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AUTOMATIC CONTINUITY OF JORDAN ALMOST MULTIPLICATIVE MAPS ON SPECIAL JORDAN BANACH ALGEBRAS

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ABSTRACT. If $T : \Lambda \to \Gamma$ is a dense range homomorphism from a Banach algebra Λ to a semisimple Banach algebra Γ, then the automatic continuity of T is being a long-standing open question. The concept of Jordan almost multiplicative maps between Jordan Banach algebras is introduced, and an extension of this open question to Jordan Banach algebras is considered. Also, a partial answer to the extended open question is derived in the sense that every dense range Jordan almost multiplicative map T from a special Jordan Banach algebra Λ^+ to a semisimple special Jordan Banach algebra Γ^+, with an additional condition on Λ^+ and Γ^+, is continuous.

1. Introduction

We provide a brief outline of definitions and known outcomes in this section. For more details, one may refer to [4]. All vector spaces are considered over the complex field, and we assume that all algebras are unital. All algebras which are to be considered are associative algebras except Jordan algebras and Jordan Banach algebras.

Definition 1.1. [4] An Banach algebra Λ is a complete normed algebra, where a normed algebra Λ is an algebra with a norm $||.||$, which also satisfies $||\rho.\vartheta|| \leq ||\rho||.||\vartheta||$, $\forall\ \rho, \vartheta \in \Lambda$.

Definition 1.2. An algebra with a Hausdorff topology is called a topological algebra, if all algebraic operations are jointly continuous.

Definition 1.3. [4] The Jacobson radical $rad(\Lambda)$ of an algebra Λ is the intersection of all maximal right(or left) ideals. An algebra is said to be semisimple, if $rad(\Lambda) = \{0\}$.

Definition 1.4. [4] The spectrum $\sigma_\Lambda(p)$ of an element p of an algebra Λ is the set of all complex numbers δ such that $\delta.1 - p$ is not invertible in Λ. The spectral radius $r_\Lambda(p)$ of an element $p \in \Lambda$ is defined by $r_\Lambda(p) = \sup\{|\delta| : \delta \in \sigma_\Lambda(p)\}$.

If $(\Lambda, ||.||)$ is a Banach algebra, then $r_\Lambda(p) = \lim_{n\to\infty} ||p^n||^{\frac{1}{n}}$. Also, for any algebra Λ, we have $rad(\Lambda) = \{p \in \Lambda : r_\Lambda(pq) = 0$, for every $q \in \Lambda\}$. See [4].

Definition 1.5. [4] If $T : \Lambda \to \Gamma$ is a linear map from a Banach algebra Λ to a Banach algebra Γ. Then the separating space of T is defined by
$S(T) = \{q \in \Gamma :$ there exists $(p_n)_{n=1}^{\infty}$ in Λ such that $p_n \to 0$ and $Tp_n \to q\}$.

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Also, $S(T)$ is a closed linear subspace of Γ and moreover, by the closed graph theorem, T is continuous if and only if $S(T) = \{0\}$. For proof, see [4].

Theorem 1.6. *[4] Let Λ and Γ be two Banach algebras and Γ be semisimple. If $T : \Lambda \to \Gamma$ is an epimorphism then $S(T) \subseteq rad(\Gamma)$ and T is continuous.*

By Theorem 1.6, every semisimple Banach algebra has a unique complete norm. This result was proved by Johnson in [9]. T. J. Ransford provided a short proof for Theorem 1.6, in [19]. There are recent articles [5,6,17,18,21,22] for automatic continuity in the theory of topological algebras.

Problem 1.7. *Let $T : \Lambda \to \Gamma$ be a dense range homomorphism from a Banach algebra Λ to a semisimple Banach algebra Γ. Is T continuous?.*

This is being an open question for more than 45 years.

