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ON MILD SOLUTIONS FOR INTEGRODIFFERENTIAL SYSTEMS WITH INTEGRAL IMPULSES IN BANACH SPACES

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Abstract. This paper deals with integrodifferential equations with integral impulses. First, we establish a new formula for the variation of the constants in order to obtain the form of the mild solution for such equations. Then, using the theory of the resolvent operator due to R. Grimmer, associated with the Burton-Kirk fixed point theorem, we give and prove a theorem of existence of solutions for these equations. At the end, we give an example to illustrate the obtained results.

1. Introduction

The theory of integrodifferential equations has gained a great development in recent years, due to its multiple applications in several fields of science and engineering. Several researchers have therefore studied various aspects, both qualitative and quantitative, of these equations. For example, in [9], Diop et al. studied the existence of mild solutions for non-Lipschitz stochastic integrodifferential evolution equations, Liu and Ezzinbi studied in [17] a mild solutions for non-autonomous integrodifferential systems with non-local conditions, Balachandran and Ravi Kumar, considered in [1] a varying delayed non-autonomous integrodifferential system, Diop et al. [11] established the existence and asymptotic behavior of solutions for neutral stochastic partial integrodifferential equations with infinite delays and in [5], Caraballo et al. considered a class of neutral stochastic delay partial functional integro-differential equations driven by a fractional Brownian motion...

However, in real life, there are situations for which, the phenomena studied undergo an abrupt change either at given instants, or during relatively short time intervals. Such phenomena are said to be impulsive and their modeling is done using so-called impulsive equations. Given the importance of the phenomena concerned, the study of impulsive equations is therefore essential.

The theory of impulsive equattions has been widely developed, especially concerning differential equations (see for example the papers of Wang and Ezzinbi [20], Mophou [18], Suganya and Arjunan [19], Hernandez et al. [15], Chadha and Pandey [8] and the book of Benchora et al. [3]).

Indeed, one of the major tools to study such equations is based on the theory of semi-groups, which greatly simplifies the form of the mild solution obtained by the formula of the variation of the constants, due to the property translation verified by the semi-groups.

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On the other hand, the case of impulsive integrodifferential equations deserves more particular attention. Indeed, the study of integrodifferential equations uses the theory of the resolvent operator developed by R. Grimmer in [13]. But unlike the semi-groups, the resolvent operator does not check the translation property, which makes it difficult to obtain the form of a mild solution by the method of variation of the constants.

Motivated by all these considerations, in this paper, we consider the class of integrodifferential equations with integral impulses given by:

$$(1)\qquad \begin{cases} z'(t) = \mathsf{A}z(t) + \int_0^t \mathsf{U}(t-s)z(s)ds + \delta\left(t, z_t, \int_0^t \varrho(t,s,z_s)ds\right), \\ \qquad\qquad t \in \mathcal{T} := [0,\mathsf{T}],\ t \neq t_i,\ i \in [\![1,m]\!] \\ z(t) = \varphi(t), \quad t \in [-\tau, 0] \\ \Delta z(t_i) = \mathsf{I}_i\left(\int_{t_i-\alpha_i}^{t_i-\beta_i} h(s,z_s)ds\right), \quad i \in [\![1,m]\!]. \end{cases}$$

where $(\mathsf{X}, \|\cdot\|)$ and $(\mathsf{Y}, \|\cdot\|_{\mathsf{Y}})$ are two Banach spaces, $\mathcal{C}(\mathcal{T},\mathsf{X})$ is the Banach space of all the functions which are continuous from $\mathcal{T}$ into X, furnished with the supremum norm. A and $\mathsf{U}(t)$, $t \in \mathcal{T}$ are closed linear operators on X, with comon domain $\mathsf{D}(\mathsf{A})$ which does not dependent on t, the functions $\delta : \mathcal{T}\times\mathcal{C}([-\tau,0],\mathsf{X})\times\mathsf{X} \to \mathsf{X}$, $\varrho : \mathcal{T}\times\mathcal{T}\times\mathcal{C}([-\tau,0],\mathsf{X}) \to \mathsf{X}$ and $h : \mathcal{T}\times\mathsf{X} \to \mathsf{X}$ are given, $\varphi \in \mathcal{C}([-\tau,0],\mathsf{X})$, $(\tau < \infty)$, $0 = t_0 < t_1 < \cdots < t_m < t_m + 1 = \mathsf{T}$, $\mathsf{I}_i \in \mathcal{C}(\mathsf{X},\mathsf{X})$ are bounded functions, $\alpha_i \leq \beta_i \leq t_i - t_{i-1}$, $i \in [\![1,m]\!]$, $\Delta z(t_i) = z(t_i^+) - z(t_i^-)$, where $z(t_i^+) = \lim\limits_{t\to t_i^+} z(t)$ and $z(t_i^-) = \lim\limits_{t\to t_i^-} z(t)$. The function $z_t : [-\tau,0] \to \mathsf{X}$ defined by $z_t(\theta) := z(t+\theta)$ represent the delay.

After having introduced some notions on the theory of the resolvent operator, we give a formula of the variation of the constants to obtain the mild solutions of impulsive integrodifferential equations, in section 2. Then, we prove in section 3 the existence of mild solutions for the impulsive integrodifferential system given in (1). In section 4, we give an example to illustrate the obtained results and we end with a conclusion.

2. Preliminaries

2.1. **The resolvent operator.**

Throughout this paper, X is a Banach space with norm $\|\cdot\|$. A, and U are closed linear operators defined on X. Let us denote by $\mathcal{BC}(\mathcal{T},\mathsf{X})$ the space of all the maps z which are continuous and bounded from $\mathcal{T}$ into X. When we endow $\mathcal{BC}(\mathcal{T},\mathsf{X})$ with the supremum norm $\|z\|_{\mathcal{BC}} = \sup\limits_{t\in\mathcal{T}} \|z(t)\|$, then $\mathcal{BC}(\mathcal{T},\mathsf{X})$ is a Banach space. This norm will also be denoted by $\|\cdot\|$ if there is no possible confusion.

We introduce the Banach space $\mathsf{Y} = (\mathsf{D}(\mathsf{A}), \|\cdot\|_{\mathsf{Y}})$, where $\|\cdot\|_{\mathsf{Y}}$ denotes the graph norm defined by $\|z\|_{\mathsf{Y}} = \|\mathsf{A}z\| + \|z\|$ for $z \in \mathsf{Y}$.

We denote by $\mathcal{C}(\mathbb{R}_+,\mathsf{Y})$, the space of all functions from $\mathbb{R}_+$ into Y which are continuous.

Now, we are interested in the problem of Cauchy below:

$$(2)\qquad \begin{cases} z'(t) = \mathsf{A}z(t) + \displaystyle\int_0^t \mathsf{U}(t-s)z(s)ds \text{ for } t \geq 0, \\ z(0) = z_0 \in \mathsf{X}. \end{cases}$$

The solution $\{z(t),\ t \geq 0\}$ satisfying the above differential system is a $\mathsf{X}-$valued process.

Definition 2.1. [13] *A bounded linear operator valued function* $\Upsilon(t) \in \mathcal{L}(\mathsf{X})$ *for* $t \geq 0$, *is said to be the resolvent operator of* (2) *if it satisfies the following conditions:*

(1) $\Upsilon(0) = \mathsf{I}$ *and* $\|\Upsilon(t)\|_{\mathcal{L}(\mathsf{X})} \leq \mathsf{M}e^{bt}$ *for some constants* M *and* b,
(2) $\forall\ z \in \mathsf{X},\ \Upsilon(t)z$ *is strongly continuous for* $t \in \mathbb{R}_+$,
(3) $\Upsilon(t) \in \mathcal{L}(\mathsf{Y})$ *for* $t \geq 0$ *and* $\Upsilon(\cdot)z \in \mathcal{C}^1(\mathbb{R}_+, \mathsf{X}) \cap \mathcal{C}(\mathbb{R}_+, \mathsf{Y})$, *for* $z \in \mathsf{Y}$.

$$\Upsilon'(t)z = \mathsf{A}\Upsilon(t)z + \int_0^t \mathsf{U}(t-s)\Upsilon(s)zds$$
$$\Upsilon'(t)z = \Upsilon(t)\mathsf{A}z + \int_0^t \Upsilon(t-s)\mathsf{U}(s)zds, \quad t \geq 0.$$

Next, we introduce the following hypotheses:

(C1) The operator A generates a strongly continuous semigroup $(\mathsf{T}(t))_{t\geq 0}$ on X.

(C2) For all $t \geq 0$, the operator $\mathsf{U}(t)$ is closed and linear from $\mathsf{D}(\mathsf{A})$ to X and $\mathsf{U}(t) \in \mathcal{L}(\mathsf{Y}, \mathsf{X})$. For any $z \in \mathsf{X}$, the map $t \mapsto \mathsf{U}(t)z$ is bounded, differentiable and the derivative $t \mapsto \mathsf{U}(t)z$ is bounded and uniformly continuous for $t \geq 0$. In addition, there is a function $\mu : \mathcal{T} \to \mathbb{R}_+$ which is integrable such that for each $z \in \mathsf{X}$, the map $t \to \mathsf{U}(t)z$ belongs to $W^{1,1}(\mathcal{T}, \mathsf{X})$ and $\left\|\frac{d\mathsf{U}(t)z}{dt}\right\| \leq \mu(t)\|z\|,\ z \in \mathsf{X},\ t \in \mathcal{T}$.

Theorem 2.1. [13] *Let* $(\mathbf{C1}) - (\mathbf{C2})$ *hold. Then the Cauchy system* (2) *possesses a resolvent operator.*

We give the following important estimate.

Lemma 2.2. [16] *Let* $(\mathbf{C1})$ *and* $(\mathbf{C2})$ *be satisfied. Then there is a constant* L *such that*

$$\|\Upsilon(t+\varepsilon) - \Upsilon(\varepsilon)\Upsilon(t)\|_{\mathcal{L}(\mathsf{X})} \leq \mathsf{L}\varepsilon$$

In the sequel, we recall some results on the existence of solutions for the following integrodifferentiel equation

$$\begin{cases} z'(t) = \mathsf{A}z(t) + \displaystyle\int_0^t \mathsf{U}(t-s)z(s)ds + q(t) \quad \text{for} \quad t \geq 0, \\ z(0) = z_0 \in \mathsf{X}. \end{cases} \tag{3}$$

where $q\ :\ [0, +\infty) \to \mathsf{X}$ is a continuous function.

Definition 2.2. [13] *A continuous function* $z : [0, +\infty) \to \mathsf{X}$ *is said to be a strict solution of the Eq.(3) if*

(1) $z \in \mathcal{C}^1([0, +\infty); \mathsf{X}) \cap \mathcal{C}([0, +\infty); \mathsf{Y})$,
(2) z *satisfies Eq.(3) for* $t \geq 0$.

