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RETARDED STOCHASTIC FRACTIONAL NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS DRIVEN BY ROSENBLATT PROCESS WITH UNBOUNDED DELAY

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ABSTRACT. Hermite processes are self-similar processes with stationary increments, the Hermite process of order 1 is fractional Brownian motion and the Hermite process of order 2 is the Rosenblatt process. In this paper, we consider a class of fractional neutral stochastic functional differential equations with infinite delay driven by Rosenblatt process with index $H \in (\frac{1}{2}, 1)$ which is a special case of a self-similar process with long-range dependence. More precisely, we prove the existence of mild solutions by using stochastic analysis and a fixed-point strategy. Finally, an illustrative example is provided to demonstrate the effectiveness of the theoretical result.

1. INTRODUCTION

In recent years the stochastic functional differential equations driven by a fractional Brownian motion have been used to model many of the physical phenomena arising in various areas of science and engineering, such as finance, economics, biology, physics, medicine and so on (see [3–5, 9, 11, 12] and references therein). On the other hand, the very large utilization of the fractional Brownian motion in practice are due to its self-similarity, stationarity of increments and long-range dependence; one prefers in general fBm before other processes because it is Gaussian and the calculus for it is easier; but in concrete situations when the gaussianity is not plausible for the model, one can use for example the Rosenblatt process. Although defined during the 60s and 70s [20, 23] due to their appearance in the Non-Central Limit Theorem, the systematic analysis of Rosenblatt processes has only been developed during the last ten years, motivated by their nice properties (self-similarity, stationarity of the increments, long-range dependence). Since they are non-Gaussian and self-similar with stationary increments, the Rosenblatt processes can also be an input in models where self-similarity is observed in empirical data which appears to be non-Gaussian. There exists a consistent literature that focuses on different theoretical aspects of the Rosenblatt processes. Let us recall some of these works. For example, the rate of convergence to the Rosenblatt process in the Non Central Limit Theorem has been given by Leonenko and Ahn [14]. Tudor [24] studied the analysis of the Rosenblatt process. The distribution of the Rosenblatt process has been given in [15].

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Our results are inspired by the one in [10, 21] where the existence and uniqueness of mild solutions for stochastic neutral functional differential equations driven by Rosenblatt process with delay is studied, as well as some results on the asymptotic behavior.

The main purpose of this paper is to prove the existence of mild solutions for fractional neutral functional stochastic differential equations driven by Rosenblatt process of the form:

$$
\left\{
\begin{array}{l}
d[J_t^{1-\alpha}(x(t)-q(t,x_t)-\varphi(0)+q(0,\varphi))]=[Ax(t)+f(t,x_t)]dt \\
\qquad\qquad\qquad\qquad +G(t)dZ^H(t),\ t\in[0,T], \\
x(t)=\varphi(t)\in L^2(\Omega,\mathcal{B}_h),\ \ for\ a.e.\ t\in(-\infty,0],
\end{array}
\right.
\tag{1.1}
$$

where $\frac{1}{2}<\alpha<1$, $J^{1-\alpha}$ is the $(1-\alpha)-$order Riemann-Liouville fractional integral operator, A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(S(t))_{t\geq 0}$, in a Hilbert space X and Z^H is a Rosenblatt process with $H>\frac{1}{2}$ on a real and separable Hilbert space Y. The history $x_t:(-\infty,0]\to X$, $x_t(\theta)=x(t+\theta)$, belongs to an abstract phase space $\mathcal{B}_h$ defined axiomatically, and $f,q:[0,T]\times\mathcal{B}_h\to X$, and $G:[0,T]\to\mathcal{L}_2^0(Y,X)$, are appropriate functions to be specified later, where $\mathcal{L}_2^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X (see section 2 below).

Fractional differential equations arise in various engineering and scientific disciplines as the mathematical modeling of phenomena in fields such as physics, finance, electrical engineering, telecommunication networks, and so on. There has been a significant development in fractional differential equations. Some authors have considered fractional stochastic equations driven by Wiener processes (we refer to Ahmed [1], Dieye et al. [7], Cui and Yan [6], Lakhel and McKibben [13]). For more details, one can see the monographs of Kilbas et al. [8], and Zhou [26], and the references therein.

To the best of the authors' knowledge, there is no work on the existence of solutions for fractional neutral stochastic differential equations driven by Rosenblatt process with infinite delay. In order to fill this gap, we will make the first attempt to study this problem in this paper. We prove the existence of mild solutions for this kind of equation with the coefficients satisfying several non-Lipschitz conditions, which include the classical Lipschitz condition as a special case.

The outline of this paper is as follows: In the next section, some necessary notations and concepts are provided. In Section 3, we derive the existence result for mild solutions for fractional neutral stochastic differential systems. Finally, in Section 4, we provide an example to illustrate the applicability of the general theory.

2. Preliminaries

In this section, we collect some definitions and lemmas on Wiener integrals with respect to an infinite dimensional Rosenblatt process and we recall some basic results about analytical semigroups and fractional powers of their infinitesimal generators, which will be used throughout the whole of this paper. For details of this section, we refer the reader to [18, 24] and references therein.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space. Selfsimilar processes are invariant in distribution under suitable scaling. They are of considerable interest in practice since aspects of the selfsimilarity appear in different phenomena like telecommunications, turbulence, hydrology or economics. A self-similar processes can be defined as limits that appear in the so-called Non-Central Limit Theorem (see [23]). We briefly recall the Rosenblatt process as well as the Wiener integral with respect to it. Let us recall the notion of Hermite rank. Denote by $H_j(x)$ the Hermite polynomial of degree j given by $H_j = (-1)^j e^{\frac{x^2}{2}} \frac{d^j}{dx^j} e^{\frac{-x^2}{2}}$ and let g be a function on $\mathbb{R}$ such that $\mathbb{E}[g(\zeta_0)] = 0$ and $\mathbb{E}[g(\zeta_0)^2] < \infty$. Assume that g has the following expansion in Hermite polynomials

$$g(x) = \sum_{j\geq 0} c_j H_j(x),$$

where $c_j = \frac{1}{j!}\mathbb{E}(g(\zeta_0 H_j(\zeta_0)))$. The Hermite rank of g is defined by

$$k = min\{j | c_j \neq 0\}.$$

Since $\mathbb{E}[g(\zeta_0)] = 0$, we have $k \geq 1$. Consider $(\zeta_n)_{n\in\mathbb{Z}}$ a stationary Gaussian sequence with mean zero and variance 1 which exhibits long range dependence in the sense that the correlation function satisfies

$$r(n) = \mathbb{E}(\zeta_0 \zeta_n) = n^{\frac{2H-2}{k}} L(n),$$

with $H \in (\frac{1}{2}, 1)$ and L is a slowly varying function at infinity. Then the following family of stochastic processes

$$\frac{1}{n^H} \sum_{j=1}^{[nt]} g(\zeta_j)$$

converges as $n \longrightarrow \infty$, in the sense of finite dimensional distributions, to the selfsimilar stochastic process with stationary increments

$$Z_H^k(t) = c(H,k) \int_{\mathbb{R}^k} \left(\int_0^t \prod_{j=1}^{k} (s - y_j)_+^{-(\frac{1}{2} + \frac{1-H}{k})} ds \right) dB(y_1)...dB(y_k), \tag{2.1}$$

where $x_+ = max(x, 0)$. The above integral is a Wiener-Itô multiple integral of order k with respect to the standard Brownian motion $(B(y))_{y\in\mathbb{R}}$ and the constant $c(H, k)$ is a normalizing constant that ensures $\mathbb{E}(Z_H^k(1))^2 = 1$.

The process $(Z_H^k(t))_{t\geq 0}$ is called the Hermite process. When $k = 1$ the process given by (2.1) is nothing else that the fractional Brownian motion (fBm) with Hurst parameter $H \in (\frac{1}{2}, 1)$. For $k = 2$ the process is not Gaussian. If $k = 2$ then the process (2.1) is known as the Rosenblatt process. It was introduced by Rosenblatt in [20] and was given its name by Taqqu in [22]. The fractional Brownian motion is of course the most studied process in the class of Hermite processes due to its significant importance in modelling. A stochastic calculus with respect to it has been intensively developed in the last decade. The Rosenblatt process is, after fBm, the most well known Hermite process. We also recall the following properties of the Rorenblatt process:

- The process Z_H^k is H-selfsimilar in the sense that for any $c > 0$,

$$(Z_H^k(ct)) =^{(d)} (c^H Z_H^k(t)), \tag{2.2}$$

  where " $=^{(d)}$ " means equivalence of all finite dimensional distributions. It has stationary increments and all moments are finite.