T. G. Honory [5] gives a partial solution to the above Problem 1.7 as follows:

Theorem 1.8. *[5] If $T : \Lambda \to \Gamma$ is a dense range homomorphism from a Banach algebra Λ to a semisimple Banach algebra Γ such that r_Γ is continuous on Γ, then $S(T) \subseteq rad(\Gamma)$, and T is continuous.*

Definition 1.9. [7] Let Λ and Γ be two Banach algebras. A linear map $T : \Lambda \to \Gamma$ is called almost multiplicative, if there exists $\epsilon \geq 0$ such that $||T(pq) - TpTq|| \leq \epsilon ||p|| \, ||q||$; $\forall p, q \in \Lambda$.

Many authors investigated almost multiplicative maps on Banach algebras. See, for example, [8, 11–13, 20].

Recently T. G. Honory et. al. obtained the following partial solution to the above open Problem 1.7 for almost multiplicative maps in [6].

Theorem 1.10. *[6] Let Λ and Γ be two Banach algebras and Γ be semisimple. If a surjective map $T : \Lambda \to \Gamma$ is an almost multiplicative map such that $r_\Gamma(Tp) \leq r_\Lambda(p)$, $\forall p \in \Lambda$, then T is continuous.*

2. Jordan Banach algebras

P. Jordan [10], a physicist, introduced Jordan algebras in an attempt to generalize a quantum mechanics formalism.

Definition 2.1. [14] A Jordan algebra Λ is a non associative algebra Λ whose product is non commutative and satisfies $(\rho.\vartheta).\rho^2 = \rho.(\vartheta.\rho^2), \forall \rho, \vartheta \in \Lambda$. A Jordan Banach algebra Λ is a Jordan algebra endowed with a complete norm $||.||$ satisfying $||\rho.\vartheta|| \leq ||\rho|| \, ||\vartheta||$, $\forall \rho, \vartheta \in \Lambda$. Let us assume that Λ is unital with unit 1 and $||1|| = 1$.

If Λ is an algebra then Λ becomes a commutative Jordan algebra for the Jordan product $p \circ q = (pq + qp)/2$. If Λ is a Banach algebra (not of characteristic 2), then Λ becomes a commutative Jordan Banach algebra for the Jordan product $p \circ q = (pq + qp)/2$. Then, it is called a special Jordan Banach algebra, which is denoted by Λ^+. Here the Jordan product is distributive over addition. There are many articles for automatic continuity in Jordan Banach algebras, see for example [1–3, 23].

An element p in an unital Jordan algebra Λ is said to be invertible if there exists q in Λ such that $p.q = 1$ and $p^2.q = p$. The spectrum $\sigma_\Lambda(p)$ of an element $p \in \Lambda$ is the set of all complex numbers δ such that $\delta.1 - p$ is not invertible in Λ. By a theorem of Jacobson, the spectrum in the Jordan sense coincides with the classical spectrum in case of special Jordan algebra. The next result is also true

Theorem 2.2. *[2] Let Λ be a Jordan Banach algebra and $p \in \Lambda$. Then the spectral radius of p is* $\lim_{n\to\infty} ||p^n||^{\frac{1}{n}}$

K. McCrimmon generalized the concept of Jacobson radical from associative algebras to Jordan algebras (see [15, 16]). An ideal I of a Jordan algebra Λ is called quasi-invertible, if for every $p \in I$, $1 - p$ is invertible. For Λ, there exists a unique quasi-invertible ideal containing every quasi-invertible ideal, which is a result proved by K. McCrimmon. By definition, this ideal is the K. McCrimmon radical of Λ and it is also denoted by $Rad(\Lambda)$. He proved that the K. McCrimmon radical coincides with the Jacobson radical in case of special Jordan algebra. If $Rad(\Lambda) = \{0\}$ in a Jordan Banach algebra Λ, then Λ is called semisimple.

