.1016/j.spl.2007.09.055.

[11] J. Bishwal, Maximum likelihood estimation in Skorohod stochastic differential equations, Proc. Amer. Math. Soc. 138 (2009) 1471–1478. https://doi.org/10.1090/s0002-9939-09-10113-2.

[12] J.P.N. Bishwal, Conditional least squares estimation in diffusion processes based on Poisson sampling, J. Appl. Probab. Stat. 5 (2010) 169-180.

[13] J.P.N. Bishwal, Minimum contrast estimation in fractional Ornstein-Uhlenbeck process: continuous and discrete sampling, Fract. Calc. Appl. Anal. 14 (2011) 375-410. https://doi.org/10.2478/s13540-011-0024-6.

[14] J.P.N. Bishwal, Hypothesis testing for fractional stochastic partial differential equations with applications to neurophysiology and finance, Asian Res. J. Math. 4 (2017) 1-24.

[15] J.P.N. Bishwal, Bernstein-von Mises theorem and small noise Bayesian asymptotics for parabolic stochastic partial differential equations, Theory Stoch. Proc. 23 (2018) 6-17.

[16] J.P.N. Bishwal, Statistics of SPDEs: From linear to nonlinear, Eur. J. Stat. 1 (2021) 1. https://doi.org/10.28924/ada/stat.1.1.

[17] J.P.N. Bishwal, Parameter estimation in stochastic volatility models, Springer Nature, Cham (in press) ISBN:978-3-031-03860-0. (2022).

[18] J.P.N. Bishwal, Mixingale estimation function for SPDEs with random sampling, Eur. J. Stat. 2 (2022) 3. https://doi.org/10.28924/ada/stat.2.3.

[19] J.P.N. Bishwal, A. Bose, Rates of convergence of approximate maximum likelihood estimators in the Ornstein-Uhlenbeck process, Computers Math. Appl. 42 (2001) 23–38. https://doi.org/10.1016/s0898-1221(01)00127-4.

[20] Z. Cheng, I. Cialenco, R. Gong, Bayesian estimations for diagonalizable bilinear SPDEs, Stoch. Proc. Appl. 130 (2020) 845–877. https://doi.org/10.1016/j.spa.2019.03.020.

[21] I. Cialenco, F. Delgado-Vences, H.-J. Kim, Drift estimation for discretely sampled SPDEs, Stoch. PDE: Anal. Comp. 8 (2020) 895–920. https://doi.org/10.1007/s40072-019-00164-4.

[22] C. Chong, High-frequency analysis of parabolic stochastic PDEs, Ann. Statist. 48 (2020) 1143-1167. https://doi.org/10.1214/19-aos1841.

[23] F. Comte, V. Genon-Catalot, A. Samson, Nonparametric estimation for stochastic differential equations with random effects, Stoch. Proc. Appl. 123 (2013) 2522–2551. https://doi.org/10.1016/j.spa.2013.04.009.

[24] M. Delattre, v. Genon-Catalot, A. Samson, Maximum likelihood estimation for stochastic differential equations with random effects, Scandinavian J. Stat. 40 (2012) 322–343. https://doi.org/10.1111/j.1467-9469.2012.00813.x.

[25] M. Delattre, V. Genon-Catalot, A. Samson, Estimation of population parameters in stochastic differential equations with random effects in the diffusion coefficient, ESAIM: PS. 19 (2015) 671–688. https://doi.org/10.1051/ps/2015006.

[26] M. Delattre, V. Genon-Catalot, A. Samson, Mixtures of stochastic differential equations with random effects: Application to data clustering, J. Stat. Plan. Inference. 173 (2016) 109–124. https://doi.org/10.1016/j.jspi.2015.12.003.

[27] M. Delattre, V. Genon-Catalot, C. Larédo, Parametric inference for discrete observations of diffusion processes with mixed effects, Stoch. Proc. Appl. 128 (2018) 1929–1957. https://doi.org/10.1016/j.spa.2017.08.016.

[28] M. Delattre, V. Genon-Catalot, C. Larédo, Approximate maximum likelihood estimation for stochastic differential equations with random effects in the drift and the diffusion, Metrika. 81 (2018) 953–983. https://doi.org/10.1007/s00184-018-0666-z.

[29] C. Dion, Nonparametric estimation in a mixed-effect Ornstein–Uhlenbeck model, Metrika. 79 (2016) 919–951. https://doi.org/10.1007/s00184-016-0583-y.

[30] C. Dion, V. Genon-Catalot, Bidimensional random effect estimation in mixed stochastic differential model, Stat. Inference Stoch. Proc. 19 (2015) 131–158. https://doi.org/10.1007/s11203-015-9122-0.

[31] S. Ditlevsen, A. De Gaetano, Mixed effects in stochastic differential equation models, REVSTAT-Stat. J. 3 (2005) 137-153. https://doi.org/10.57805/REVSTAT.V3I2.22.

[32] S. Donnet, A. Samson, Parametric inference for mixed models defined by stochastic differential equations, ESAIM: PS. 12 (2008) 196–218. https://doi.org/10.1051/ps:2007045.

[33] K. Es-Sebaiy, M. Al-Foraih, F. Alazemi, Wasserstein bounds in the CLT of the MLE for the drift coefficient of a stochastic partial differential equation, Fractal Fract. 5 (2021) 187. https://doi.org/10.3390/fractalfract5040187.

[34] H. Follmer, W. Cheung, M.A.H. Dempster, Stock price fluctuation as a diffusion in a random environment, Phil. Trans. R. Soc. London: Phys. Sci. Eng. 347 (1684) 471-483.