Remark 2.1. *From this definition we deduce that* $z(t) \in \mathsf{D}(\mathsf{A})$, *and the function* $\mathsf{U}(t-s)z(s)$ *is integrable, for all* $0 < s < t < \infty$.

Theorem 2.3. [13] *Assume that hypotheses* $(\mathbf{C1})$ *and* $(\mathbf{C2})$ *hold. If* z *is a strict solution of the Eq.(3), then the following variation of constant formula holds*

$$z(t) = \Upsilon(t)z_0 + \int_0^t \Upsilon(t-s)q(s)ds \quad \textit{for} \quad t \geq 0. \tag{4}$$

Accordingly, we can establish the following definiton.

Definition 2.3. [13] *A function $z : [0, +\infty) \to \mathsf{X}$ is called mild solution of the Eq.(3), for $z_0 \in \mathsf{X}$, if z satisfies the variation of constants formula (4).*

The next theorem provides sufficient conditions ensuring the regularity of solutions of the Eq.(3).

Theorem 2.4. [13] *Let $q \in \mathcal{C}^1([0, +\infty); \mathsf{X})$ and z be defined by (4). If $z_0 \in \mathsf{D}(\mathsf{A})$, then z is a strict solution of the Eq.(3).*

2.2. **Definition of the mild solution for impulsive integrodifferential systems.** To define the concept of mild solution for the considered integrodifferential system, we need some spaces and notations.
So, we introduce the space $\Gamma := \{z : [-\tau, \mathsf{T}] \to \mathsf{X} : \ z_i \in \mathcal{C}(\mathsf{I}_i, \mathsf{X}),\ i \in [\![0, m]\!]$ and $z(t_i^+)$ and $z(t_i^-)$ exist with $z(t_i^-) = z(t_i),\ i \in [\![1, m]\!],\ z(t) = \varphi(t)$ in $[-\tau, 0]\}$ endowed with the norm $\|z\|_\Gamma := \max_{i \in [\![1,m]\!]} \{\|z_i\|_{\mathsf{J}_i}\}$, where z_i denote the restriction of z to $\mathsf{J}_i := [t_i, t_{i+1}],\ i \in [\![0, m]\!]$. It is not so difficult to establish that $(\Gamma, \|\cdot\|_\Gamma)$ is a Banach space.

Now, it is very important to note that, the concept of mild solution for integrodifferential impulsive systems is not easy to establish. This is due to the use of the theory of the resolvent operator in the studying of such systems. Indeed, the resolvent operator introduced by R. Grimmer [13] is not a semigroup and then don't check the translation property which is in general checked by the semigroups. So, in what follows, we introduce the defintion of mild solution for the impulsive integrodifferential systems.

Let's denote by $z \in \Gamma$, the mild solution of the system (1). Using the formula of the variation of constants, we get:

For $t \in [0, t_1]$,

$$z(t) = \Upsilon(t)\varphi(0) + \int_0^t \Upsilon(t-s)\delta\left(s, z_s, \int_0^s \varrho(s, \xi, z_\xi)d\xi\right) ds.$$

For $t \in (t_1, t_2]$,

$$\begin{aligned}
z(t) &= \Upsilon(t-t_1)z(t_1^+) + \int_{t_1}^t \Upsilon(t-s)\delta\left(s, z_s, \int_0^s \varrho(s, \xi, z_\xi)d\xi\right) ds \\
&= \Upsilon(t-t_1)\left(z(t_1) + \mathsf{I}_1\left(\int_{t_1-\alpha_i}^{t_1-\beta_i} h(s, z_s)ds\right)\right) + \int_{t_1}^t \Upsilon(t-s)\delta\left(s, z_s, \int_0^s \varrho(s, \xi, z_\xi)d\xi\right) ds \\
&= \Upsilon(t-t_1)\Upsilon(t_1)\varphi(0) + \int_0^{t_1} \Upsilon(t-t_1)\Upsilon(t_1-s)\delta\left(s, z_s, \int_0^s \varrho(s, \xi, z_\xi)d\xi\right) ds \\
&+ \Upsilon(t-t_1)\mathsf{I}_1\left(\int_{t_1-\alpha_1}^{t_1-\beta_1} h(s, z_s)ds\right) + \int_{t_1}^t \Upsilon(t-s)\delta\left(s, z_s, \int_0^s \varrho(s, \xi, z_\xi)d\xi\right) ds.
\end{aligned}$$

For $t \in (t_2, t_3]$,

$$\begin{aligned}
z(t) &= \Upsilon(t-t_2)z(t_2^+) + \int_{t_2}^{t} \Upsilon(t-s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)ds \\
&= \Upsilon(t-t_2)\left(z(t_2) + \mathsf{I}_2\left(\int_{t_2-\alpha_2}^{t_2-\beta_2} h(s,z_s)ds\right)\right) + \int_{t_2}^{t} \Upsilon(t-s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)ds \\
&= \Upsilon(t-t_2)\Upsilon(t_2-t_1)\Upsilon(t_1)\varphi(0) + \Upsilon(t-t_2)\int_0^{t_1} \Upsilon(t_2-t_1)\Upsilon(t_1-s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)ds \\
&+ \Upsilon(t-t_2)\Upsilon(t_2-t_1)\mathsf{I}_1\left(\int_{t_1-\alpha_1}^{t_1-\beta_1} h(s,z_s)ds\right) + \Upsilon(t-t_2)\mathsf{I}_2\left(\int_{t_2-\alpha_2}^{t_2-\beta_2} h(s,z_s)ds\right) \\
&+ \int_{t_1}^{t_2} \Upsilon(t-t_2)\Upsilon(t_2-s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)ds + \int_{t_2}^{t} \Upsilon(t-s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)ds.
\end{aligned}$$

Reiterating these procedures, one can prove that

For $t \in (t_i, t_{i+1}],\ i \in [\![1, m]\!]$,

$$\begin{aligned}
z(t) &= \Upsilon(t-t_i)\left(\prod_{j=1}^{i} \Upsilon(t_j - t_{j-1})\right)\varphi(0) + \Upsilon(t-t_i)\sum_{j=1}^{i-1}\left(\prod_{k=j}^{i-1}\Upsilon(t_{k+1}-t_k)\right)\mathsf{I}_j\left(\int_{t_j-\alpha_j}^{t_j-\beta_j} h(s,z_s)ds\right) \\
&+ \Upsilon(t-t_i)\mathsf{I}_i\left(\int_{t_i-\alpha_i}^{t_i-\beta_i} h(s,z_s)ds\right) + \Upsilon(t-t_i)\sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j}\left(\prod_{k=j}^{i-1}\Upsilon(t_{k+1}-t_k)\right) \\
&\times \Upsilon(t_j-s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)ds + \int_{t_{i-1}}^{t_i} \Upsilon(t-t_i)\Upsilon(t_i-s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)ds \\
&+ \int_{t_i}^{t} \Upsilon(t-s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)ds.
\end{aligned}$$

Accordingly, we deduce the definition of mild solution of the integrodifferential system (1) as follow:

Definition 2.4. *A mild solution for the impulsive integrodifferential system* (1) *is a function* $z : [-\tau, \mathsf{T}] \longrightarrow \mathsf{X}$ *satisfying* $z(t) = \varphi(t)$ *on* $[-\tau, 0]$*, and*

$$z(t) = \begin{cases} \Upsilon(t)\varphi(0) + \int_0^t \Upsilon(t-s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)ds, & t \in [0, t_1] \\[2ex] \Upsilon(t-t_i)\left(\prod\limits_{j=1}^{i}\Upsilon(t_j-t_{j-1})\right)\varphi(0) & \\ +\Upsilon(t-t_i)\sum\limits_{j=1}^{i-1}\int_{t_{j-1}}^{t_j}\left(\prod\limits_{k=j}^{i-1}\Upsilon(t_{k+1}-t_k)\right)\Upsilon(t_j-s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)ds & \\ +\displaystyle\int_{t_{i-1}}^{t_i}\Upsilon(t-t_i)\Upsilon(t_i-s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)ds & \\ +\displaystyle\int_{t_i}^{t}\Upsilon(t-s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)ds + \Upsilon(t-t_i)\mathsf{I}_i\left(\int_{t_i-\alpha_i}^{t_i-\beta_i} h(s,z_s)ds\right) & \\ +\Upsilon(t-t_i)\sum\limits_{j=1}^{i-1}\left(\prod\limits_{k=j}^{i-1}\Upsilon(t_{k+1}-t_k)\right)\mathsf{I}_j\left(\int_{t_j-\alpha_j}^{t_j-\beta_j} h(s,z_s)ds\right), & t \in (t_i, t_{i+1}],\ i \in [\![1, m]\!]. \end{cases} \tag{5}$$

The followwing lemma is a criteria of compactness.

Lemma 2.5. [7] *Let us consider the subspace $\mathcal{C}$ of $\mathcal{BC}(\mathcal{T},\mathsf{X})$ satisfying:*

(i) *$\mathcal{C}$ is bounded in $\mathcal{BC}(\mathcal{T},\mathsf{X})$;*
(ii) *the functions which belong to $\mathcal{C}$ are equicontinuous on any compact of $\mathcal{T}$;*
(iii) *the set $\mathcal{C}(t) := \{z(t) : z \in \mathcal{C}\}$ is relatively compact on any compact of $\mathcal{T}$;*
(iv) *the functions from $\mathcal{C}$ are equiconvergent; this is: for $\epsilon > 0$, there is $\Sigma(\epsilon) > 0$ such that $\|z(t) - z(+\infty)\| < \epsilon$ for any $t \geq \Sigma(\epsilon)$ and $z \in \mathcal{C}$.*

Then $\mathcal{C}$ is relatively compact in $\mathcal{BC}(\mathcal{T},\mathsf{X})$.

We give the Burton-Kirk's fixed-point theorem, which is very useful in the proof of the existence theorem.

Theorem 2.6. [4] *In the Banach space X, let us consider the operators Ξ and Λ defined from X into X such that Ξ is a compact operator and Λ a contraction.*

Then either

(i) *$z = \lambda\Lambda\left(\frac{z}{\lambda}\right) + \lambda\Xi z$ admits a solution for $\lambda = 1$, or*
(ii) *the set $\{z \in \mathsf{X} : z = \delta\Xi\left(\frac{z}{\lambda}\right) + \lambda\Lambda z,\ \lambda \in (0,1)\}$ is unbounded.*

Throughout this work, we suppose that the assumptions **(C1)** and **(C2)** hold.

3. Existence results of mild solution for the integrodifferential system (1)

Before discussing the result of existence of mild solution for the integrodifferential system (1), we introduce the following hypothesis.