- From the stationarity of increments and the self-similarity, it follows that, for any $p \geq 1$

$$\mathbb{E}|Z_H(t) - Z_H(s)|^p \leq |\mathbb{E}(Z_H(1))|^p |t-s|^{pH}.$$

  As a consequence the Rosenblatt process has Hölder continuous paths of order γ with $0 < \gamma < H$.

Self-similarity and long-range dependence make this process a useful driving noise in models arising in physics, telecommunication networks, finance and other fields.

Consider a time interval $[0,T]$ with arbitrary fixed horizon T and let $\{Z_H(t), t \in [0,T]\}$ the one-dimensional Rosenblatt process with parameter $H \in (1/2, 1)$. By Tudor [24], it is well known that Z_H has the following integral representation:

$$Z_H(t) = d(H) \int_0^t \int_0^t \left[\int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \right] dB(y_1) dB(y_2), \tag{2.3}$$

where $B = \{B(t) : \ t \in [0,T]\}$ is a Wiener process, $H' = \frac{H+1}{2}$ and $K^H(t,s)$ is the kernel given by

$$K^H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

for $t > s$, where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ and $\beta(,)$ denotes the Beta function. We put $K^H(t,s) = 0$ if $t \leq s$ and $d(H) = \frac{1}{H+1}\sqrt{\frac{H}{2(2H-1)}}$ is a normalizing constant.

The covariance of the Rosenblatt process $\{Z_H(t), t \in [0,T]\}$ satisfies, for every $s, t \geq 0$,

$$R_H(s,t) := \mathbb{E}(Z_H(t) Z_H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

The basic observation is the fact that the covariance structure of the Rosenblatt process is similar to the one of the fractional Brownian motion and this allows the use of the same classes of deterministic integrands as in the fractional Brownian motion case whose properties are known.

Now, we introduce Wiener integrals with respect to the Rosenblatt process. We refer to [24] for additional details on the Rosenblatt process .
By formula (2.3) we can write

$$Z_H(t) = \int_0^t \int_0^t I(\mathbf{1}_{[0,t]})(y_1, y_2) dB(y_1) dB(y_2),$$

where by I we denote the mapping on the set of functions $f : [0,T] \longrightarrow \mathbb{R}$ to the set of functions $f : [0,T]^2 \longrightarrow \mathbb{R}$

$$I(f)(y_1, y_2) = d(H) \int_{y_1 \vee y_2}^T f(u) \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du.$$

Let us denote by $\mathcal{E}$ the class of elementary functions on R of the form

$$f(.) = \sum_{j=1}^n a_j \mathbf{1}_{(t_j, t_{j+1}]}(.), \qquad 0 \leq t_j < t_{j+1} \leq T, \quad a_j \in \mathbb{R}, \quad i = 1, ..., n.$$

For $f \in \mathcal{E}$ as above, it is natural to define its Wiener integral with respect to the Rosenblatt process Z_H by

$$\int_0^T f(s)dZ_H(s) := \sum_{j=1}^{n} a_j \left[Z_H(t_{j+1}) - Z_H(t_j)\right] = \int_0^T \int_0^T I(f)(y_1, y_2)dB(y_1)dB(y_2). \tag{2.4}$$

Let $\mathcal{H}$ be the set of functions f such that

$$\mathcal{H} = \left\{ f : [0,T] \longrightarrow \mathbb{R} : \quad \|f\|_{\mathcal{H}} := \int_0^T \int_0^T \left(I(f)(y_1, y_2)\right)^2 dy_1 dy_2 < \infty \right\}.$$

It hold that (see Maejima and Tudor [16])

$$\|f\|_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T f(u)f(v)|u-v|^{2H-2} du dv,$$

and, the mapping

$$f \longrightarrow \int_0^T f(u)dZ_H(u) \tag{2.5}$$

provides an isometry from $\mathcal{E}$ to $L^2(\Omega)$. On the other hand, it has been proved in [19] that the set of elementary functions $\mathcal{E}$ is dense in $\mathcal{H}$. As a consequence the mapping (2.5) can be extended to an isometry from $\mathcal{H}$ to $L^2(\Omega)$. We call this extension as the Wiener integral of $f \in \mathcal{H}$ with respect to Z_H.

Let us consider the operator K_H^* from $\mathcal{E}$ to $\mathbb{L}^2([0,T])$ defined by

$$(K_H^* \varphi)(y_1, y_2) = \int_{y_1 \vee y_2}^T \varphi(r) \frac{\partial K}{\partial r}(r, y_1, y_2) dr,$$

where $K(.,.,.)$ is the kernel of Rosenblatt process in representation (2.3)

$$K(r, y_1, y_2) = \mathbf{1}_{[0,t]}(y_1)\mathbf{1}_{[0,t]}(y_2) \int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du.$$

We refer to [24] for the proof of the fact that K_H^* is an isometry between $\mathcal{H}$ and $L^2([0,T])$. It follows from [24] that $\mathcal{H}$ contains not only functions but its elements could be also distributions. In order to obtain a space of functions contained in $\mathcal{H}$, we consider the linear space $|\mathcal{H}|$ generated by the measurable functions ψ such that

$$\|\psi\|^2_{|\mathcal{H}|} := \alpha_H \int_0^T \int_0^T |\psi(s)||\psi(t)||s-t|^{2H-2} ds dt < \infty,$$

where $\alpha_H = H(2H-1)$. The space $|\mathcal{H}|$ is a Banach space with the norm $\|\psi\|_{|\mathcal{H}|}$ and we have the following inclusions (see [24]).

Lemma 2.1.

$$\mathbb{L}^2([0,T]) \subseteq \mathbb{L}^{1/H}([0,T]) \subseteq |\mathcal{H}| \subseteq \mathcal{H},$$

and for any $\psi \in \mathbb{L}^2([0,T])$, we have

$$\|\psi\|^2_{|\mathcal{H}|} \leq 2HT^{2H-1} \int_0^T |\psi(s)|^2 ds.$$

Let X and Y be two real, separable Hilbert spaces and let $\mathcal{L}(Y,X)$ be the space of bounded linear operator from Y to X. For the sake of convenience, we shall use the same notation to denote the norms in X,Y and $\mathcal{L}(Y,X)$. Let $Q \in \mathcal{L}(Y,Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty} \lambda_n < \infty$. where $\lambda_n \geq 0$ $(n = 1,2...)$ are non-negative real numbers and $\{e_n\}$ $(n = 1,2...)$ is a complete orthonormal basis in Y. We define the infinite dimensional $Q-$Rosenblatt process on Y as

$$Z_H(t) = Z_Q(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n z_n(t), \tag{2.6}$$

where $(z_n)_{n\geq 0}$ is a family of real independent Rosenblatt process.

Note that the series (2.6) is convergent in $L^2(\Omega)$ for every $t \in [0,T]$, since

$$\mathbb{E}|Z_Q(t)|^2 = \sum_{n=1}^{\infty} \lambda_n \mathbb{E}(z_n(t))^2 = t^{2H} \sum_{n=1}^{\infty} \lambda_n < \infty.$$

Note also that Z_Q has covariance function in the sense that

$$E\langle Z_Q(t), x\rangle\langle Z_Q(s), y\rangle = R(s,t)\langle Q(x), y\rangle \text{ for all } x,y \in Y \text{ and } t,s \in [0,T].$$

In order to define Wiener integrals with respect to the Q-Rosenblatt process, we introduce the space $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y,X)$ of all Q-Hilbert-Schmidt operators $\psi : Y \to X$. We recall that $\psi \in \mathcal{L}(Y,X)$ is called a Q-Hilbert-Schmidt operator, if

$$\|\psi\|^2_{\mathcal{L}_2^0} := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2 < \infty,$$

and that the space $\mathcal{L}_2^0$ equipped with the inner product $\langle \varphi, \psi\rangle_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n\rangle$ is a separable Hilbert space.