[35] H. Föllmer, M. Schweizer, A microeconomic approach to diffusion models for stock prices, Math. Finance. 3 (1993) 1–23. https://doi.org/10.1111/j.1467-9965.1993.tb00035.x.

[36] V. Genon-Catalot, C. Larédo, Estimation for stochastic differential equations with mixed effects, Statistics. 50 (2016) 1014–1035. https://doi.org/10.1080/02331888.2016.1141910.

[37] N. Hofmann, T. Müller-Gronbach, K. Ritter, The optimal discretization of stochastic differential equations, J. Complexi. 17 (2001) 117–153. https://doi.org/10.1006/jcom.2000.0570.

[38] J. Jacod, On continuous conditional Gaussian martingales and stable convergence in law. In: Azéma, J., Yor, M., Emery, M. (eds) Séminaire de Probabilités XXXI. Lecture Notes in Mathematics, vol 1655. Springer, Berlin, Heidelberg. (1997). https://doi.org/10.1007/BFb0119308

[39] Y.T. Kim, H.S. Park, Convergence rate of maximum likelihood estimator of parameter in stochastic partial differential equation, J. Korean Stat. Soc. 44 (2015) 312–320. https://doi.org/10.1016/j.jkss.2015.01.001.

[40] M.L. Kleptsyna, A. Le Breton, A. statistical analysis of the fractional Ornstein–Uhlenbeck type process, Stat. Inference Stoch. Proc. 5 (2002) 229–248. https://doi.org/10.1023/a:1021220818545.

[41] T. Koski, W. Loges, Asymptotic statistical inference for a stochastic heat flow problem, Stat. Probab. Lett. 3 (1985) 185–189. https://doi.org/10.1016/0167-7152(85)90015-x.

[42] T. Koski, W. Loges, On minimum-contrast estimation for hilbert space-valued stochastic differential equations, Stochastics. 16 (1986) 217–225. https://doi.org/10.1080/17442508608833374.

[43] W. Loges, Girsanov's theorem in Hilbert space and an application to the statistics of Hilbert space- valued stochastic differential equations, Stoch. Proc. Appl. 17 (1984) 243–263. https://doi.org/10.1016/0304-4149(84)90004-8.

[44] T. Maitra, S. Bhattacharya, On Bayesian asymptotics in stochastic differential equations with random effects, Stat. Probab. Lett. 103 (2015) 148–159. https://doi.org/10.1016/j.spl.2015.04.009.

[45] T. Maitra, S. Bhattacharya, On asymptotics related to classical inference in stochastic differential equations with random effects, Stat. Probab. Lett. 110 (2016) 278–288. https://doi.org/10.1016/j.spl.2015.10.001.

[46] T. Maitra, S. Bhattacharya, On asymptotic inference in stochastic differential equations with time-varying covariates, Can. J. Stat. 46 (2018) 635–655. https://doi.org/10.1002/cjs.11471.

[47] T. Maitra, T. Bhattacharya, Increasing domain infill asymptotics for stochastic differential equations driven by fractional Brownian motion, ArXiv:2005.09577v1, (2020). https://doi.org/10.48550/arXiv.2005.09577.

[48] J. Mohapl, On estimation in the planar ornstein-unlenbeck process, Commun. Stat. Stoch. Models. 13 (1997) 435–455. https://doi.org/10.1080/15326349708807435.

[49] I. Norros, E. Valkeila, J. Virtamo, An elementary approach to a girsanov formula and other analytical results on fractional Brownian motions, Bernoulli. 5 (1999) 571. https://doi.org/10.2307/3318691.

[50] M. Peligrad, S. Utev, Central limit theorem for linear processes, Ann. Probab. 25 (1997) 443-456. https://doi.org/10.1214/aop/1024404295.

[51] U. Picchini, A. De Gaetano, S. Ditlevsen, Stochastic differential mixed-effect models, Scandinavian J. Stat. 37 (2010) 67-90.

[52] U. Picchini, S. Ditlevsen, Practical estimation of high dimensional stochastic differential mixed-effects models, Comput. Stat. Data Anal. 55 (2011) 1426–1444. https://doi.org/10.1016/j.csda.2010.10.003.

[53] U. Picchini, S. Ditlevsen, A. De Gaetano, P. Lansky, Parameters of the diffusion leaky integrate-and-fire neuronal model for a slowly fluctuating signal, Neural Comput. 20 (2008) 2696–2714. https://doi.org/10.1162/neco.2008.11-07-653.

[54] U. Picchini, J.L. Forman, Bayesian inference for stochastic differential equation mixed effects models of a tumour xenography study, J. R. Stat. Soc. C. 68 (2019) 887–913. https://doi.org/10.1111/rssc.12347.

[55] M.G. Ruse, A. Samson, S. Ditlevsen, Inference for biomedical data by using diffusion models with covariates and mixed effects, J. R. Stat. Soc. C. 69 (2019) 167–193. https://doi.org/10.1111/rssc.12386.

[56] M.J. Schervish, Theory of Statistics, Springer-Verlag, New York, (1995).

[57] F. Solomon, Random walks in a random environment, Ann. Probab. 3 (1975) 1-31.

[58] C.A. Tudor, F.G. Viens, Statistical aspects of the fractional stochastic calculus, Ann. Statist. 35 (2007) 1183-1212. https://doi.org/10.1214/009053606000001541.

[59] G.A. Whitaker, A. Golightly, R.J. Boys, C. Sherlock, Bayesian inference for diffusion-driven mixed-effects models, Bayesian Anal. 12 (2017) 435-463. https://doi.org/10.1214/16-ba1009.