(D1) $(\Upsilon(t))_{t\geq 0}$ is compact and there is a constant $\mathsf{M} \geq 1$ and $b > 0$ satisfying

$$\|\Upsilon(t)\|_{\mathcal{L}(X)} \leq \mathsf{M}e^{-bt} \text{ for every } t \geq 0.$$

(D2) There is $\zeta_\delta,\ \zeta_\varrho \in \mathcal{C}([0,\mathsf{T}],\mathbb{R}_+)$ and $\gamma_h > 0$ such that for all $t,\ s \in [0,\mathsf{T}],\ u_1,\ u_2 \in \mathcal{C}$ and $v_1,\ v_2 \in \mathsf{X}$.

(a) $\|\delta(t,u_1,v_1) - \delta(t,u_2,v_2)\| \leq \zeta_\delta(t)(\|u_1 - u_2\|_{\mathcal{C}} + \|v_1 - v_2\|)$;
(b) $\|\varrho(t,s,u_1) - \varrho(t,s,u_2)\| \leq \zeta_\varrho(t)\|u_1 - u_2\|_{\mathcal{C}}$;
(c) $\|h(t,u_1) - h(t,u_2)\| \leq \gamma_h\|u_1 - u_2\|_{\mathcal{C}}$;
(d) There is some constants $\sigma_i,\ i \in [\![1,m]\!]$ such that $\|\mathsf{I}_i(v_1) - \mathsf{I}_i(v_2)\| \leq \sigma_i\|v_1 - v_2\|;\ i \in [\![1,m]\!]$.

To prove the existence of mild solution for the system (1), we introduce the following operators:

$\Lambda : \Gamma \to \Gamma$, defined by:

$$(\Lambda z)(t) = \begin{cases} 0, \qquad t \in [-\tau, t_1] \\ \Upsilon(t-t_i)\displaystyle\sum_{j=1}^{i-1}\left(\prod_{k=j}^{i-1}\Upsilon(t_{k+1}-t_k)\right)\mathsf{I}_j\left(\int_{t_j-\alpha_j}^{t_j-\beta_j} h(s,z_s)ds\right) \\ +\Upsilon(t-t_i)\mathsf{I}_i\left(\int_{t_i-\alpha_i}^{t_i-\beta_i} h(s,z_s)ds\right),\ t \in (t_i, t_{i+1}],\ i \in [\![1,m]\!]. \end{cases} \tag{6}$$

and

$\Xi : \Gamma \to \Gamma$, defined by:

$$(7)\quad (\Xi z)(t) = \begin{cases} \varphi(t), \quad t \in [-\tau, 0] \\ \Upsilon(t)\varphi(0) + \int_0^t \Upsilon(t-s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right) ds, \quad t \in [0, t_1] \\ \Upsilon(t-t_i)\left(\prod_{j=1}^{i} \Upsilon(t_j - t_{j-1})\right)\varphi(0) \\ +\Upsilon(t-t_i)\sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j}\left(\prod_{k=j}^{i-1}\Upsilon(t_{k+1}-t_k)\right)\Upsilon(t_j - s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right) ds \\ +\int_{t_{i-1}}^{t_i}\Upsilon(t-t_i)\Upsilon(t_i - s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right) ds \\ +\int_{t_i}^{t}\Upsilon(t-s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right) ds, \quad t \in (t_i, t_{i+1}],\ i \in [\![1, m]\!]. \end{cases}$$

We give and prove the following theorem.

Theorem 3.1. *Under the conditions* **(D1)** *and* **(D2)**, *the impulsive integrodifferential system* (1) *has a mild solution on* Γ, *provided*

$$(8)\qquad 2\mathsf{T}\mathsf{M}\zeta_h\left(\sum_{j=1}^{i-1}\left(\mathsf{M}^{i-j}\right)\sigma_j + \sigma_i\right) < 1.$$

Proof. We give the proof in the following six steps.

Step 1: The operator Ξ is continuous. For any $z \in \Gamma$ and any sequence $\{z_n\}_{n\in\mathbb{N}} \subset \Gamma$, there is $\varsigma > 0$, such that $\forall\, n \in \mathbb{N}$, $\|z_n\|_\Gamma \le \varsigma$ and hence $\|z\|_\Gamma \le \varsigma$. It follows that $\{Z_n\}_{n\in\mathbb{N}} \subseteq \mathsf{B}_\varsigma$ and $z \in \mathsf{B}_\varsigma$.

Using the Lebesgue dominated convergence theorem, we prove that

For $t \in [0, t_1]$,

$$(9)\quad \begin{aligned} \|(\Xi z_n)(t) - (\Xi z)(t)\|_\Gamma &\le \int_0^t \|\Upsilon(t-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s, z_{ns}, \int_0^s \varrho(s,\xi,z_{n\xi})d\xi\right) - \delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)\right\| ds \\ &\xrightarrow[n\to\infty]{} 0. \end{aligned}$$

For $t \in (t_i, t_{i+1}]$,

$$(10)\quad \begin{aligned} &\|(\Xi z_n)(t) - (\Xi z)(t)\|_\Gamma \\ &\le \sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j}\left\|\Upsilon(t-t_i)\Upsilon(t_j - s)\prod_{k=j}^{i-1}\Upsilon(t_{k+1}-t_k)\right\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s, z_{ns}, \int_0^s \varrho(s,\xi,z_{n\xi})d\xi\right) - \delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)\right\| ds \\ &+ \int_{t_{i-1}}^{t_i}\|\Upsilon(t-t_i)\Upsilon(t_i - s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s, z_{ns}, \int_0^s \varrho(s,\xi,z_{n\xi})d\xi\right) - \delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)\right\| ds \\ &+ \int_{t_i}^{t}\|\Upsilon(t-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s, z_{ns}, \int_0^s \varrho(s,\xi,z_{n\xi})d\xi\right) - \delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)\right\| ds \\ &\xrightarrow[n\to\infty]{} 0. \end{aligned}$$

Thus Ξ is continuous.

Step 2: The operator Ξ maps bounded sets into bounded sets in Γ. For $z \in \mathsf{B}_\varsigma$, we will prove that there is $\eta > 0$ such that $\|\Xi(z)\|_\Gamma \leq \eta$.

For $t \in [-\tau, 0]$,

$$\begin{aligned} \|(\Xi z)(t)\| &\leq \|\varphi(t)\| \\ &\leq \|\varphi\|_{\mathcal{C}} \end{aligned} \tag{11}$$

For $t \in [0, t_1]$,

$$\begin{aligned} \|(\Xi z)(t)\| &\leq \|\Upsilon(t)\|_{\mathcal{L}(\mathsf{X})}\|\varphi(0)\| + \int_0^t \|\Upsilon(t-s)\|_{\mathcal{L}(\mathsf{X})} \left\|\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)\right\| ds \\ &\leq \|\Upsilon(t)\|_{\mathcal{L}(\mathsf{X})}\|\varphi\|_{\mathcal{C}} + \int_0^t \|\Upsilon(t-s)\|_{\mathcal{L}(\mathsf{X})} \left\|\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right) - \delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds \\ &+ \int_0^t \|\Upsilon(t-s)\|_{\mathcal{L}(\mathsf{X})} \left\|\delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds \\ &\leq \mathsf{M}\|\varphi\|_{\mathcal{C}} + \int_0^t \mathsf{M}\zeta_\delta(s)\left(\|z_s\|_{\mathcal{C}} + \int_0^s \zeta_\varrho(\xi)\|z_\xi\|_{\mathcal{C}}d\xi\right)ds + \int_0^t \mathsf{M}\left\|\delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds \\ &\leq \mathsf{M}\|\varphi\|_{\mathcal{C}} + \int_0^{\mathsf{T}} \mathsf{M}\zeta_\delta(s)\varsigma\left(1 + \int_0^s \zeta_\varrho(\xi)d\xi\right)ds + \int_0^{\mathsf{T}} \mathsf{M}\left\|\delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds := \Theta. \end{aligned} \tag{12}$$

For $t \in (t_i, t_{i+1}],\ i \in [\![1, m]\!]$,

$$\begin{aligned} \|(\Xi z)(t)\| &\leq \left\|\Upsilon(t-t_i)\left(\prod_{j=1}^{i}\Upsilon(t_j - t_{j-1})\right)\right\|_{\mathcal{L}(\mathsf{X})}\|\varphi(0)\| \\ &+ \sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j}\left\|\Upsilon(t-t_i)\left(\prod_{k=j}^{i-1}\Upsilon(t_{k+1}-t_k)\right)\Upsilon(t_j - s)\right\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)\right\| ds \\ &+ \int_{t_{i-1}}^{t_i}\|\Upsilon(t-t_i)\Upsilon(t_i - s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)\right\| ds \\ &+ \int_{t_i}^{t}\|\Upsilon(t-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)\right\| ds \\ &\leq \mathsf{M}^{i+1}\|\varphi\|_{\mathcal{C}} + \sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j}\mathsf{M}^{i-j+1}\left\|\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right) - \delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds \\ &+ \sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j}\mathsf{M}^{i-j+1}\left\|\delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds + \int_{t_{i-1}}^{t_i}\mathsf{M}^2\left\|\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)\right. \\ &\left. -\delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds + \int_{t_{i-1}}^{t_i}\mathsf{M}^2\left\|\delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds \\ &+ \int_{t_i}^{t}\mathsf{M}\left\|\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right) - \delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds + \int_{t_i}^{t}\mathsf{M}\left\|\delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds \\ &\leq \mathsf{M}^{i+1}\|\varphi\|_{\mathcal{C}} + \sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j}\mathsf{M}^{i-j+1}\left\|\zeta_\delta(s)\left(\|z_s\|_{\mathcal{C}} + \int_0^s \zeta_\varrho(\xi)\|z_\xi\|_{\mathcal{C}}d\xi\right)\right\| ds \\ &+ \sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j}\mathsf{M}^{i-j+1}\left\|\delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds + \int_{t_{i-1}}^{t_i}\mathsf{M}^2\left\|\zeta_\delta(s)\left(\|z_s\|_{\mathcal{C}} + \int_0^s \zeta_\varrho(\xi)\|z_\xi\|_{\mathcal{C}}d\xi\right)\right\| ds \\ &+ \int_{t_{i-1}}^{t_i}\mathsf{M}^2\left\|\delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds + \int_{t_i}^{t}\mathsf{M}\left\|\zeta_\delta(s)\left(\|z_s\|_{\mathcal{C}} + \int_0^s \zeta_\varrho(\xi)\|z_\xi\|_{\mathcal{C}}d\xi\right)\right\| ds \\ &+ \int_{t_i}^{t}\mathsf{M}\left\|\delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds \end{aligned}$$