Now, let $\phi(s);\ s \in [0,T]$ be a function with values in $\mathcal{L}_2^0(Y,X)$, such that $\sum_{n=1}^{\infty} \|K^*\phi Q^{\frac{1}{2}} e_n\|^2_{\mathcal{L}_2^0} < \infty$. The Wiener integral of ϕ with respect to Z_Q is defined by

(2.7)

$$\int_0^t \phi(s)dZ_Q(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\phi(s)e_n dz_n(s) = \sum_{n=1}^{\infty} \int_0^t \int_0^t \sqrt{\lambda_n} K_H^*(\phi e_n)(y_1,y_2)dB(y_1)dB(y_2).$$

Now, we end this subsection by stating the following result which is fundamental to prove our result.

Lemma 2.2. *If $\psi : [0,T] \to \mathcal{L}_2^0(Y,X)$ satisfies $\int_0^T \|\psi(s)\|^2_{\mathcal{L}_2^0} ds < \infty$ then the above sum in (2.7) is well defined as a X-valued random variable and we have*

$$\mathbb{E}\|\int_0^t \psi(s)dZ_H(s)\|^2 \leq 2Ht^{2H-1} \int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0} ds.$$

Proof. By Lemma 2.1, we have

$$\begin{aligned}
\mathbb{E}\|\int_0^t \psi(s)dZ_H(s)\|^2 &= \sum_{n=1}^{\infty} \mathbb{E}\|\int_0^t \int_0^t \sqrt{\lambda_n} K_H^*(\psi e_n)(y_1,y_2)dB_n(y_1)dB_n(y_2)\|^2 \\
&\leq \sum_{n=1}^{\infty} 2Ht^{2H-1} \int_0^t \lambda_n \|\psi(s)e_n\|^2 ds \\
&= 2Ht^{2H-1} \int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0} ds.
\end{aligned}$$

□

It is known that the study of theory of differential equation with infinite delay depends on a choice of the abstract phase space. We assume that the phase space $\mathcal{B}_h$ is a linear space of functions mapping $(-\infty, 0]$ into X, endowed with a norm $\|.\|_{\mathcal{B}_h}$. We shall introduce some basic definitions, notations and lemma used in this paper. First, we present the abstract phase space $\mathcal{B}_h$. Assume that $h : (-\infty, 0] \longrightarrow [0, +\infty)$ is a continuous function with $l = \int_{-\infty}^{0} h(s)ds < +\infty$.

We define the abstract phase space $\mathcal{B}_h$ by

$$\begin{aligned}\mathcal{B}_h = \ & \{\psi : (-\infty, 0] \longrightarrow X \text{ for any } \tau > 0, (\mathbb{E}\|\psi\|^2)^{\frac{1}{2}} \text{ is bounded and measurable} \\ & \text{function on } [-\tau, 0] \text{ and } \textstyle\int_{-\infty}^{0} h(t) \sup_{t\le s\le 0}(\mathbb{E}\|\psi(s)\|^2)^{\frac{1}{2}}dt < +\infty\}.\end{aligned}$$

If we equip this space with the norm

$$\|\psi\|_{\mathcal{B}_h} := \int_{-\infty}^{0} h(t) \sup_{t\le s\le 0} (\mathbb{E}\|\psi(s)\|^2)^{\frac{1}{2}}dt,$$

then it is clear that $(\mathcal{B}_h, \|.\|_{\mathcal{B}_h})$ is a Banach space.

Next, We consider the space $\mathcal{B}_T$, given by

$$\mathcal{B}_T = \{x : x \in \mathcal{C}((-\infty, T], X), \text{ with } x_0 = \varphi \in \mathcal{B}_h\},$$

where $\mathcal{C}((-\infty, T], X)$ denotes the space of all continuous $X-$valued stochastic processes $\{x(t),\ t \in (-\infty, T]\}$. The function $\|.\|_{\mathcal{B}_T}$ to be a semi-norm in $\mathcal{B}_T$, it is defined by

$$\|x\|_{\mathcal{B}_T} = \|x_0\|_{\mathcal{B}_h} + \sup_{0\le t\le T} (\mathbb{E}\|x(t)\|^2)^{\frac{1}{2}}.$$

The following lemma is a common property of phase spaces.

Lemma 2.3. *[17] Suppose $x \in \mathcal{B}_T$, then for all $t \in [0, T]$, $x_t \in \mathcal{B}_h$ and*

$$l(\mathbb{E}\|x(t)\|^2)^{\frac{1}{2}} \le \|x_t\|_{\mathcal{B}_h} \le l \sup_{0\le s\le t} (\mathbb{E}\|x(s)\|^2)^{\frac{1}{2}} + \|x_0\|_{\mathcal{B}_h},$$

where $l = \int_{-\infty}^{0} h(s)ds < \infty$.

Let us give the following well-known definitions related to fractional order differentiation and integration.

Definition 2.4. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : \mathbb{R}^+ \longrightarrow X$ is defined by*

$$J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}ds,$$

where $\Gamma(.)$ is the Gamma function.

Definition 2.5. *The Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$ of a function $f : \mathbb{R}^+ \longrightarrow X$ is defined by*

$$D_t^\alpha f(t) = \frac{d}{dt} J_t^{1-\alpha} f(t).$$

Definition 2.6. *The Caputo fractional derivative of order $\alpha \in (0, 1)$ of $f : \mathbb{R}^+ \longrightarrow X$ is defined by*

$${}^C D_t^\alpha f(t) = D_t^\alpha (f(t) - f(0)).$$

For more details on fractional calculus, one can see [8].

We suppose $0 \in \rho(A)$, the resolvent set of A, and the semigroup, $(S(t))_{t\geq 0}$, is uniformly bounded; that is, there exists $M \geq 1$ such that $\|S(t)\| \leq M$ for every $t \geq 0$. It is possible to define the fractional power $(-A)^\alpha$ for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X, and the expression

$$\|h\|_\alpha = \|(-A)^\alpha h\|$$

defines a norm in $D(-A)^\alpha$. If X_α represents the space $D(-A)^\alpha$ endowed with the norm $\|.\|_\alpha$, then the following properties hold (see [18], p. 74).

Lemma 2.7. *Suppose that A, X_α, and $(-A)^\alpha$ are as described above.*

(i) *For $0 < \alpha \leq 1$, X_α is a Banach space.*

(ii) *If $0 < \beta \leq \alpha$, then the injection $X_\alpha \hookrightarrow X_\beta$ is continuous.*

(iii) *For every $0 < \alpha \leq 1$, there exists $M_\alpha > 0$ such that*

$$\|(-A)^\alpha S(t)\| \leq M_\alpha t^{-\alpha} e^{-\lambda t}, \quad t > 0, \quad \lambda > 0.$$

3. Existence Result

Before starting and proving the main result, we present the definition of mild solutions for fractional neutral stochastic functional differential equation (1.1).

Definition 3.1. *An X-valued process $\{x(t) : t \in (-\infty, T]\}$ is a mild solution of (1.1) if*

(1) *$x(t) = \varphi(t)$ on $(-\infty, 0]$ satisfying $\|\varphi\|^2_{\mathcal{B}_h} < \infty$,*

(2) *$x(t)$ is continuous on $[0, T]$ almost surely and for each $s \in [0, t)$ and $\alpha \in (0, 1)$ the function $(t-s)^{\alpha-1} A S_\alpha(t-s) q(s, x_s)$ is integrable, such that the following stochastic integral equation is verified:*

$$\begin{aligned} x(t) &= T_\alpha(t)(\varphi(0) - q(0,\varphi)) + q(t, x_t) \\ &\quad + \textstyle\int_0^t (t-s)^{\alpha-1} A S_\alpha(t-s) q(s, x_s) ds + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s, x_s) ds \\ &\quad + \textstyle\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) G(s) dZ^H(s), \ \mathbb{P} - a.s. \end{aligned} \tag{3.1}$$

where

$$T_\alpha(t)x = \int_0^\infty \eta_\alpha(\theta) S(t^\alpha \theta) x d\theta, \ t \geq 0, \ x \in X.$$

$$S_\alpha(t)x = \alpha \int_0^\infty \theta \eta_\alpha(\theta) S(t^\alpha \theta) x d\theta, \ t \geq 0, \ x \in X,$$

where

$$\eta_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \omega_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0,$$

$$\omega_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-\alpha n - 1} \frac{\Gamma(n\alpha + 1)}{n!} \sin(n\alpha\pi), \quad \theta \in]0, \infty[,$$

η_α is a probability density function defined on $(0, \infty)$.