Theorem 2.3. *[16] Let Λ be an associative algebra. Then the McCrimmon radical $Rad(\Lambda^+)$ of the Jordan algebra Λ^+ coincides with the Jacobson radical $Rad(\Lambda)$ of Λ. That is, $Rad(\Lambda^+) = Rad(\Lambda)$.*

For two Jordan Banach algebras Λ and Γ, a linear map $T : \Lambda \to \Gamma$ is called Jordan homomorphism if T preserves the Jordan product. Similarly, we can extend the concept of almost multiplicative mapping given in Definition 1.9 as a concept of Jordan almost multiplicative mapping between two Jordan Banach algebras. Now, we can extend the above open Problem 1.7 to Jordan Banach algebras as an open question, for Jordan almost multiplicative mappings.

Problem 2.4. *Let $T : \Lambda \to \Gamma$ be a dense range Jordan almost multiplicative map between a Jordan Banach algebra Λ and a semisimple Jordan Banach algebra Γ. Is T continuous?.*

Also, we derive a partial solution to the extended open Problem 2.4. More specifically, we prove that every dense range Jordan almost multiplicative map T from a special Jordan Banach algebra Λ^+ to a semisimple special Jordan Banach algebra Γ^+, with an additional condition on Λ^+ and Γ^+, is continuous.

3. Main Result

In this section we assume that all Jordan Banach algebras are special Jordan Banach algebras.

Proposition 3.1. *Let Λ and Γ be two Banach algebras. Every almost multiplicative mapping T from Λ to Γ is also a Jordan almost multiplicative mapping from Λ^+ to Γ^+, where Λ^+ and Γ^+ are special Jordan Banach algebras of Λ and Γ. This result is true even when multiplication in Λ and Γ are not associative.*

Proof. Since T is almost multiplicative map, for $\rho, \vartheta \in \Lambda$ we have

$$\begin{aligned} ||T(\rho \circ \vartheta) - T\rho \circ T\vartheta|| &= ||T(\frac{1}{2}(\rho\vartheta + \vartheta\rho)) - \frac{1}{2}(T\rho T\vartheta + T\vartheta T\rho)|| \\ &= \frac{1}{2}||T(\rho\vartheta) + T(\vartheta\rho) - T\rho T\vartheta - T\vartheta T\rho|| \\ &\leq \frac{1}{2}(||T(\rho\vartheta) - T\rho T\vartheta|| + ||T(\vartheta\rho) - T\vartheta T\rho||) \\ &\leq \frac{1}{2}(\epsilon||\rho|| \, ||\vartheta|| + \epsilon||\vartheta|| \, ||\rho||) \\ &= \epsilon||\rho|| \, ||\vartheta||. \end{aligned}$$

So, we have $||T(\rho \circ \vartheta) - T\rho \circ T\vartheta|| \leq \epsilon||\rho|| \, ||\vartheta||$; for every $\rho, \vartheta \in \Lambda$. □

Let us use the arguments used in proof of Theorem 3.4 in [6], to establish the next Theorem 3.2.

Theorem 3.2. *Let Λ^+ and Γ^+ be two special Jordan Banach algebras. If $T : \Lambda^+ \to \Gamma^+$ is a dense range Jordan almost multiplicative map, then the separating space $S(T)$ is a closed (two sided) ideal in Γ^+.*

Proof. Obviously $S(T)$ is a closed linear subspace of Γ^+.

First we prove that $S(T)$ is an ideal in $T(\Lambda^+)$. Let $b \in S(T)$ and $c \in T(\Lambda^+)$. Then there exists $(a_n)_{n=1}^{\infty}$ in Λ^+ such that $a_n \to 0, Ta_n \to b$, and $Tx = c$, for some $x \in \Lambda^+$. Also we have $x \circ a_n \to 0$. Since T is Jordan almost multiplicative map, we have

$$\begin{aligned} ||T(x \circ a_n) - c \circ b|| &\leq ||T(x \circ a_n) - Tx \circ Ta_n|| + ||c \circ Ta_n - c \circ b|| \\ &\leq \epsilon||x|| \, ||a_n|| + ||c|| \, ||Ta_n - b||. \end{aligned}$$

Since $||Ta_n - b|| \to 0$, and $||a_n|| \to 0$ we have $||T(x \circ a_n) - c \circ b|| \to 0$, and hence $T(x \circ a_n) \to c \circ b$, when $x \circ a_n \to 0$. Therefore, we conclude that $c \circ b \in S(T)$. It was mentioned that the Jordan product $\circ$ is commutative. Hence $S(T)$ is a two sided ideal in $T(\Lambda^+)$.