$$\begin{aligned}
&\leq \mathsf{M}^{i+1}\|\varphi\|_{\mathcal{C}}+\sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j}\mathsf{M}^{i-j+1}\left\|\zeta_\delta(s)\varsigma\left(1+\int_0^s\zeta_\varrho(\xi)d\xi\right)\right\|ds\\
&+\sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j}\mathsf{M}^{i-j+1}\left\|\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds\\
&+\int_{t_{i-1}}^{t_i}\mathsf{M}^2\left\|\zeta_\delta(s)\varsigma\left(1+\int_0^s\zeta_\varrho(\xi)d\xi\right)\right\|ds+\int_{t_{i-1}}^{t_i}\mathsf{M}^2\left\|\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds\\
&+\int_{t_i}^{t}\mathsf{M}\left\|\zeta_\delta(s)\varsigma\left(1+\int_0^s\zeta_\varrho(\xi)d\xi\right)\right\|ds+\int_{t_i}^{t}\mathsf{M}\left\|\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds\\
&\leq \mathsf{M}^{m+1}\|\varphi\|_{\mathcal{C}}+\sum_{j=1}^{m-1}\int_{t_{j-1}}^{t_j}\mathsf{M}^{m-j+1}\left\|\zeta_\delta(s)\varsigma\left(1+\int_0^s\zeta_\varrho(\xi)d\xi\right)\right\|ds\\
&+\sum_{j=1}^{m-1}\int_{t_{j-1}}^{t_j}\mathsf{M}^{m-j+1}\left\|\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds\\
&+\int_0^{\mathsf{T}}\mathsf{M}^2\left\|\zeta_\delta(s)\varsigma\left(1+\int_0^s\zeta_\varrho(\xi)d\xi\right)\right\|ds+\int_0^{\mathsf{T}}\mathsf{M}^2\left\|\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds\\
&+\int_0^{\mathsf{T}}\mathsf{M}\left\|\zeta_\delta(s)\varsigma\left(1+\int_0^s\zeta_\varrho(\xi)d\xi\right)\right\|ds+\int_0^{\mathsf{T}}\mathsf{M}\left\|\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds:=\Theta_m.
\end{aligned}\tag{13}$$

Further, it is easy to see that $\Theta \leq \Theta_m$. Thus, for $z \in \mathsf{B}_\varsigma$, we obtain:

$$\|\Xi z\|_\Gamma=\sup_{t\in[-\tau,\mathsf{T}]}\|(\Xi z)(t)\|\leq \max\{\|\varphi\|_{\mathcal{C}},\Theta_m\}:=\eta.$$

Step 3: The operator Ξ maps bounded sets into equicontinuous sets of Γ. Let $z \in \mathsf{B}_\varsigma := \{z \in \Gamma : \|z\|_\Gamma \leq \varsigma\}$, we have:

For $\varepsilon_1 < \varepsilon_2$ in $[0, t_1]$,

$$\begin{aligned}
&\|(\Xi z)(\varepsilon_2)-(\Xi z)(\varepsilon_1)\|\\
&\leq \|\Upsilon(\varepsilon_2-\varepsilon_1)\|_{\mathcal{L}(\mathsf{X})}\|\varphi(0)\|+\int_0^{\varepsilon_1}\|\Upsilon(\varepsilon_2-s)-\Upsilon(\varepsilon_1-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,z_s,\int_0^s\varrho(s,\xi,z_\xi)d\xi\right)\right\|ds\\
&+\int_{\varepsilon_1}^{\varepsilon_2}\|\Upsilon(\varepsilon_2-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,z_s,\int_0^s\varrho(s,\xi,z_\xi)d\xi\right)\right\|ds\\
&\leq \|\Upsilon(\varepsilon_2)-\Upsilon(\varepsilon_1)\|_{\mathcal{L}(\mathsf{X})}\|\varphi\|_{\mathcal{C}}\\
&+\int_0^{\varepsilon_1}\|\Upsilon(\varepsilon_2-s)-\Upsilon(\varepsilon_1-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,z_s,\int_0^s\varrho(s,\xi,z_\xi)d\xi\right)-\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds\\
&+\int_0^{\varepsilon_1}\|\Upsilon(\varepsilon_2-s)-\Upsilon(\varepsilon_1-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds\\
&+\int_{\varepsilon_1}^{\varepsilon_2}\|\Upsilon(\varepsilon_2-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,z_s,\int_0^s\varrho(s,\xi,z_\xi)d\xi\right)-\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds\\
&+\int_{\varepsilon_1}^{\varepsilon_2}\|\Upsilon(\varepsilon_2-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds\\
&\leq \|\Upsilon(\varepsilon_2)-\Upsilon(\varepsilon_1)\|_{\mathcal{L}(\mathsf{X})}\|\varphi\|_{\mathcal{C}}+\int_0^{\varepsilon_1}\|\Upsilon(\varepsilon_2-s)-\Upsilon(\varepsilon_1-s)\|_{\mathcal{L}(\mathsf{X})}\zeta_\delta(s)\left(\|z_s\|_{\mathcal{C}}+\int_0^s\zeta_\varrho(\xi)\|z_\xi\|_{\mathcal{C}}d\xi\right)ds\\
&+\int_0^{\varepsilon_1}\|\Upsilon(\varepsilon_2-s)-\Upsilon(\varepsilon_1-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds\\
&+\int_{\varepsilon_1}^{\varepsilon_2}\|\Upsilon(\varepsilon_2-s)\|_{\mathcal{L}(\mathsf{X})}\zeta_\delta(s)\left(\|z_s\|_{\mathcal{C}}+\int_0^s\zeta_\varrho(\xi)\|z_\xi\|_{\mathcal{C}}d\xi\right)ds\\
&+\int_{\varepsilon_1}^{\varepsilon_2}\|\Upsilon(\varepsilon_2-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds
\end{aligned}$$

$$\le \|\Upsilon(\varepsilon_2)-\Upsilon(\varepsilon_1)\|_{\mathcal{L}(\mathsf{X})}\|\varphi\|_{\mathcal{C}}+\int_0^{\varepsilon_1}\|\Upsilon(\varepsilon_2-s)-\Upsilon(\varepsilon_1-s)\|_{\mathcal{L}(\mathsf{X})}\zeta_\delta(s)\varsigma\left(1+\int_0^s\zeta_\varrho(\xi)d\xi\right)ds$$
$$+\int_0^{\varepsilon_1}\|\Upsilon(\varepsilon_2-s)-\Upsilon(\varepsilon_1-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds$$
$$+\int_{\varepsilon_1}^{\varepsilon_2}\|\Upsilon(\varepsilon_2-s)\|_{\mathcal{L}(\mathsf{X})}\zeta_\delta(s)\varsigma\left(1+\int_0^s\zeta_\varrho(\xi)d\xi\right)ds+\int_{\varepsilon_1}^{\varepsilon_2}\|\Upsilon(\varepsilon_2-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds.$$

For $\varepsilon_1<\varepsilon_2$ in $(t_i,t_{i+1}]$, $i\in[\![1,m]\!]$,

$$\|(\Xi z)(\varepsilon_2)-(\Xi z)(\varepsilon_1)\|$$
$$\le\left\|\Upsilon(\varepsilon_2-t_i)\left(\prod_{j=1}^{i}\Upsilon(t_j-t_{j-1})\right)-\Upsilon(\varepsilon_1-t_i)\left(\prod_{j=1}^{i}\Upsilon(t_j-t_{j-1})\right)\right\|_{\mathcal{L}(\mathsf{X})}\|\varphi(0)\|$$
$$+\sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j}\left\|(\Upsilon(\varepsilon_2-t_i)-\Upsilon(\varepsilon_1-t_i))\left(\prod_{k=j}^{i-1}\Upsilon(t_{k+1}-t_k)\right)\Upsilon(t_j-s)\right\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,z_s,\int_0^s\varrho(s,\xi,z_\xi)d\xi\right)\right\|ds$$
$$+\int_{t_{i-1}}^{t_i}\|(\Upsilon(\varepsilon_2-t_i)-\Upsilon(\varepsilon_1-t_i))\Upsilon(t_i-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,z_s,\int_0^s\varrho(s,\xi,z_\xi)d\xi\right)\right\|ds$$
$$+\int_{t_i}^{\varepsilon_1}\|\Upsilon(\varepsilon_2-s)-\Upsilon(\varepsilon_1-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,z_s,\int_0^s\varrho(s,\xi,z_\xi)d\xi\right)\right\|ds$$
$$+\int_{\varepsilon_1}^{\varepsilon_2}\|\Upsilon(\varepsilon_1-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,z_s,\int_0^s\varrho(s,\xi,z_\xi)d\xi\right)\right\|ds$$
$$\le\left\|(\Upsilon(\varepsilon_2-t_i)-\Upsilon(\varepsilon_1-t_i))\left(\prod_{j=1}^{i}\Upsilon(t_j-t_{j-1})\right)\right\|_{\mathcal{L}(\mathsf{X})}\|\varphi\|_{\mathcal{C}}$$
$$+\sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j}\left\|(\Upsilon(\varepsilon_2-t_i)-\Upsilon(\varepsilon_1-t_i))\left(\prod_{k=j}^{i-1}\Upsilon(t_{k+1}-t_k)\right)\Upsilon(t_j-s)\right\|_{\mathcal{L}(\mathsf{X})}$$
$$\times\left\|\delta\left(s,z_s,\int_0^s\varrho(s,\xi,z_\xi)d\xi\right)-\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds$$
$$+\sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j}\left\|(\Upsilon(\varepsilon_2-t_i)-\Upsilon(\varepsilon_1-t_i))\left(\prod_{k=j}^{i-1}\Upsilon(t_{k+1}-t_k)\right)\Upsilon(t_j-s)\right\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds$$
$$+\int_{t_{i-1}}^{t_i}\|(\Upsilon(\varepsilon_2-t_i)-\Upsilon(\varepsilon_1-t_i))\Upsilon(t_i-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,z_s,\int_0^s\varrho(s,\xi,z_\xi)d\xi\right)-\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds$$
$$+\int_{t_{i-1}}^{t_i}\|(\Upsilon(\varepsilon_2-t_i)-\Upsilon(\varepsilon_1-t_i))\Upsilon(t_i-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds$$
$$+\int_{t_i}^{\varepsilon_1}\|\Upsilon(\varepsilon_2-s)-\Upsilon(\varepsilon_1-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,z_s,\int_0^s\varrho(s,\xi,z_\xi)d\xi\right)-\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds$$
$$+\int_{t_i}^{\varepsilon_1}\|\Upsilon(\varepsilon_2-s)-\Upsilon(\varepsilon_1-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds$$
$$+\int_{\varepsilon_1}^{\varepsilon_2}\|\Upsilon(\varepsilon_1-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,z_s,\int_0^s\varrho(s,\xi,z_\xi)d\xi\right)-\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds$$
$$+\int_{\varepsilon_1}^{\varepsilon_2}\|\Upsilon(\varepsilon_1-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds$$
$$\le\left\|\Upsilon(\varepsilon_2-t_i)\left(\prod_{j=1}^{i}\Upsilon(t_j-t_{j-1})\right)-\Upsilon(\varepsilon_1-t_i)\left(\prod_{j=1}^{i}\Upsilon(t_j-t_{j-1})\right)\right\|_{\mathcal{L}(\mathsf{X})}\|\varphi\|_{\mathcal{C}}$$
$$+\sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j}\left\|(\Upsilon(\varepsilon_2-t_i)-\Upsilon(\varepsilon_1-t_i))\left(\prod_{k=j}^{i-1}\Upsilon(t_{k+1}-t_k)\right)\Upsilon(t_j-s)\right\|_{\mathcal{L}(\mathsf{X})}$$