Remark 3.2. *(see [25])*

$$\int_0^\infty \theta \eta_\alpha(\theta) d\theta = \frac{1}{\Gamma(1+\alpha)}. \tag{3.2}$$

The following properties of T_α and S_α appeared in [25] are useful in what follows.

Lemma 3.3. *Under the previous assumptions on $S(t)$, $t \geq 0$ and A, the operators $T_\alpha(t)$ and $S_\alpha(t)$ have the following properties:*

(i) *For any $x \in X$, $\|T_\alpha(t)x\| \leq M\|x\|$, $\|S_\alpha(t)x\| \leq \frac{M}{\Gamma(\alpha)}\|x\|$.*
(ii) *$\{T_\alpha(t),\ t \geq 0\}$ and $\{S_\alpha(t),\ t \geq 0\}$ are strongly continuous.*
(iii) *For any $t > 0$, $T_\alpha(t)$ and $S_\alpha(t)$ are also compact operators if $S(t)$ is compact.*
(iv) *For any $x \in X$, $\beta \in (0,1)$ and $\delta \in (0,1]$, we have*

$$AS_\alpha(t)x = A^{1-\beta} S_\alpha A^\beta x, \ \textit{and}\ \|A^\delta S_\alpha(t)\| \leq \frac{\alpha M_\delta}{t^{\alpha\delta}} \frac{\Gamma(2-\delta)}{\Gamma(1+\alpha(1-\delta))},\ t \in (0,T].$$

The proof of the main result makes use of the following fixed point theorem.

Lemma 3.4. *(Sadovskii's fixed point theorem) Let Φ be a condensing operator on a Banach space X, that is, Φ is continuous and takes bounded sets into bounded sets, and $\mu(\Phi(B)) \leq \mu(B)$ for every bounded set B of X with $\mu(B) > 0$. If $\Phi(N) \subset N$ for a convex, closed set of X, then Φ has a fixed point in X (where $\mu(.)$ denotes Kuratowski's measure of noncompactness).*

In this paper, we impose the following conditions on the data of the problem:

($\mathcal{H}$.1) The analytic semigroup, $(S(t))_{t\geq 0}$, generated by A is compact for $t > 0$, and there exists $M \geq 1$ such that

$$\sup_{t\geq 0} \|S(t)\| \leq M, \qquad \text{and } c_1 = \|(-A)^{-\beta}\|.$$

($\mathcal{H}$.2) The map $f : [0,T] \times \mathcal{B}_h \to X$ satisfies the following conditions:

(i) The function $t \longmapsto f(t,x)$ is measurable for each $x \in \mathcal{B}_h$, the function $x \longmapsto f(t,x)$ is continuous for almost all $t \in [0,T]$,
(ii) there exists a nonnegative function $p \in L^1([0,T], \mathbb{R}^+)$, and a continuous nondecreasing function $\vartheta : \mathbb{R}^+ \longrightarrow (0,+\infty)$ such that for $\delta > \frac{1}{2\alpha-1}$, $\quad (\alpha \in (\frac{1}{2},1))$,

$$\int_0^T (\vartheta(s))^\delta ds < \infty, \qquad \liminf_{k\longrightarrow+\infty} \frac{\vartheta(k)}{k} = \gamma < \infty,$$

and

$$\mathbb{E}\|f(t,x)\|^2 \leq p(t)\vartheta(\|x\|^2_{\mathcal{B}_h}), \ \text{ for all } x \in \mathcal{B}_h \text{ and for a.e. } t \in [0,T].$$

($\mathcal{H}$.3) The function $q : [0,T] \times \mathcal{B}_h \longrightarrow X$ is continuous. For $\beta \in (0,1)$, satisfied with $\alpha\beta > \frac{1}{2}$, the function q is X_β-valued and there exists positive constant M_q, such that

$$\mathbb{E}\|(-A)^\beta q(t,x) - (-A)^\beta q(t,y)\|^2 \leq M_q\|x-y\|^2_{\mathcal{B}_h}, \ \text{ for all } x \in \mathcal{B}_h \text{ and for a.e. } t \in [0,T],$$

$$\mathbb{E}\|(-A)^\beta q(t,x)\|^2 \leq M_q[\|x\|^2_{\mathcal{B}_h} + 1], \ \text{ for all } x \in \mathcal{B}_h \text{ and for a.e. } t \in [0,T],$$

$(\mathcal{H}.4)$ There exists a constant $p > \frac{1}{2\alpha-1}$ such that the function $G : [0,\infty) \to \mathcal{L}_2^0(Y,X)$ satisfies

$$\int_0^T \|G(s)\|_{\mathcal{L}_2^0}^{2p} ds < \infty, \quad \forall T > 0.$$

The main result of this chapter is the following.

Theorem 3.5. *Suppose that* $(\mathcal{H}.1) - (\mathcal{H}.4)$ *hold. Then, there exists a mild solution to the system (1.1) on* $(-\infty, T]$ *provided that*

$$20l^2\{M_q[c_1^2 + \frac{T^{2\alpha\beta}\alpha^2 M_{1-\beta}^2 \Gamma^2(\beta+1)}{(2\alpha\beta-1)\Gamma^2(\alpha\beta+1)}] + \frac{\gamma M^2 T}{\Gamma^2(\alpha)} \int_0^T (T-s)^{2\alpha-2} p(s) ds\} < 1. \tag{3.3}$$

Proof. Transform the problem (1.1) into a fixed-point problem. To do this, define operator Ψ on $\mathcal{B}_T$ by

$$\Psi(x)(t) = \begin{cases} \varphi(t), \quad \text{if } \ t \in (-\infty, 0], \\ \\ T_\alpha(t)(\varphi(0) - q(0,\varphi)) + q(t, x_t) + \int_0^t (t-s)^{\alpha-1} A S_\alpha(t-s) q(s, x_s) ds \\ \\ + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) G(s) dZ^H(s), \quad \text{if } \ t \in [0, T]. \end{cases}$$

It is clear that to prove the existence of mild solutions to equation (1.1) is equivalent to find a fixed point for the operator Ψ.

Let $y : (-\infty, T] \longrightarrow X$ be the function defined by

$$y(t) = \begin{cases} \varphi(t), & \text{if } \ t \in (-\infty, 0], \\ S(t)\varphi(0), & \text{if } \ t \in [0, T], \end{cases}$$

then, $y_0 = \varphi$. For each function $z \in \mathcal{B}_T$, set

$$x(t) = z(t) + y(t).$$

It is obvious that x satisfies the stochastic control system (3.1) if and only if z satisfies $z_0 = 0$ and

$$\begin{aligned} z(t) = \ & q(t, z_t + y_t) - T_\alpha(t) q(0,\varphi) + \int_0^t (t-s)^{\alpha-1} A S_\alpha(t-s) q(s, z_s + y_s) ds \\ & + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) G(s) dZ^H(s). \end{aligned} \tag{3.4}$$

Set

$$\mathcal{B}_T^0 = \{z \in \mathcal{B}_T : z_0 = 0\};$$

for any $z \in B_T^0$, we have

$$\|z\|_{\mathcal{B}_T^0} = \|z_0\|_{\mathcal{B}_h} + \sup_{t\in[0,T]} (\mathbb{E}\|z(t)\|^2)^{\frac{1}{2}} = \sup_{t\in[0,T]} (\mathbb{E}\|z(t)\|^2)^{\frac{1}{2}}.$$

Then, $(\mathcal{B}_T^0, \|.\|_{\mathcal{B}_T^0})$ is a Banach space. Define the operator $\Pi : \mathcal{B}_T^0 \longrightarrow \mathcal{B}_T^0$ by

$$(3.5)\qquad (\Pi z)(t)=\begin{cases} 0 \ \text{ if } \ t\in(-\infty,0], \\ q(t,z_t+y_t)-T_\alpha(t)q(0,\varphi))+\int_0^t(t-s)^{\alpha-1}AS_\alpha(t-s)q(s,z_s+y_s)ds \\ +\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f(s,z_s+y_s)ds \\ +\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)G(s)dZ^H(s), \qquad \text{if } \ t\in[0,T]. \end{cases}$$