Next we prove that $S(T)$ is an ideal in $\overline{T(\Lambda^+)}$. Let $b \in S(T)$ and $c \in \overline{T(\Lambda^+)}$, then there exists $(c_n)_{n=1}^{\infty}$ in $T(\Lambda^+)$ such that $c_n \to c$, which implies $c_n \circ b \to c \circ b$. By the above argument, both $c_n \circ b$ and $b \circ c_n$ are in $S(T)$ so that $c \circ b \in \overline{S(T)} = S(T)$ and $b \circ c \in \overline{S(T)} = S(T)$. Therefore, $S(T)$ is an ideal in Γ^+, because $\overline{T(\Lambda^+)} = \Gamma^+$. □

Theorem 3.3. *Let Λ^+ and Γ^+ be two special Jordan Banach algebras such that Γ^+ is semisimple, and r_{Γ^+} is continuous on Γ^+. If $T : \Lambda^+ \to \Gamma^+$ is a dense range Jordan almost multiplicative map with $r_{\Gamma^+}(Ta) \leq r_{\Lambda^+}(a), \forall a \in \Lambda^+$. Then T is continuous.*

Proof. Let $b \in S(T)$. Then there exists $(a_n)_{n=1}^{\infty}$ in Λ^+ such that $a_n \to 0$ and $Ta_n \to b$. Since $r_{\Gamma^+}(Ta) \leq r_{\Lambda^+}(a)$ and $r_{\Lambda^+}(a_n) \to 0$, we have $r_{\Gamma^+}(Ta_n) \to 0$. Also, we have $r_{\Gamma^+}(Ta_n) \to r_{\Gamma^+}(b)$. So, we conclude that $r_{\Gamma^+}(b) = 0$. By Theorem 3.2, $S(T)$ is an ideal in Γ^+. For every $c \in \Gamma^+, b \circ c \in S(T)$. Therefore $r_{\Gamma^+}(b \circ c) = 0$. By Theorem 2.3, $rad(\Gamma^+) = \{b \in \Gamma^+ : r_{\Gamma^+}(b \circ c) = 0, \forall c \in \Gamma^+\}$, and hence $b \in rad(\Gamma^+)$. So, $S(T) \subseteq rad(\Gamma^+)$. Since Γ^+ is semisimple, $S(T) = \{0\}$. Therefore T is continuous, by the closed graph theorem. □

From this Theorem 3.3, we get a partial solution to the open Problem 2.4 in case of special Jordan Banach algebras.

Corollary 3.4. *Let Λ^+ and Γ^+ be two special Jordan Banach algebras such that Γ^+ is semisimple, and r_{Γ^+} is continuous on Γ^+. If $T : \Lambda^+ \to \Gamma^+$ is a dense range Jordan homomorphism, then T is continuous.*

Proof. The map T is Jordan almost multiplicative, because every Jordan homomorphism is Jordan almost multiplicative, for every $\epsilon \geq 0$. Since T is a dense range Jordan homomorphism, we have $r_{\Gamma^+}(Ta) \leq r_{\Lambda^+}(a), \forall a \in \Lambda^+$. □

This Corollary 3.4 is a generalization of Theorem 2.2 in [3].

4. Conclusion

We extended the open Problem 1.7 in Banach algebras to Jordan Banach algebras as an open Problem 2.4 to every dense range Jordan almost multiplicative map from a Jordan Banach algebra to a semisimple Jordan Banach algebra for automatic continuity. If we are able to find a positive solution to this open Problem 2.4, then we would be able to get a positive solution to the open Problem 1.7 in Banach algebras. We obtained a partial solution to Problem 2.4 in Theorem 3.3

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