$$
\begin{aligned}
&\times \left\| \zeta_\delta(s) \left(\|z_s\|_{\mathcal{C}} + \int_0^s \zeta_\varrho(\xi) \|z_\xi\|_{\mathcal{C}} d\xi \right) \right\| ds \\
&+ \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} \left\| (\Upsilon(\varepsilon_2 - t_i) - \Upsilon(\varepsilon_1 - t_i)) \left(\prod_{k=j}^{i-1} \Upsilon(t_{k+1} - t_k) \right) \Upsilon(t_j - s) \right\|_{\mathcal{L}(\mathsf{X})} \left\| \delta\left(s, 0, \int_0^s \varrho(s,\xi,0) d\xi \right) \right\| ds \\
&+ \int_{t_{i-1}}^{t_i} \|(\Upsilon(\varepsilon_2 - t_i) - \Upsilon(\varepsilon_1 - t_i)) \Upsilon(t_i - s)\|_{\mathcal{L}(\mathsf{X})} \zeta_\delta(s) \left(\|z_s\|_{\mathcal{C}} + \int_0^s \zeta_\varrho(\xi) \|z_\xi\|_{\mathcal{C}} d\xi \right) ds \\
&+ \int_{t_{i-1}}^{t_i} \|(\Upsilon(\varepsilon_2 - t_i) - \Upsilon(\varepsilon_1 - t_i)) \Upsilon(t_i - s)\|_{\mathcal{L}(\mathsf{X})} \left\| \delta\left(s, 0, \int_0^s \varrho(s,\xi,0) d\xi \right) \right\| ds \\
&+ \int_{t_i}^{\varepsilon_1} \|\Upsilon(\varepsilon_2 - s) - \Upsilon(\varepsilon_1 - s)\|_{\mathcal{L}(\mathsf{X})} \zeta_\delta(s) \left(\|z_s\|_{\mathcal{C}} + \int_0^s \zeta_\varrho(\xi) \|z_\xi\|_{\mathcal{C}} d\xi \right) ds \\
&+ \int_{t_i}^{\varepsilon_1} \|\Upsilon(\varepsilon_2 - s) - \Upsilon(\varepsilon_1 - s)\|_{\mathcal{L}(\mathsf{X})} \left\| \delta\left(s, 0, \int_0^s \varrho(s,\xi,0) d\xi \right) \right\| ds \\
&+ \int_{\varepsilon_1}^{\varepsilon_2} \|\Upsilon(\varepsilon_1 - s)\|_{\mathcal{L}(\mathsf{X})} \zeta_\delta(s) \left(\|z_s\|_{\mathcal{C}} + \int_0^s \zeta_\varrho(\xi) \|z_\xi\|_{\mathcal{C}} d\xi \right) ds \\
&+ \int_{\varepsilon_1}^{\varepsilon_2} \|\Upsilon(\varepsilon_1 - s)\|_{\mathcal{L}(\mathsf{X})} \left\| \delta\left(s, 0, \int_0^s \varrho(s,\xi,0) d\xi \right) \right\| ds \\
&\le \left\| \Upsilon(\varepsilon_2 - t_i) \left(\prod_{j=1}^{i} \Upsilon(t_j - t_{j-1}) \right) - \Upsilon(\varepsilon_1 - t_i) \left(\prod_{j=1}^{i} \Upsilon(t_j - t_{j-1}) \right) \right\|_{\mathcal{L}(\mathsf{X})} \|\varphi\|_{\mathcal{C}} \\
&+ \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} \left\| (\Upsilon(\varepsilon_2 - t_i) - \Upsilon(\varepsilon_1 - t_i)) \left(\prod_{k=j}^{i-1} \Upsilon(t_{k+1} - t_k) \right) \Upsilon(t_j - s) \right\|_{\mathcal{L}(\mathsf{X})} \zeta_\delta(s) \varsigma \left(1 + \int_0^s \zeta_\varrho(\xi) d\xi \right) ds \\
&+ \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} \left\| (\Upsilon(\varepsilon_2 - t_i) - \Upsilon(\varepsilon_1 - t_i)) \left(\prod_{k=j}^{i-1} \Upsilon(t_{k+1} - t_k) \right) \Upsilon(t_j - s) \right\|_{\mathcal{L}(\mathsf{X})} \left\| \delta\left(s, 0, \int_0^s \varrho(s,\xi,0) d\xi \right) \right\| ds \\
&+ \int_{t_{i-1}}^{t_i} \|(\Upsilon(\varepsilon_2 - t_i) - \Upsilon(\varepsilon_1 - t_i)) \Upsilon(t_i - s)\|_{\mathcal{L}(\mathsf{X})} \zeta_\delta(s) \varsigma \left(1 + \int_0^s \zeta_\varrho(\xi) d\xi \right) ds \\
&+ \int_{t_{i-1}}^{t_i} \|(\Upsilon(\varepsilon_2 - t_i) - \Upsilon(\varepsilon_1 - t_i)) \Upsilon(t_i - s)\|_{\mathcal{L}(\mathsf{X})} \left\| \delta\left(s, 0, \int_0^s \varrho(s,\xi,0) d\xi \right) \right\| ds \\
&+ \int_{t_i}^{\varepsilon_1} \|\Upsilon(\varepsilon_2 - s) - \Upsilon(\varepsilon_1 - s)\|_{\mathcal{L}(\mathsf{X})} \zeta_\delta(s) \varsigma \left(1 + \int_0^s \zeta_\varrho(\xi) d\xi \right) ds \\
&+ \int_{t_i}^{\varepsilon_1} \|\Upsilon(\varepsilon_2 - s) - \Upsilon(\varepsilon_1 - s)\|_{\mathcal{L}(\mathsf{X})} \left\| \delta\left(s, 0, \int_0^s \varrho(s,\xi,0) d\xi \right) \right\| ds \\
&+ \int_{\varepsilon_1}^{\varepsilon_2} \|\Upsilon(\varepsilon_1 - s)\|_{\mathcal{L}(\mathsf{X})} \zeta_\delta(s) \varsigma \left(1 + \int_0^s \zeta_\varrho(\xi) d\xi \right) ds + \int_{\varepsilon_1}^{\varepsilon_2} \|\Upsilon(\varepsilon_1 - s)\|_{\mathcal{L}(\mathsf{X})} \left\| \delta\left(s, 0, \int_0^s \varrho(s,\xi,0) d\xi \right) \right\| ds
\end{aligned}
$$

As $\varepsilon_1 \to \varepsilon_2$, the right hand side of the above inequality tends to zero. For the cases $-\tau < \varepsilon_1 < \varepsilon_2 \le 0$ and $-\tau < \varepsilon_1 < 0 < \varepsilon_2 < \mathsf{T} \le 0$ the equicontinuity one can be obtain easily via uniform continuity of the function Υ and above case. Thus, $\|(\Xi z)(\varepsilon_2) - (\Xi z)(\varepsilon_1)\| \le \Psi|\varepsilon_2 - \varepsilon_1|$; with constant $\Psi > 0$, $\forall t \in [-\tau, \mathsf{T}]$ and we conclude that B B q is an equicontinuous family of functions. The equicontinuity and uniform boundedness of the set $\{(\Xi z)(t) : \|z\|_\Gamma \le \varsigma, \ -\tau \le t \le \mathsf{T}\}$ is as of now illustrated.

Step 4: The operator Ξ maps B_σ into a precompact set in Γ. Let $0 < t < T_1$ be fixed and ϵ be a real number satisfying $0 < \epsilon < t$. For $z \in \mathsf{B}_\varsigma$, we define the operators

$$
(\Xi^\epsilon z)(t) = \Upsilon(t)\varphi(0) + \Upsilon(\epsilon) \int_0^{t-\epsilon} \Upsilon(t - s - \epsilon) \delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi) d\xi \right) ds
$$

and

$$(\widetilde{\Xi}^{\epsilon} z)(t) = \Upsilon(t)\varphi(0) + \int_0^{t-\epsilon} \Upsilon(t-s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right) ds.$$

By the compactness of the operator $\Upsilon(\epsilon)$, the set $\{(\Xi^{\epsilon} z)(t): \ z \in \mathsf{B}_{\varsigma}\}$ is relatively compact in γ, for every $\epsilon \in (0,t_1)$. Moreover, by Hölder's inequality, for each $z \in \mathsf{B}_{\varsigma}$, we obtain

$$\begin{aligned}\|(\Xi^{\epsilon} z)(t) - (\widetilde{\Xi}^{\epsilon} z)(t)\| &\leq \int_0^{t-\epsilon} \|\Upsilon(\epsilon)\Upsilon(t-s-\epsilon) - \Upsilon(t-s)\|_{\mathcal{L}(\mathsf{X})} \left\|\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)\right\| ds \\ &\leq (\mathsf{L}\epsilon)\int_0^{t-\epsilon} \left\|\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)\right\| ds. \qquad (14)\end{aligned}$$

So, we can conclude that for $t \in [0,t_1]$, the set $\{(\widetilde{\Xi}^{\epsilon} z)(t): \ z \in \mathsf{B}_{\varsigma}\}$ is precompact in Γ by using the total boundedness.

Applying this idea again, we obtain

$$\begin{aligned}&\|(\Xi z)(t) - (\widetilde{\Xi}^{\epsilon} z)(t)\| \\ &\leq \int_{t-\epsilon}^{t} \|\Upsilon(t-s)\|_{\mathcal{L}(\mathsf{X})} \left\|\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)\right\| ds \\ &\leq \int_{t-\epsilon}^{t} \|\Upsilon(t-s)\|_{\mathcal{L}(\mathsf{X})} \left\|\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right) - \delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds \\ &+ \int_{t-\epsilon}^{t} \|\Upsilon(t-s)\|_{\mathcal{L}(\mathsf{X})} \left\|\delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds \\ &\leq \mathsf{M}\int_{t-\epsilon}^{t} \zeta_{\delta}(s)\left(\|z_s\|_{\mathcal{C}} + \int_0^s \zeta_{\varrho}(\xi)\|z_\xi\|_{\mathcal{C}} d\xi\right) ds + \mathsf{M}\int_{t-\epsilon}^{t} \left\|\delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds \\ &\leq \mathsf{M}\int_{t-\epsilon}^{t} \zeta_{\delta}(s)\varsigma\left(1 + \int_0^s \zeta_{\varrho}(\xi)d\xi\right) ds + \mathsf{M}\int_{t-\epsilon}^{t} \left\|\delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds \\ &\xrightarrow[\epsilon\to 0]{} 0, \qquad (15)\end{aligned}$$

and there are precompact sets arbitrarily close to the set $\{(\Xi z)(t): \ z \in \mathsf{B}_{\varsigma}\}$ and consequently, this set is precompact in Γ.