Set

$$\mathcal{B}_k=\{z\in\mathcal{B}_T^0:\|z\|^2_{\mathcal{B}_T^0}\le k\},\qquad \text{for some } k\ge 0,$$

then $\mathcal{B}_k\subseteq\mathcal{B}_T^0$ is a bounded closed convex set, and for $z\in\mathcal{B}_k$, we have

$$(3.6)\qquad \begin{aligned} \|z_t+y_t\|^2_{\mathcal{B}_h} &\le 2(\|z_t\|^2_{\mathcal{B}_h}+\|y_t\|^2_{\mathcal{B}_h}) \\ &\le 4(l^2\sup\nolimits_{0\le s\le t}\mathbb{E}\|z(s)\|^2+\|z_0\|^2_{\mathcal{B}_h} \\ &+l^2\sup\nolimits_{0\le s\le t}\mathbb{E}\|y(s)\|^2+\|y_0\|^2_{\mathcal{B}_h}) \\ &\le 4l^2(k+M^2\mathbb{E}\|\varphi(0)\|^2)+4\|y\|^2_{\mathcal{B}_h} \\ &:=q'. \end{aligned}$$

It is clear that the operator Ψ has a fixed point if and only if Π has one. So, we show that Π has a fixed point. To this end, we decompose Π as $\Pi=\Pi_1+\Pi_2$, where Π_1 and Π_2 are defined on $\mathcal{B}_T^0$, respectively by

$$(3.7)\qquad (\Pi_1 z)(t)=\begin{cases} 0 \ \text{ if } \ t\in(-\infty,0], \\ q(t,z_t+y_t)-T_\alpha(t)q(0,\varphi))+\int_0^t(t-s)^{\alpha-1}AS_\alpha(t-s)q(s,z_s+y_s)ds \\ +\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)G(s)dZ^H(s), \quad \text{if } \ t\in[0,T], \end{cases}$$

and

$$(3.8)\qquad (\Pi_2 z)(t)=\begin{cases} 0 \ \text{ if } \ t\in(-\infty,0], \\ \int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f(s,z_s+y_s)ds, \ \text{if } \ t\in[0,T]. \end{cases}$$

For convenience, the proof will be given in several steps.

Step 1. We claim that there exists a positive number k, such that $\Pi_1(x)+\Pi_2(x)\in\mathcal{B}_k$ whenever $x\in\mathcal{B}_k$. If it is not true, then for each positive number k, there is a function $z^k(.)\in\mathcal{B}_k$, but $\Pi_1(z^k)+\Pi_2(z^k)\notin\mathcal{B}_k$, that is $\mathbb{E}\|\Pi_1(z^k)(t)+\Pi_2(z^k)(t)\|^2>k$ for some $t\in[0,T]$. On the other hand, we have

$$
\begin{aligned}
k < \mathbb{E}\|\Pi_1(z^k)(t)+\Pi_2(z^k)(t)\|^2 &\leq 5\{\mathbb{E}\|T_\alpha(t)q(0,\varphi)\|^2+\mathbb{E}\|q(t,z_t^k+y_t)\|^2 \\
&+\mathbb{E}\|\int_0^t(t-s)^{\alpha-1}AS_\alpha(t-s)q(s,z_s^k+y_s)ds\|^2 \\
&+\mathbb{E}\|\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f(s,z_s^k+y_s)ds\|^2 \\
&+\mathbb{E}\|\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)G(s)dZ^H(s)\|^2\} \\
&\leq 5\sum_{i=1}^5 I_i.
\end{aligned} \tag{3.9}
$$

By $(\mathcal{H}.3)$, (i) of Lemma 3.3, we have

$$
\begin{aligned}
I_1 &\leq \mathbb{E}\|T\alpha(t)q(0,\varphi)\|^2 \\
&\leq M^2\|(-A)^{-\beta}\|^2\|(-A)^\beta q(0,\varphi)\|^2 \\
&\leq M^2c_1^2M_q[\|\varphi\|_{\mathcal{B}_h}^2+1].
\end{aligned} \tag{3.10}
$$

By $(\mathcal{H}.3)$, (3.6), we have

$$
\begin{aligned}
I_2 &\leq \|(-A)^{-\beta}\|^2\mathbb{E}\|(-A)^\beta q(t,z_t^k+y_t)\|^2 \\
&\leq c_1^2M_q[\|z_t^k+y_t\|_{\mathcal{B}_h}^2+1] \\
&\leq c_1^2M_q[4l^2(k+M^2\mathbb{E}\|\varphi(0)\|^2)+4\|y\|_{\mathcal{B}_h}^2+1].
\end{aligned} \tag{3.11}
$$

By (iv) of Lemma 3.3, $(\mathcal{H}.3)$ and Hölder's inequality, we have

$$
\begin{aligned}
I_3 &\leq \mathbb{E}\|\int_0^t(t-s)^{\alpha-1}AS_\alpha(t-s)q(s,z_s^k+y_s)ds\|^2 \\
&\leq \mathbb{E}\|(\int_0^t(t-s)^{\alpha-1}(-A)^{1-\beta}S_\alpha(t-s)(-A)^\beta q(s,z_s^k+y_s)ds\|^2 \\
&\leq \mathbb{E}(\int_0^t(t-s)^{\alpha-1}\|(-A)^{1-\beta}S_\alpha(t-s)(-A)^\beta q(s,z_s^k+y_s)\|ds)^2 \\
&\leq \frac{\alpha^2M_{1-\beta}^2\Gamma^2(\beta+1)}{\Gamma^2(\alpha\beta+1)}\mathbb{E}(\int_0^t(t-s)^{\alpha-1}\|(t-s)^{\alpha\beta-\alpha}(-A)^\beta q(s,z_s^k+y_s)\|ds)^2 \\
&\leq \frac{\alpha^2M_{1-\beta}^2\Gamma^2(\beta+1)}{\Gamma^2(\alpha\beta+1)}\int_0^t(t-s)^{2\alpha\beta-2}ds\int_0^t\mathbb{E}\|(-A)^\beta q(s,z_s^k+y_s)\|^2ds \\
&\leq \frac{T^{2\alpha\beta-1}\alpha^2M_{1-\beta}^2\Gamma^2(\beta+1)}{(2\alpha\beta-1)\Gamma^2(\alpha\beta+1)}\int_0^t M_q(4l^2(k+M^2\mathbb{E}\|\varphi(0)\|^2)+4\|y\|_{\mathcal{B}_h}^2+1)ds \\
&\leq \frac{T^{2\alpha\beta}\alpha^2M_{1-\beta}^2\Gamma^2(\beta+1)}{(2\alpha\beta-1)\Gamma^2(\alpha\beta+1)}M_q[4l^2(k+M^2\mathbb{E}\|\varphi(0)\|^2)+4\|y\|_{\mathcal{B}_h}^2+1].
\end{aligned} \tag{3.12}
$$

From $(\mathcal{H}.2)$ and Hölder's inequality, we have

$$
\begin{aligned}
I_4 &\leq \mathbb{E}\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s, z_s^k + y_s) ds\|^2 \\
&\leq \tfrac{M^2 T}{\Gamma^2(\alpha)} \mathbb{E} \int_0^t \|(t-s)^{\alpha-1} f(s, z_s^k + y_s)\|^2 ds \\
&\leq \tfrac{M^2 T}{\Gamma^2(\alpha)} \int_0^T (T-s)^{2\alpha-2} \mathbb{E}\|f(s, z_s^k + y_s)\|^2 ds \\
&\leq \tfrac{M^2 T}{\Gamma^2(\alpha)} \int_0^T (T-s)^{2\alpha-2} p(s) \vartheta(\|z_s^k + y_s\|_{\mathcal{B}_h}^2) ds \\
&\leq \tfrac{M^2 T}{\Gamma^2(\alpha)} \vartheta(4l^2(k + M^2 \mathbb{E}\|\varphi(0)\|^2) + 4\|y\|_{\mathcal{B}_h}^2) \int_0^T (T-s)^{2\alpha-2} p(s) ds
\end{aligned} \tag{3.13}
$$