Otherwise, for $z \in \mathsf{B}_{\varsigma}$, $t_i < t < t_{i+1}$, fixed and $\epsilon \in (0, t-t_i)$, $i \in [\![1,m]\!]$, we define the operators

$$\begin{aligned}(\Xi^{\epsilon} z)(t) &= \Upsilon(t-t_i)\left(\prod_{j=1}^{i}\Upsilon(t_j-t_{j-1})\right)\varphi(0) \\ &+ \Upsilon(t-t_i)\sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j}\left(\prod_{k=j}^{i-1}\Upsilon(t_{k+1}-t_k)\right)\Upsilon(t_j-s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right) ds \\ &+ \int_{t_{i-1}}^{t_i}\Upsilon(t-t_i)\Upsilon(t_i-s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right) ds \\ &+ \Upsilon(\epsilon)\int_{t_i}^{t-\epsilon}\Upsilon(t-s-\epsilon)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right) ds\end{aligned}$$

and

$$
\begin{aligned}
(\widetilde{\Xi}^{\epsilon} z)(t) &= \Upsilon(t-t_i)\left(\prod_{j=1}^{i}\Upsilon(t_j-t_{j-1})\right)\varphi(0) \\
&+ \Upsilon(t-t_i)\sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j}\left(\prod_{k=j}^{i-1}\Upsilon(t_{k+1}-t_k)\right)\Upsilon(t_j-s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)ds \\
&+ \int_{t_{i-1}}^{t_i}\Upsilon(t-t_i)\Upsilon(t_i-s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)ds \\
&+ \int_{t_i}^{t-\epsilon}\Upsilon(t-s)\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)ds.
\end{aligned}
$$

By the compactness of the operator $\Upsilon(\epsilon)$, the set $\{(\Xi^{\epsilon} z)(t) : \ z \in \mathsf{B}_{\varsigma}\}$ is relatively compact in γ, for every $\epsilon \in (0, t_1)$. Moreover, by Hölder's inequality, for each $z \in \mathsf{B}_{\varsigma}$, we obtain

$$
\begin{aligned}
\|(\Xi^{\epsilon} z)(t) - (\widetilde{\Xi}^{\epsilon} z)(t)\| &\leq \int_{t_i}^{t-\epsilon}\|\Upsilon(\epsilon)\Upsilon(t-s-\epsilon)-\Upsilon(t-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)\right\|ds \\
&\leq (\mathsf{L}\epsilon)\int_{t_i}^{t-\epsilon}\left\|\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)\right\|ds. \qquad (16)
\end{aligned}
$$

So, we can conclude that for $t \in (t_i, t_{i+1}]$, the set $\{(\widetilde{\Xi}^{\epsilon} z)(t) : \ z \in \mathsf{B}_{\varsigma}\}$ is precompact in Γ by using the total boundedness.

Applying this idea again, we obtain

$$
\begin{aligned}
\|(\Xi z)(t) - (\widetilde{\Xi}^{\epsilon} z)(t)\| &\leq \int_{t-\epsilon}^{t}\|\Upsilon(t-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)\right\|ds \\
&\leq \int_{t-\epsilon}^{t}\|\Upsilon(t-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi\right)-\delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\|ds \\
&+ \int_{t-\epsilon}^{t}\|\Upsilon(t-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\|ds \\
&\leq \mathsf{M}\int_{t-\epsilon}^{t}\zeta_{\delta}(s)\left(\|z_s\|_{\mathcal{C}}+\int_0^s \zeta_{\varrho}(\xi)\|z_\xi\|_{\mathcal{C}}d\xi\right)ds + \mathsf{M}\int_{t-\epsilon}^{t}\left\|\delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\|ds \\
&\leq \mathsf{M}\int_{t-\epsilon}^{t}\zeta_{\delta}(s)\varsigma\left(1+\int_0^s \zeta_{\varrho}(\xi)d\xi\right)ds + \mathsf{M}\int_{t-\epsilon}^{t}\left\|\delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\|ds \\
&\xrightarrow[\epsilon\to 0]{} 0, \qquad (17)
\end{aligned}
$$

and there are precompact sets arbitrarily close to the set $\{(\Xi z)(t) : \ z \in \mathsf{B}_{\varsigma}\}$ and consequently, this set is precompact in Γ.

Finally, by the Arzelà–Ascoli theorem, we can conclude that the operator Ξ is completely continuous.

Step 5: The operator Λ is a contraction. Let $z_1, \ z_2 \in \Gamma$. By using the hypothesis **(D1)** and **(D2)**, we obtain, for $t \in (t_i, t_{i+1}]$,

$$
\begin{aligned}
&\|(\Lambda z_1)(t) - (\Lambda z_2)(t)\| \\
&\leq \left\|\Upsilon(t-t_i)\sum_{j=1}^{i-1}\left(\prod_{k=j}^{i-1}\Upsilon(t_{k+1}-t_k)\right)\right\|_{\mathcal{L}(\mathsf{X})}\left\|\mathsf{I}_j\left(\int_{t_j-\alpha_j}^{t_j-\beta_j}h(s,z_{1s})ds\right)-\mathsf{I}_j\left(\int_{t_j-\alpha_j}^{t_j-\beta_j}h(s,z_{2s})ds\right)\right\| \\
&+ \|\Upsilon(t-t_i)\|_{\mathcal{L}(\mathsf{X})}\left\|\mathsf{I}_i\left(\int_{t_i-\alpha_i}^{t_i-\beta_i}h(s,z_{1s})ds\right)-\mathsf{I}_i\left(\int_{t_i-\alpha_i}^{t_i-\beta_i}h(s,z_{2s})ds\right)\right\| \\
&\leq \mathsf{M}\sum_{j=1}^{i-1}\left(\mathsf{M}^{i-j}\right)\sigma_j\left\|\int_{t_j-\alpha_j}^{t_j-\beta_j}h(s,z_{1s})ds-\int_{t_j-\alpha_j}^{t_j-\beta_j}h(s,z_{2s})ds\right\| + \mathsf{M}\sigma_i\left\|\int_{t_i-\alpha_i}^{t_i-\beta_i}h(s,z_{1s})ds-\int_{t_i-\alpha_i}^{t_i-\beta_i}h(s,z_{2s})ds\right\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \mathsf{M}\sum_{j=1}^{i-1}\left(\mathsf{M}^{i-j}\right)\sigma_j\zeta_h\int_{t_j-\alpha_j}^{t_j-\beta_j}\|z_{1s}-z_{2s}\|_{\mathcal{C}}\,ds+\mathsf{M}\sigma_i\zeta_h\int_{t_i-\alpha_i}^{t_i-\beta_i}\|z_{1s}-z_{2s}\|_{\mathcal{C}}\,ds\\
&\leq \mathsf{M}\sum_{j=1}^{i-1}\left(\mathsf{M}^{i-j}\right)\sigma_j\zeta_h\int_{t_j-\alpha_j}^{t_j-\beta_j}\|z_1-z_2\|_{\Gamma}\,ds+\mathsf{M}\sigma_i\zeta_h\int_{t_i-\alpha_i}^{t_i-\beta_i}\|z_1-z_2\|_{\Gamma}\,ds\\
&\leq 2\mathsf{T}\mathsf{M}\sum_{j=1}^{i-1}\left(\mathsf{M}^{i-j}\right)\sigma_j\zeta_h\|z_1-z_2\|_{\Gamma}+2\mathsf{T}\mathsf{M}\sigma_i\zeta_h\|z_1-z_2\|_{\Gamma}\\
&\leq 2\mathsf{T}\mathsf{M}\zeta_h\left(\sum_{j=1}^{i-1}\left(\mathsf{M}^{i-j}\right)\sigma_j+\sigma_i\right)\|z_1-z_2\|_{\Gamma}. \qquad (18)
\end{aligned}
$$

Step 6: The set $\Sigma := \left\{z\in\Gamma:\ z=\lambda\Lambda\left(\frac{z}{\lambda}\right)+\lambda\Xi(z),\ 0<\lambda<1\right\}$ is bounded. First, we consider the equation

$$z=\lambda\Lambda\left(\frac{z}{\lambda}\right)+\lambda\Xi(z),\ z\in\Sigma \text{ and } 0<\lambda<1.$$

For $t\in[0,t_1]$,

$$
\begin{aligned}
\|z(t)\| &\leq |\lambda|\|\Upsilon(t)\|_{\mathcal{L}(\mathsf{X})}\|\varphi(0)\|+|\lambda|\int_0^t\|\Upsilon(t-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,z_s,\int_0^s\varrho(s,\xi,z_\xi)d\xi\right)\right\|ds\\
&\leq |\lambda|\|\Upsilon(t)\|_{\mathcal{L}(\mathsf{X})}\|\varphi\|_{\mathcal{C}}+|\lambda|\int_0^t\|\Upsilon(t-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,z_s,\int_0^s\varrho(s,\xi,z_\xi)d\xi\right)-\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds\\
&+|\lambda|\int_0^t\|\Upsilon(t-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds\\
&\leq \mathsf{M}|\lambda|\|\varphi\|_{\mathcal{C}}+|\lambda|\int_0^t\mathsf{M}\zeta_\delta(s)\left(\|z_s\|_{\mathcal{C}}+\int_0^s\zeta_\varrho(\xi)\|z_\xi\|_{\mathcal{C}}d\xi\right)ds+|\lambda|\int_0^t\mathsf{M}\left\|\delta\left(s,0,\int_0^s\varrho(s,\xi,0)d\xi\right)\right\|ds.
\end{aligned}
$$

For $t\in(t_i,t_{i+1}],\ i\in[\![1,m]\!]$,

$$
\begin{aligned}
\|z(t)\| &\leq |\lambda|\left\|\Upsilon(t-t_i)\left(\prod_{j=1}^{i}\Upsilon(t_j-t_{j-1})\right)\right\|_{\mathcal{L}(\mathsf{X})}\|\varphi(0)\|\\
&+|\lambda|\sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j}\left\|\Upsilon(t-t_i)\left(\prod_{k=j}^{i-1}\Upsilon(t_{k+1}-t_k)\right)\Upsilon(t_j-s)\right\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,z_s,\int_0^s\varrho(s,\xi,z_\xi)d\xi\right)\right\|ds\\
&+|\lambda|\int_{t_{i-1}}^{t_i}\|\Upsilon(t-t_i)\Upsilon(t_i-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,z_s,\int_0^s\varrho(s,\xi,z_\xi)d\xi\right)\right\|ds\\
&+|\lambda|\int_{t_i}^{t}\|\Upsilon(t-s)\|_{\mathcal{L}(\mathsf{X})}\left\|\delta\left(s,z_s,\int_0^s\varrho(s,\xi,z_\xi)d\xi\right)\right\|ds\\
&+|\lambda|\left\|\Upsilon(t-t_i)\sum_{j=1}^{i-1}\left(\prod_{k=j}^{i-1}\Upsilon(t_{k+1}-t_k)\right)\right\|_{\mathcal{L}(\mathsf{X})}\left\|\mathsf{I}_j\left(\int_{t_j-\alpha_j}^{t_j-\beta_j}h(s,\frac{z_s}{\lambda})ds\right)-\mathsf{I}_j\left(\int_{t_j-\alpha_j}^{t_j-\beta_j}h(s,0)ds\right)\right\|\\
&+|\lambda|\left\|\Upsilon(t-t_i)\sum_{j=1}^{i-1}\left(\prod_{k=j}^{i-1}\Upsilon(t_{k+1}-t_k)\right)\right\|_{\mathcal{L}(\mathsf{X})}\left\|\mathsf{I}_j\left(\int_{t_j-\alpha_j}^{t_j-\beta_j}h(s,0)ds\right)\right\|\\
&+|\lambda|\,\|\Upsilon(t-t_i)\|_{\mathcal{L}(\mathsf{X})}\left\|\mathsf{I}_i\left(\int_{t_i-\alpha_i}^{t_i-\beta_i}h(s,\frac{z_s}{\lambda})ds\right)-\mathsf{I}_i\left(\int_{t_i-\alpha_i}^{t_i-\beta_i}h(s,0)ds\right)\right\|
\end{aligned}
$$

$$+ |\lambda| \, \|\Upsilon(t-t_i)\|_{\mathcal{L}(\mathsf{X})} \left\| \mathsf{I}_i \left(\int_{t_i-\alpha_i}^{t_i-\beta_i} h(s,0)ds \right) \right\|$$

$$\leq \mathsf{M}^{i+1}|\lambda| \|\varphi\|_{\mathcal{C}} + |\lambda| \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} \mathsf{M}^{i-j+1} \left\| \delta \left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi \right) - \delta \left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi \right) \right\| ds$$

$$+ |\lambda| \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} \mathsf{M}^{i-j+1} \left\| \delta \left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi \right) \right\| ds$$

$$+ |\lambda| \int_{t_{i-1}}^{t_i} \mathsf{M}^2 \left\| \delta \left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi \right) - \delta \left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi \right) \right\| ds$$

$$+ |\lambda| \int_{t_{i-1}}^{t_i} \mathsf{M}^2 \left\| \delta \left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi \right) \right\| ds$$

$$+ |\lambda| \int_{t_i}^{t} \mathsf{M} \left\| \delta \left(s, z_s, \int_0^s \varrho(s,\xi,z_\xi)d\xi \right) - \delta \left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi \right) \right\| ds + |\lambda| \int_{t_i}^{t} \mathsf{M} \left\| \delta \left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi \right) \right\| ds$$

$$+ |\lambda| \mathsf{M} \sum_{j=1}^{i-1} \mathsf{M}^{i-j+1} \sigma_j \left\| \int_{t_j-\alpha_j}^{t_j-\beta_j} \left[h\left(s, \frac{z_s}{\lambda}\right) - h(s,0) \right] ds \right\| + |\lambda| \mathsf{M} \sum_{j=1}^{i-1} \mathsf{M}^{i-j+1} \left\| \mathsf{I}_j \left(\int_{t_j-\alpha_j}^{t_j-\beta_j} h(s,0)ds \right) \right\|$$

$$+ |\lambda| \mathsf{M} \sigma_i \left\| \int_{t_i-\alpha_i}^{t_i-\beta_i} \left[h\left(s, \frac{z_s}{\lambda}\right) - h(s,0) \right] ds \right\| + |\lambda| \mathsf{M} \left\| \mathsf{I}_i \left(\int_{t_i-\alpha_i}^{t_i-\beta_i} h(s,0)ds \right) \right\|$$

$$\leq \mathsf{M}^{i+1}|\lambda| \|\varphi\|_{\mathcal{C}} + |\lambda| \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} \mathsf{M}^{i-j+1} \left\| \zeta_\delta(s) \left(\|z_s\|_{\mathcal{C}} + \int_0^s \zeta_\varrho(\xi) \|z_\xi\|_{\mathcal{C}} d\xi \right) \right\| ds$$

$$+ |\lambda| \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} \mathsf{M}^{i-j+1} \left\| \delta \left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi \right) \right\| ds$$

$$+ |\lambda| \int_{t_{i-1}}^{t_i} \mathsf{M}^2 \left\| \zeta_\delta(s) \left(\|z_s\|_{\mathcal{C}} + \int_0^s \zeta_\varrho(\xi) \|z_\xi\|_{\mathcal{C}} d\xi \right) \right\| ds + |\lambda| \int_{t_{i-1}}^{t_i} \mathsf{M}^2 \left\| \delta \left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi \right) \right\| ds$$

$$+ |\lambda| \int_{t_i}^{t} \mathsf{M} \left\| \zeta_\delta(s) \left(\|z_s\|_{\mathcal{C}} + \int_0^s \zeta_\varrho(\xi) \|z_\xi\|_{\mathcal{C}} d\xi \right) \right\| ds + |\lambda| \int_{t_i}^{t} \mathsf{M} \left\| \delta \left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi \right) \right\| ds$$

$$+ \mathsf{M} \sum_{j=1}^{i-1} \mathsf{M}^{i-j+1} \sigma_j \gamma_h \int_{t_j-\alpha_j}^{t_j-\beta_j} \|z_s\|_{\mathcal{C}} \, ds + |\lambda| \mathsf{M} \sum_{j=1}^{i-1} \mathsf{M}^{i-j+1} \left\| \mathsf{I}_j \left(\int_{t_j-\alpha_j}^{t_j-\beta_j} h(s,0)ds \right) \right\|$$

$$+ \mathsf{M} \sigma_i \gamma_h \int_{t_i-\alpha_i}^{t_i-\beta_i} \|z_s\|_{\mathcal{C}} \, ds + |\lambda| \mathsf{M} \left\| \mathsf{I}_i \left(\int_{t_i-\alpha_i}^{t_i-\beta_i} h(s,0)ds \right) \right\|$$

$$\leq \mathsf{M}^{i+1} \|\varphi\|_{\mathcal{C}} + \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} \mathsf{M}^{i-j+1} \left\| \zeta_\delta(s) \left(\|z_s\|_{\mathcal{C}} + \int_0^s \zeta_\varrho(\xi) \|z_\xi\|_{\mathcal{C}} d\xi \right) \right\| ds$$

$$+ \sum_{j=1}^{i-1} \int_{t_{j-1}}^{t_j} \mathsf{M}^{i-j+1} \left\| \delta \left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi \right) \right\| ds$$

$$+ \int_{t_{i-1}}^{t_i} \mathsf{M}^2 \left\| \zeta_\delta(s) \left(\|z_s\|_{\mathcal{C}} + \int_0^s \zeta_\varrho(\xi) \|z_\xi\|_{\mathcal{C}} d\xi \right) \right\| ds + \int_{t_{i-1}}^{t_i} \mathsf{M}^2 \left\| \delta \left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi \right) \right\| ds$$

$$+ \int_{t_i}^{t} \mathsf{M} \left\| \zeta_\delta(s) \left(\|z_s\|_{\mathcal{C}} + \int_0^s \zeta_\varrho(\xi) \|z_\xi\|_{\mathcal{C}} d\xi \right) \right\| ds + \int_{t_i}^{t} \mathsf{M} \left\| \delta \left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi \right) \right\| ds$$

$$+ \mathsf{M} \sum_{j=1}^{i-1} \mathsf{M}^{i-j+1} \sigma_j \gamma_h \int_{t_j-\alpha_j}^{t_j-\beta_j} \|z_s\|_{\mathcal{C}} \, ds + \mathsf{M} \sum_{j=1}^{i-1} \mathsf{M}^{i-j+1} \left\| \mathsf{I}_j \left(\int_{t_j-\alpha_j}^{t_j-\beta_j} h(s,0)ds \right) \right\|$$

(19)

$$+ \mathsf{M} \sigma_i \gamma_h \int_{t_i-\alpha_i}^{t_i-\beta_i} \|z_s\|_{\mathcal{C}} \, ds + \mathsf{M} \left\| \mathsf{I}_i \left(\int_{t_i-\alpha_i}^{t_i-\beta_i} h(s,0)ds \right) \right\|.$$

Finally, we conclude that $\forall\ t \in [-\tau, \mathsf{T}]$, $\|z_t\|_{\mathcal{C}} \leq \Psi(t) := \sup\{\|z(s)\| : \ s \in [-\tau, t]\}$ and there is $\mu \in [-\tau, t]$ such that $\Psi(t) = \|z(\mu)\|$.

So, we have:

For $\mu \in [-\tau, 0]$,

$$\Psi(t) = \|\varphi(t)\| \leq \|\varphi\|_{\mathcal{C}}.$$

For $\mu \in [0, t_1]$,

$$\begin{aligned}\Psi(t) &\leq \mathsf{M}|\lambda|\|\varphi\|_{\mathcal{C}} + |\lambda| \int_0^{\mu} \mathsf{M}\zeta_\delta(s) \left(\|z_s\|_{\mathcal{C}} + \int_0^s \zeta_\varrho(\xi)\|z_\xi\|_{\mathcal{C}} d\xi \right) ds + |\lambda| \int_0^t \mathsf{M} \left\| \delta\left(s, 0, \int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds \\ &\leq \mathsf{M}\|\varphi\|_{\mathcal{C}} + \int_0^{\mu} \mathsf{M}\zeta_\delta(s)\left(\Psi(s) + \int_0^s \zeta_\varrho(\xi)\Psi(\xi)d\xi\right) ds + \int_0^t \mathsf{M}\left\|\delta\left(s,0,\int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds. \qquad (20)\end{aligned}$$