From (ii) of $(\mathcal{H}.2)$ and Hölder's inequality, it follows that for $\delta > \frac{1}{2\alpha-1}$,

$$
\begin{aligned}
\int_0^T (T-s)^{2\alpha-2} p(s) ds &\leq \left(\int_0^T (T-s)^{\frac{(2\alpha-2)\delta}{\delta-1}} ds \right)^{\frac{\delta-1}{\delta}} \left(\int_0^T (p(s))^\delta ds \right)^{\frac{1}{\delta}} \\
&\leq T^{\frac{(2\alpha-1)\delta-1}{\delta}} \left(\int_0^T (p(s))^\delta ds \right)^{\frac{1}{\delta}} \\
&< \infty.
\end{aligned}
$$

On the other hand, for $p > \frac{1}{2\alpha-1}$, we have

$$
\begin{aligned}
\int_0^T (T-s)^{(2\alpha-2)} \|G(s)\|_{\mathcal{L}_2^0}^2 ds &\leq \left(\int_0^T (T-s)^{\frac{(2\alpha-2)p}{p-1}} ds \right)^{\frac{p-1}{p}} \left(\int_0^T \|G(s)\|_{\mathcal{L}_2^0}^{2p} ds \right)^{\frac{1}{p}} \\
&\leq T^{\frac{(2\alpha-1)p-1}{p}} \left(\int_0^T \|G(s)\|_{\mathcal{L}_2^0}^{2p} ds \right)^{\frac{1}{p}} \\
&< \infty.
\end{aligned} \tag{3.14}
$$

By Lemma 2.2, Lemma 3.3 and (3.14), we have for $p > \frac{1}{2\alpha-1}$,

$$
\begin{aligned}
I_5 &\leq \mathbb{E}\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) G(s) dZ^H(s)\|^2 \\
&\leq \frac{2M^2 T^{2H-1}}{\Gamma^2(\alpha)} \int_0^T (T-s)^{(2\alpha-2)} \|G(s)\|_{\mathcal{L}_2^0}^2 ds \\
&\leq \frac{2M^2 T^{2H-1}}{\Gamma^2(\alpha)} T^{\frac{(2\alpha-1)p-1}{p}} \left(\int_0^T \|G(s)\|_{\mathcal{L}_2^0}^{2p} ds \right)^{\frac{1}{p}}.
\end{aligned} \tag{3.15}
$$

By (3.9), (3.10), (3.11), (3.12), (3.13) and (3.15), we have

$$
\begin{aligned}
k < \ & \mathbb{E}\|\Pi_1(z^k)(t) + \Pi_2(z^k)(t)\|^2 \leq \overline{K} + 20 M_q l^2 k [c_1^2 + \tfrac{T^{2\alpha\beta}\alpha^2 M_{1-\beta}^2 \Gamma^2(\beta+1)}{(2\alpha\beta-1)\Gamma^2(\alpha\beta+1)}] \\
& + 5 \tfrac{M^2 T}{\Gamma^2(\alpha)} \vartheta(4l^2(k + M^2 \mathbb{E}\|\varphi(0)\|^2) + 4\|y\|_{\mathcal{B}_h}^2) \int_0^T (T-s)^{2\alpha-2} p(s) ds
\end{aligned}
$$

where

$$\overline{K} = 5c_1^2 M_q\{M^2[\|\varphi\|^2_{\mathcal{B}_h}+1] + [4l^2M^2\mathbb{E}\|\varphi(0)\|^2 + 4\|y\|^2_{\mathcal{B}_h}+1]\}$$
$$+5\tfrac{T^{2\alpha\beta}\alpha^2M^2_{1-\beta}\Gamma^2(\beta+1)}{(2\alpha\beta-1)\Gamma^2(\alpha\beta+1)}M_q[4l^2M^2\mathbb{E}\|\varphi(0)\|^2 + 4\|y\|^2_{\mathcal{B}_h}+1]$$
$$+\tfrac{10M^2T^{2H-1}}{\Gamma^2(\alpha)}T^{\frac{(2\alpha-1)p-1}{p}}\left(\int_0^T\|G(s)\|^{2p}_{\mathcal{L}_2^0}ds\right)^{\frac{1}{p}}.$$

Noting that $\overline{K}$ is independent of k, dividing both sides by k and taking the lower limit as $k\longrightarrow\infty$, yields

$$q' = 4l^2(k + M\mathbb{E}\|\varphi(0)\|^2) + 4\|y\|_{\mathcal{B}_h}\longrightarrow\infty \text{ as } k\longrightarrow\infty,$$
$$\liminf_{k\longrightarrow\infty}\frac{\vartheta(q')}{k} = \liminf_{k\longrightarrow\infty}\frac{\vartheta(q')}{q'}.\frac{q'}{k} = 4l^2\gamma,$$

Thus, we have

$$1 \le 20l^2\{M_q[c_1^2 + \tfrac{T^{2\alpha\beta}\alpha^2M^2_{1-\beta}\Gamma^2(\beta+1)}{(2\alpha\beta-1)\Gamma^2(\alpha\beta+1)}] + \tfrac{\gamma M^2T}{\Gamma^2(\alpha)}\int_0^T(T-s)^{2\alpha-2}p(s)ds\}.$$

This contradicts (3.3). Hence for some positive k,

$$(\Pi_1+\Pi_2)(\mathcal{B}_k)\subseteq\mathcal{B}_k.$$

Step 2. We will show that $\Pi_1+\Pi_2$ has a fixed point in B_k. We divide this proof into four claims.

Claim 1. Π_1 is a contraction.
Let $t\in[0,T]$ and $z^1,z^2\in\mathcal{B}_T^0$

$$\begin{aligned}\mathbb{E}\|(\Pi_1z^1)(t)-(\Pi_1z^2)(t)\|^2 &\le 2\mathbb{E}\|q(t,z_t^1+y_t)-q(t,z_t^2+y_t)\|^2\\
&+2\mathbb{E}\|\int_0^t(t-s)^{\alpha-1}AS_\alpha(t-s)(q(s,z_s^1+y_s)-q(s,z_s^2+y_s))ds\|^2\\
&\le 2M_q\|(-A)^{-\beta}\|^2\|z_s^1-z_s^2\|^2_{\mathcal{B}_h}\\
&+2\int_0^t(t-s)^{\alpha-1}(-A)^{1-\beta}S_\alpha(t-s)(-A)^\beta(q(s,z_s^1+y_s)-q(s,z_s^2+y_s))ds\|^2\\
&\le 2M_q\|(-A)^{-\beta}\|^2\|z_s^1-z_s^2\|^2_{\mathcal{B}_h}\\
&+\tfrac{2\alpha^2M^2_{1-\beta}\Gamma^2(\beta+1)}{\Gamma^2(\alpha\beta+1)}\int_0^t(t-s)^{2\alpha\beta-2}ds\int_0^tM_q\|z_s^1-z_s^2\|^2_{\mathcal{B}_h}ds\\
&\le 2M_q\left\{\|(-A)^{-\beta}\|^2+\tfrac{2\alpha^2M^2_{1-\beta}\Gamma^2(\beta+1)}{\Gamma^2(\alpha\beta+1)}\tfrac{T^{2\alpha\beta}}{2\alpha\beta-1}\right\}(2l^2\sup_{0\le s\le T}\\
&\mathbb{E}\|z^1(s)-z^2(s)\|^2+2(\|z_0^1\|^2_{\mathcal{B}_h}+\|z_0^2\|^2_{\mathcal{B}_h})\\
&\le\kappa\sup_{0\le s\le T}\mathbb{E}\|z^1(s)-z^2(s)\|^2)\quad(\text{ since }\ z_0^1=z_0^2=0)\end{aligned}$$

Taking supremum over t, yields

$$\|(\Pi_1z^1)(t)-(\Pi_1z^2)(t)\|_{\mathcal{B}_T^0}\le\kappa\|z^1-z^2\|_{\mathcal{B}_T^0},$$

where

$$\kappa = 4M_q l^2 \left\{ c_1^2 + \frac{2\alpha^2 M_{1-\beta}^2 \Gamma^2(\beta+1)}{\Gamma^2(\alpha\beta+1)} \frac{T^{2\alpha\beta}}{2\alpha\beta - 1} \right\}.$$

By (3.3), we have $\kappa < 1$. Thus Π_1 is a contraction on $\mathcal{B}_T^0$.

Next, we will show that Π_2 is compact operator. Let k satisfy $\Pi_2(B_k) \subset B_k$.

Claim 2. Π_2 maps bounded sets into bounded sets in B_k. For each $t \in [0, T]$, $z \in B_k$, from (3.6), it follows that

$$\|z_t + y_t\|_{\mathcal{B}_h}^2 \leq 4l^2(k + M^2\mathbb{E}\|\varphi(0)\|^2) + 4\|y\|_{\mathcal{B}_h}^2 := q'.$$

By the similar argument as above, we get

$$\begin{aligned} \mathbb{E}\|\Pi_2 z(t)\|^2 &\leq \mathbb{E}\| \textstyle\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s, z_s + y_s) ds\|^2 \\ &\leq \tfrac{M^2 T}{\Gamma^2(\alpha)} \vartheta(q') \textstyle\int_0^T (T-s)^{2\alpha-2} p(s) ds \\ &:= \Delta, \end{aligned}$$

which implies that for each $z \in B_k$, $\|\Pi_2 z\|_{\mathcal{B}_T^0}^2 \leq \Delta$.

Claim 2. Π_2 maps $\mathcal{B}_k$ into equicontinuous family. Let $z \in \mathcal{B}_k$ and $|h|$ be sufficiently small, we have

$$\begin{aligned} \mathbb{E}\| \quad &(\Pi_2 z)(t+h) - (\Pi_2 z)(t)\|^2 \leq \mathbb{E}\| \textstyle\int_0^{t+h} (t+h-s)^{\alpha-1} S_\alpha(t+h-s) f(s, z_s + y_s) ds \\ &- \textstyle\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s, z_s + y_s) ds\|^2 \\ &\leq 3\mathbb{E}\| \textstyle\int_0^t \left((t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\right) S_\alpha(t+h-s) f(s, z_s + y_s) ds\|^2 \\ &+ 3\mathbb{E}\| \textstyle\int_t^{t+h} (t+h-s)^{\alpha-1} S_\alpha(t+h-s) f(s, z_s + y_s) ds\|^2 \\ &+ 3\mathbb{E}\| \textstyle\int_0^t (t-s)^{\alpha-1} \left(S_\alpha(t+h-s) - S_\alpha(t-s)\right) f(s, z_s + y_s) ds\|^2. \end{aligned}$$

From (iii) of Lemma 3.3, we have $S_\alpha(t)$ is compact for any $t > 0$. Let $0 < \varepsilon < t < T$, and $\delta > 0$ such that $\|S_\alpha(\tau_1) - S_\alpha(\tau_2)\| \leq \epsilon$ for every $\tau_1, \tau_2 \in [0, T]$ with $|\tau_1 - \tau_2| \leq \delta$. Applying Lemma 3.3 together with Hölder inequality, it follows that

$$\begin{aligned} &\mathbb{E}\|(\Pi_2 z)(t+h) - (\Pi_2 z)(t)\|^2 \\ &\leq \tfrac{3M^2 T \vartheta(q')}{\Gamma^2(\alpha)} \textstyle\int_0^t \left((t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\right)^2 p(s) ds \\ &+ \tfrac{3M^2 T \vartheta(q')}{\Gamma^2(\alpha)} \textstyle\int_t^{t+h} (t+h-s)^{2(\alpha-1)} p(s) ds \\ &+ \tfrac{3M^2 T}{2\alpha-1} \epsilon \textstyle\int_0^t (t-s)^{2(\alpha-1)} p(s) ds. \end{aligned} \tag{3.16}$$

From (ii) of $(\mathcal{H}.2)$ and Hölder's inequality, it follows that for $\delta > \frac{1}{2\alpha-1}$,

$$\begin{aligned}\int_0^t (t-s)^{2\alpha-2} p(s)ds &\leq \left(\int_0^t (t-s)^{\frac{(2\alpha-2)\delta}{\delta-1}} ds\right)^{\frac{\delta-1}{\delta}} \left(\int_0^T (p(s))^\delta ds\right)^{\frac{1}{\delta}} \\ &\leq T^{\frac{(2\alpha-1)\delta-1}{\delta}} \left(\int_0^T (p(s))^\delta ds\right)^{\frac{1}{\delta}} \\ &< \infty.\end{aligned}$$

Similarly, we have

$$\int_0^t (t+h-s)^{2(\alpha-1)} p(s)ds < \infty.$$

By the dominated convergence theorem, we have

$$\int_0^t \left((t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\right)^2 p(s)ds \longrightarrow 0, \text{ as } h \longrightarrow 0.$$

Therefore, for sufficiently small positive number ϵ, we have from (3.16) that

$$\mathbb{E}\|(\Pi_2 z)(t+h) - (\Pi_2 z)(t)\|^2 \longrightarrow 0 \text{ as } h \longrightarrow 0.$$

Thus, Π_2 maps $\mathcal{B}_k$ into an equicontinuous family of functions.

Claim 3. $(\Pi_2\mathcal{B}_k)(t)$ is precompact set in X.

Let $0 < t \leq T$ be fixed, and $0 < \epsilon < t$. For $\delta > 0$ and $z \in \mathcal{B}_k$, we define

$$\begin{aligned}(\Pi_{2,\epsilon}^\delta z)(t) &= \alpha \int_0^{t-\epsilon} \int_\delta^\infty \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^\alpha\theta) f(s, z_s + y_s) d\theta ds \\ &= S(\epsilon^\alpha\delta)\alpha \int_0^{t-\epsilon} \int_\delta^\infty \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^\alpha\theta - \epsilon^\alpha\delta) f(s, z_s + y_s) d\theta ds.\end{aligned}$$

From the compactness of $S(t)$ $(t > 0)$, we obtain that the set $V_\epsilon^\delta(t) = \{(\Pi_{2,\epsilon}^\delta z)(t) : \ z \in \mathcal{B}_k\}$ is relative compact in X for every ϵ, $0 < \epsilon < t$ and $\delta > 0$. Moreover, for every $z \in \mathcal{B}_k$, we have

(3.17)

$$\begin{aligned}\mathbb{E}\|\Pi_2 z)(t) \ &-\Pi_{2,\epsilon}^\delta z)(t)\|^2 \leq 2\alpha^2 \mathbb{E}\| \int_0^t \int_0^\delta \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^\alpha\theta) f(s, z_s + y_s) d\theta ds\|^2 \\ &+2\alpha^2 \mathbb{E}\| \int_{t-\epsilon}^t \int_\delta^\infty \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^\alpha\theta) f(s, z_s + y_s) d\theta ds\|^2 \\ &= 2\left(J_1 + J_2\right).\end{aligned}$$

A similar argument as before, can be used to show that

$$\begin{aligned}J_1 &\leq \alpha^2 M^2 T \mathbb{E} \int_0^t \| \int_0^\delta \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) f(s, z_s + y_s) d\theta\|^2 ds \\ &\leq \alpha^2 M^2 T \| \int_0^\delta \theta\eta_\alpha(\theta) d\theta\|^2 \int_0^t (t-s)^{2\alpha-2} \mathbb{E}\|f(s, z_s + y_s)\|^2 ds \qquad (3.18) \\ &\leq \alpha^2 M^2 T \vartheta(q') \| \int_0^\delta \theta\eta_\alpha(\theta) d\theta\|^2 \int_0^t (t-s)^{2\alpha-2} p(s) ds.\end{aligned}$$

For J_2, by (3.2), we have

$$
\begin{aligned}
J_2 &\le \alpha^2 M^2 T\vartheta(q')\| \textstyle\int_0^\infty \theta\eta_\alpha(\theta)d\theta\|^2 \int_{t-\epsilon}^t (t-s)^{2\alpha-2}p(s)ds \\
&\le \tfrac{\alpha^2 M^2 T\vartheta(q')}{\Gamma^2(1+\alpha)} \textstyle\int_{t-\epsilon}^t (t-s)^{2\alpha-2}p(s)ds \\
&\le \tfrac{\alpha^2 M^2 T\vartheta(q')}{\Gamma^2(1+\alpha)} \left(\textstyle\int_{t-\epsilon}^t (t-s)^{\frac{(2\alpha-2)\delta}{\delta-1}}ds\right)^{\frac{\delta-1}{\delta}} \left(\textstyle\int_{t-\epsilon}^t (p(s))^\delta ds\right)^{\frac{1}{\delta}} \\
&\le \tfrac{\alpha^2 M^2 T\vartheta(q')}{\Gamma^2(1+\alpha)} \epsilon^{\frac{(2\alpha-1)\delta-1}{\delta}} \left(\textstyle\int_{t-\epsilon}^t (p(s))^\delta ds\right)^{\frac{1}{\delta}},
\end{aligned} \tag{3.19}
$$

where $\delta > \frac{1}{2\alpha-1}$.