For $\mu \in (t_i, t_{i+1}],\ i \in [\![1, m]\!]$,

$$\begin{aligned}\Psi(t) &\leq \mathsf{M}^{i+1}\|\varphi\|_{\mathcal{C}} + \sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j} \mathsf{M}^{i-j+1}\left\|\zeta_\delta(s)\left(\|z_s\|_{\mathcal{C}} + \int_0^s \zeta_\varrho(\xi)\|z_\xi\|_{\mathcal{C}}d\xi\right)\right\| ds \\ &+ \sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j} \mathsf{M}^{i-j+1}\left\|\delta\left(s,0,\int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds \\ &+ \int_{t_{i-1}}^{t_i} \mathsf{M}^2\left\|\zeta_\delta(s)\left(\|z_s\|_{\mathcal{C}} + \int_0^s \zeta_\varrho(\xi)\|z_\xi\|_{\mathcal{C}}d\xi\right)\right\| ds + \int_{t_{i-1}}^{t_i} \mathsf{M}^2\left\|\delta\left(s,0,\int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds \\ &+ \int_{t_i}^{\mu} \mathsf{M}\left\|\zeta_\delta(s)\left(\|z_s\|_{\mathcal{C}} + \int_0^s \zeta_\varrho(\xi)\|z_\xi\|_{\mathcal{C}}d\xi\right)\right\| ds + \int_{t_i}^{\mathsf{T}} \mathsf{M}\left\|\delta\left(s,0,\int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds \\ &+ \mathsf{M}\sum_{j=1}^{i-1}\mathsf{M}^{i-j+1}\sigma_j\gamma_h\int_{t_j-\alpha_j}^{t_j-\beta_j}\|z_s\|_{\mathcal{C}}\,ds + \mathsf{M}\sum_{j=1}^{i-1}\mathsf{M}^{i-j+1}\left\|\mathsf{I}_j\left(\int_{t_j-\alpha_j}^{t_j-\beta_j} h(s,0)ds\right)\right\| \\ &+ \mathsf{M}\sigma_i\gamma_h\int_{t_i-\alpha_i}^{t_i-\beta_i}\|z_s\|_{\mathcal{C}}\,ds + \mathsf{M}\left\|\mathsf{I}_i\left(\int_{t_i-\alpha_i}^{t_i-\beta_i} h(s,0)ds\right)\right\| \\ &\leq \mathsf{M}^{i+1}\|\varphi\|_{\mathcal{C}} + \sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j} \mathsf{M}^{i-j+1}\left\|\zeta_\delta(s)\left(\Psi(s) + \int_0^s \zeta_\varrho(\xi)\Psi(\xi)d\xi\right)\right\| ds \\ &+ \sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j} \mathsf{M}^{i-j+1}\left\|\delta\left(s,0,\int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds \\ &+ \int_{t_{i-1}}^{t_i} \mathsf{M}^2\left\|\zeta_\delta(s)\left(\Psi(s) + \int_0^s \zeta_\varrho(\xi)\Psi(\xi)d\xi\right)\right\| ds + \int_{t_{i-1}}^{t_i} \mathsf{M}^2\left\|\delta\left(s,0,\int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds \\ &+ \int_{t_i}^{\mu} \mathsf{M}\left\|\zeta_\delta(s)\left(\Psi(s) + \int_0^s \zeta_\varrho(\xi)\Psi(\xi)d\xi\right)\right\| ds + \int_{t_i}^{\mathsf{T}} \mathsf{M}\left\|\delta\left(s,0,\int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds \\ &+ \mathsf{M}\sum_{j=1}^{i-1}\mathsf{M}^{i-j+1}\sigma_j\gamma_h\int_{t_j-\alpha_j}^{t_j-\beta_j}\Psi(s)ds + \mathsf{M}\sum_{j=1}^{i-1}\mathsf{M}^{i-j+1}\left\|\mathsf{I}_j\left(\int_{t_j-\alpha_j}^{t_j-\beta_j} h(s,0)ds\right)\right\| \\ &+ \mathsf{M}\sigma_i\gamma_h\int_{t_i-\alpha_i}^{t_i-\beta_i}\Psi(s)ds + \mathsf{M}\left\|\mathsf{I}_i\left(\int_{t_i-\alpha_i}^{t_i-\beta_i} h(s,0)ds\right)\right\|\end{aligned}$$

$$\leq \mathsf{M}^{m+1}\|\varphi\|_{\mathcal{C}} + \sum_{j=1}^{m-1}\int_{t_{j-1}}^{t_j} \mathsf{M}^{m-j+1}\left\|\zeta_\delta(s)\left(\Psi(s)+\int_0^s \zeta_\varrho(\xi)\Psi(\xi)d\xi\right)\right\| ds$$

$$+\sum_{j=1}^{m-1}\int_{t_{j-1}}^{t_j} \mathsf{M}^{m-j+1}\left\|\delta\left(s,0,\int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds$$

$$+\int_0^\mu \mathsf{M}^2\left\|\zeta_\delta(s)\left(\Psi(s)+\int_0^s \zeta_\varrho(\xi)\Psi(\xi)d\xi\right)\right\| ds + \int_0^{\mathsf{T}} \mathsf{M}^2\left\|\delta\left(s,0,\int_0^s \varrho(s,\xi,0)d\xi\right)\right\| ds$$

$$+\mathsf{M}\sum_{j=1}^{m-1}\mathsf{M}^{m-j+1}\sigma_j\gamma_h\int_{t_j-\alpha_j}^{t_j-\beta_j}\Psi(s)ds + \mathsf{M}\sum_{j=1}^{m-1}\mathsf{M}^{m-j+1}\left\|\mathsf{I}_j\left(\int_{t_j-\alpha_j}^{t_j-\beta_j} h(s,0)ds\right)\right\|$$

$$+\mathsf{M}\gamma_h\max_{i\in[\![1,m]\!]}\sigma_i\int_{t_i-\alpha_i}^{t_i-\beta_i}\Psi(s)ds + \mathsf{M}\left\|\mathsf{I}_i\left(\int_{t_i-\alpha_i}^{t_i-\beta_i} h(s,0)ds\right)\right\| := \mathcal{O}. \tag{21}$$

Moreover, it is easy to see that $\forall\ \mu \in [-\tau, \mathsf{T}]$, $\|z\|_\Gamma \leq \mathcal{O}$.

This implies that the set Σ is bounded and using the Burton-Kirk fixed point theorem, we conclude that the operator $\Lambda + \Xi$ admits at least a fixed point which is a solution for the impulsive integrodifferential system (1).

□

4. Application

In what follows, we illustrate applicability of the obtained results.

So, we consider the integrodifferential system defined bellow by:

$$\begin{cases} \frac{\partial}{\partial t}z(t,\xi) &= \frac{\partial^2}{\partial \xi^2}z(t,\xi) + \int_0^t e^{-b(t-s)}\frac{\partial^2}{\partial \xi^2}z(s,\xi)ds + \ln 2 - \sin\left(z(t-4,\xi)\right) \\ & \quad + \int_0^t \cos\left(z(t-4,\xi)\right)ds, \qquad t\in\mathcal{T},\ t\neq t_1 = 2 \\ z(t,\xi) &= t, \qquad t\in[-7,0] \\ \Delta z(t_1) &= \sin\left(\int_{t_1-\frac{1}{3}}^{t_1-\frac{1}{5}} h(s,z(t-4,\xi))ds\right) \end{cases} \tag{22}$$

where $\mathcal{T} := [0, 10]$, h is any Lipschitz continuous function with constant γ_h and $b \in \mathbb{R}$.

In order to rewrite (22) in the abstract form, we choose the state space as $\mathsf{X} = \mathsf{L}^2([0,\pi],\mathbb{R})$ be furnished with the usual norm $\|\cdot\|_{\mathsf{L}^2}$.

Then we set:

- $$\begin{cases} \mathsf{A} : \mathsf{D}(\mathsf{A}) \subset \mathsf{X} \to \mathsf{X} \\ \mathsf{D}(\mathsf{A}) = \{z\in\mathsf{X} : z,\ z' \text{ are absolutely continuous, } z''\in\mathsf{X} \text{ and } z(0)=z(\pi)=0\}; \\ \forall z\in\mathsf{D}(\mathsf{A}),\ \mathsf{A}z = \frac{\partial^2 z}{\partial \xi^2} \end{cases}$$
- $z(t)(\xi) = z(t,\xi)$ for $t \in \mathcal{T}$ and $\xi \in [0,\pi]$.

In fact A can be written as

$$\mathsf{A}z = -\sum_{n=1}^{\infty} n^2 \langle z, e_n \rangle e_n, \ z \in \mathsf{D}(\mathsf{A}),$$

and generates a compact C_0-semigroup $(\mathsf{T}(t))_{t\geq 0}$ given by

$$\mathsf{T}(t)z = \sum_{n=1}^{\infty} e^{-n^2 t} \langle z, e_n \rangle e_n, \ z \in \mathsf{X},$$

where $e_n(z) = \left(\frac{2}{\pi}\right)^{1/2} \sin(nz), \ 0 \leq z \leq \pi, \ n = 1, 2, \cdots$ denotes the completed orthonormal basis in X.

In addition, the function $t \mapsto e^{-b(t-s)}$ is bounded on the compact $\mathcal{T}$ and it is a $\mathcal{C}^1$ function which derivative is bounded and uniformly continuous.

So the hypothesis **(C1)** and **(C2)** are satisfied and we get the existence of the resolvent operator $\{\Upsilon(t), \ t \geq 0\}$ for the system (22).

Next, we set

(1) $z(t)(\cdot) = z(t, \cdot), \ t \in \mathcal{T},$
(2) $\delta(t, u, v) = \ln 2 - \sin u + v, \ t \in \mathcal{T}, \ u, \ v \in \mathsf{X},$
(3) $\varrho(t, s, v) = \cos v, \ t, \ s \in \mathcal{T}, \ v \in \mathsf{X},$
(4) $\mathsf{I}_1(t, v) = \sin(v), \ t \in \mathcal{T}, \ v \in \mathsf{X}.$

Then the system (22) can take the abstract form (1).

Moreover, $\forall \ t, \ s \in \mathcal{T}, \ u_1, \ u_2, \ v_1, \ v_2 \in \mathsf{X}$, we have

$$\begin{aligned} \|\delta(t, u_1, v_1) - \delta(t, u_2, v_2)\| &\leq \|\sin u_1 - \sin u_2\| + \|v_1 - v_2\| \\ &\leq \|u_1 - u_2\| + \|v_1 - v_2\| \end{aligned}$$

$$\begin{aligned} \|\varrho(t, s, v_1) - \varrho(t, s, v_2)\| &\leq \|\cos v_1 - \cos v_2\| \\ &\leq \|v_1 - v_2\| \end{aligned}$$

$$\|h(t, v_1) - h(t, v_2)\| \leq \gamma_h \|v_1 - v_2\|.$$

Finally, all the assumptions of our results hold and then using the Theorem 3.1, we get that the system (22) has a mild solution.

5. Conclusion

In this work, we considered an integrodifferential system with integral impulses. Using the formula of variation of constants, we give a form for the mild solutions for such systems. Then, under relatively simple conditions, we established that the considered impulsive integrodifferential system has a mild solution. Indeed, few papers deal with this type of impulsive integrodifferential equations, chiefly in the case where the implusions are in an integral form. In perspective, we intend to extend this work to the stochastic case.

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