Substitute (3.18) and (3.19) into (3.17) to obtain

$$\mathbb{E}\|\Pi_2 z)(t) - \Pi_{2,\epsilon}^{\delta} z)(t)\|^2 \longrightarrow 0, \qquad \text{as } \epsilon \longrightarrow 0^+,\ \delta \longrightarrow 0^+.$$

Therefore, there are precompact sets arbitrarily close to the set $V(t) = \{(\Pi_2 z)(t) :\ z \in B_k\}$, hence the set $V(t)$ is also precompact in X.

Thus, by Arzela-Ascoli theorem Π_2 is a compact operator.

Then, $\Pi = \Pi_1 + \Pi_2$ is a condensing operator in B_k. By Lemma 3.4, there exists a fixed point $z(.)$ for Π on $\mathcal{B}_k$. If we define $x(t) = z(t) + y(t)$, $-\infty < t \le T$, it is easy to see that $x(.)$ is a mild solution of (1.1). This completes the proof.

□

4. Example

To illustrate the above abstract result, we consider the following fractional neutral stochastic partial differential equation with infinite delays driven by a fractional Brownian motion of the form

$$
\begin{cases}
dJ_t^{1-\alpha}[v(t,\xi) - q(t, v(t-r,\xi)) - \varphi(0,\xi) + q(0, v(-r,\xi))] = [\frac{\partial^2}{\partial^2\xi} v(t,\xi) + +f(t, t-r, \xi)]dt \\
+G(t)\frac{dZ^H(t)}{dt}, \quad 0 \le t \le T,\ r > 0,\ 0 \le \xi \le 1 \\
v(t,0) = v(t,1) = 0, \qquad 0 \le t \le T, \\
v(s,\xi) = \varphi(s,\xi), \ ; -\infty < s \le 0 \quad 0 \le \xi \le 1,
\end{cases} \tag{4.1}
$$

where $Z^H(t)$ is cylindrical fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. g, f, G are appropriate functions and $\varphi : (-\infty, 0] \times [0,1] \longrightarrow \mathbb{R}$ is a given measurable and satisfies $\|\varphi\|^2_{\mathcal{B}_h} < \infty$.

We rewrite (4.1) into abstract form of (1.1). We take $X = Y = U = L^2([0,1])$. Define the operator $A : D(A) \subset X \longrightarrow X$ given by $A = \frac{\partial^2}{\partial^2\xi}$ with

$$D(A) = \{y \in X :\ y' \text{ is absolutely continuous}, y'' \in X, \quad y(0) = y(1) = 0\},$$

then we get

$$Ax = \sum_{n=1}^{\infty} n^2 < x, e_n >_X e_n, \quad x \in D(A),$$

where $e_n := \sqrt{\frac{2}{\pi}} \sin nx,\ n = 1, 2, ...$ is an orthogonal set of eigenvector of $-A$.

The bounded linear operator $(-A)^{\frac{2}{3}}$ is given by

$$(-A)^{\frac{2}{3}} x = \sum_{n=1}^{\infty} n^{\frac{4}{3}} < x, e_n >_X e_n,$$

with domain

$$D((-A)^{\frac{2}{3}}) = \{x \in X, \sum_{n=1}^{\infty} n^{\frac{4}{3}} < x, e_n >_X e_n \in X\}.$$

It is known that A generates a compact analytic semigroup $\{S(t)\}_{t\geq 0}$ in X, and is given by (see [18])

$$S(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} < x, e_n > e_n,$$

for $x \in X$ and $t \geq 0$. Since the semigroup $\{S(t)\}_{t\geq 0}$ is analytic, there exists a constant $M > 0$ such that $\|S(t)\|^2 \leq M$ for every $t \geq 0$. In other words, the condition $(\mathcal{H}.1)$ holds.

If we choose $\alpha \in (\frac{3}{4}, 1)$,

$$S_\alpha(t)x = \int_0^\infty \alpha\theta\eta_\alpha(\theta) S(\theta t^\alpha) d\theta, \quad x \in X.$$

In order to define the operator $Q : Y := L^2([0, 1], \mathbb{R}) \longrightarrow Y$, we choose a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$, set $Qe_n = \lambda_n e_n$, and assume that

$$tr(Q) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} < \infty.$$

Define the fractional Brownian motion in Y by

$$Z^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta^H(t) e_n,$$

where $H \in (\frac{1}{2}, 1)$ and $\{\beta_n^H\}_{n\in\mathbb{N}}$ is a sequence of one-dimensional fractional Brownian motions mutually independent. Let us assume the function $G : [0, +\infty) \to \mathcal{L}_2^0(L^2([0, 1]), L^2([0, 1]))$ satisfies

$$\int_0^T \|G(s)\|_{\mathcal{L}_2^0}^{2p} ds < \infty, \quad \text{for some } p > \frac{1}{2\alpha - 1}.$$

We choose the phase function $h(s) = e^{2s},\ s < 0$, then $l = \int_{-\infty}^0 h(s)ds = \frac{1}{2} < \infty$, and the abstract phase space $\mathcal{B}_h$ is Banach space with the norm

$$\|\varphi\|_{\mathcal{B}_h} = \int_{-\infty}^0 h(s) \sup_{\theta\in[s,0]} (\mathbb{E}\|\varphi(\theta)\|^2)^{\frac{1}{2}} ds.$$

To rewrite the initial-boundary value problem (4.1) in the abstract form (1.1), we assume the following:

For $(t, \varphi) \in [0, T] \times \mathcal{B}_h$, where $\varphi(\theta)(\xi) = \varphi(\theta, \xi)$, $(\theta, \xi) \in (-\infty, 0] \times [0, 1]$, we put $v(t)(\xi) = v(t, \xi)$. Define $q : [0, T] \times \mathcal{B}_h \longrightarrow X$, $f : [0, T] \times \mathcal{B}_h \longrightarrow X$ by

$$(-A)^{\frac{2}{3}} q(t, \varphi)(\xi) = \int_{-\infty}^0 e^{-4\theta} \varphi(\theta)(\xi) d\theta,$$

$$f(t, \varphi)(\xi) = \int_{-\infty}^0 \mu_1(t, \xi, \theta) f_1(\varphi(\theta)(\xi)) d\theta,$$

where

(i) the function $\mu_1(t,\xi,\theta) \geq 0$ is continuous in $[0,T] \times [0,1] \times (-\infty, 0)$,

$$\int_{-\infty}^{0} \mu_1(t,\xi,\theta)d\theta = p_1(t,\xi) < \infty, \qquad \text{and} \qquad \left(\int_0^1 p_1^2(t,\xi)\right)\frac{1}{2} = p(t) < \infty;$$

(ii) the function $f_1(.)$ is continuous, $0 \leq f_1(v(\theta,\xi)) \leq \vartheta(\|v(\theta,.)\|_{L^2})$ for $(\theta,\xi) \in (-\infty,0) \times (0,1)$, where $\vartheta(.) : [0,\infty) \longrightarrow (0,\infty)$ is continuous and nondecreasing.

By the similar method as in Balasubramaniyam and Ntouyas [2], we can show that all the assumptions of Theorem 3.5 are satisfied. Therefore, there exists a mild solution for the system (4.1).

5. Conclusion

In this paper, we have studied the existence of mild solutions for a class of fractional neutral stochastic functional differential equations with infinite delay driven by Rosenblatt process in a separable Hilbert space. The results are obtained by using the stochastic analysis theory and Sadovskii's fixed point theorem. Also, an example is provided to illustrate the applicability of the obtained result. Upon making some appropriate assumptions , by employing the ideas and techniques same as in this paper, one can establish the existence results with impulses and nonlocal conditions. Our future work will be concerned on the transportation inequalities, with respect to the uniform distance, for the law of the mild solution of fractional neutral differential systems driven by Rosenblatt process.

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