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SQUARE-MEAN PSEUDO ALMOST PERIODIC SOLUTIONS OF CLASS r UNDER THE LIGHT OF MEASURE THEORY

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ABSTRACT. The aim of this work is to present new concept of square-mean pseudo almost periodic of class r using the measure theory. We use the (μ,ν)-ergodic process to define the spaces of (μ,ν)-pseudo almost periodic processes of class r in the square-mean sense. We present many interesting results on those spaces like completeness and composition theorems and we study the existence and the uniqueness of the square-mean (μ,ν)-pseudo almost periodic solutions of class r for the stochastic evolution equation.

1. INTRODUCTION

In this work, we study some properties of the square-mean (μ,ν)-pseudo almost periodic process using the measure theory and we used those results to study the following stochastic evolution equations in a Hilbert space H,

$$dx(t) = [Ax(t) + L(x_t) + f(t)]dt + g(t)dW(t), \tag{1.1}$$

where $A : D(A) \subset H$ is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on H, $f, g : \mathbb{R} \to L^2(P, H)$ are two stochastic processes and $W(t)$ is a two-sided and standard one-dimensional Brownian notion defined on the filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ with $\mathcal{F}_t = \sigma\{W(u) - W(v) \mid u, v \leq t\}$ and L is a bounded linear operator from C into $L^2(P, H)$. $C = C([-r, 0]; L^2(P, H))$ denotes the space of continuous functions from $[-r, 0]$ to $L^2(P, H)$ endowed with the uniform topology norm. For every $t \geq 0$, x_t denotes the history function of C defined by $x_t(\theta) = x(t + \theta)$ for $-r \leq \theta \leq 0$.

We assume $(H, \|,\|)$ is a real separable Hilbert space and $L^2(P, H)$ is the space of all H-valued random variables x such that

$$\mathbb{E}\|x\|^2 = \int_\Omega \|x\|^2 dP < +\infty.$$

This work is an extension of [14] whose authors have studied equation (1.1) in the deterministic case. Some recent contributions concerning square-mean pseudo almost periodic solutions for abstract differential equations similar to equation (1.1) have been made. For example in [10] the authors studied equation(1.1) without the operator L. They showed that the equation has a unique square-mean μ-pseudo almost periodic mild solution on $\mathbb{R}$ when f and g are square

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mean pseudo almost periodic functions.
In [5] the authors studied the square-mean almost periodic solutions to a class of nonautonomous stochastic differential equations without our operator L and without delay on a separable real Hilbert space. They established the existence and uniqueness of a square-mean almost periodic mild solution to those nonautonomous stochastic differential equations with the 'Acquistapace-Terreni' conditions.
In [9] The authors established the existence, uniqueness and stability of square-mean μ-pseudo almost periodic (resp. automorphic) mild solution to a linear and semilinear case of the stochastic evolution equations in case when the functions forcing are both continuous and $S^2-\mu-$pseudo almost periodic (resp. automorphic) and verify some suitable assumptions.
This work is organized as follow, in section 2, we give spectral decomposition of phase space in section 3 we study square-mean (μ,ν)-ergodic process of class r, in section 4 we study square-mean (μ,ν)-pseudo almost process functions and properties and last section is devoted to an application.

2. Spectral decomposition

To equation (1.1), associate the following initial value problem

$$\begin{cases} du_t = [Au_t + L(u_t) + f(t)]dt + g(t)dW(t) \text{ for } t \geq 0 \\ \\ u_0 = \varphi \in C = C([-r,0], L^2(P,H)), \end{cases} \tag{2.1}$$

where $f:\mathbb{R}^+ \to L^2(P,H)$ and $g:\mathbb{R}^+ \to L^2(P,H)$ are stochastic processes continuous.

Definition 2.1. We say that a continuous function u from $[-r,+\infty[$ into $L^2(P,H)$ is an integral solution of equation, if the following conditions hold:

(1) $\int_0^t u(s)ds \in D(A)$ for $t \geq 0$,
(2) $u(t) = \phi(0) + A\int_0^t u(s)ds + \int_0^t (L(u_s)+f(s))ds + \int_0^t g(s)dW(s)$, for $t \geq 0$,
(3) $u_0 = \phi$.

If $\overline{D(A)} = L^2(P,H)$, the integral solution coincide with the known mild solutions. One can see that if $u(t)$ is an integral of equation 2.1, then $u(t) \in \overline{D(A)}$ for all $t \geq 0$, in particular $\phi(0) \in \overline{D(A)}$

Let us introduce the part A_0 of the operator A in $\overline{D(A)}$ which defined by

$$\begin{cases} D(A_0) = \{x \in D(A): \ Ax \in \overline{D(A)}\} \\ A_0x = Ax \text{ for } \ x \in D(A_0) \end{cases}$$

The following assumption is supposed:
$(\mathbf{H_0})$ A satisfies the Hille-Yosida condition.

Proposition 2.2. *[2] A_0 generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$.*

The phase space $\mathcal{C}_0$ of equation (2.1) is defined by

$$C_0 = \{\varphi \in C: \ \varphi(0) \in \overline{D(A)}\}.$$

For each $t \geq 0$, the linear operator $\mathcal{U}(t)$ on C_0 is defined by

$$\mathcal{U}(t) = v_t(.,\varphi)$$

where $v(.,\varphi)$ is the solution of the following homogeneous equation

$$\begin{cases} \dfrac{d}{dt}v_t = Av_t + L(v_t) \text{ for } t \geq 0 \\ \\ v_0 = \varphi \in C. \end{cases}$$

Proposition 2.3. *[3] $(\mathcal{U}(t))_{t\geq 0}$ is a strongly continuous semigroup of linear operators on C_0. Moreover, $(\mathcal{U}(t))_{t\geq 0}$ satisfies, for $t \geq 0$ and $\theta \in [-r,0]$, the following translation property*

$$(\mathcal{U}(t)\varphi)(\theta) = \begin{cases} (\mathcal{U}(t+\theta)\varphi)(0) \; for \, t+\theta \geq 0 \\ \\ \varphi(t+\theta) \; for \, t+\theta \leq 0. \end{cases}$$

Theorem 2.4. *[3] Let $\mathcal{A}_{\mathcal{U}}$ defined on C_0 by*

$$\begin{cases} D(\mathcal{A}_{\mathcal{U}}) = \Big\{\varphi \in C^1([-r,0];X);\ \varphi(0) \in \overline{D(A)} \ \text{ and } \ \varphi'(0) = A\varphi(0) + L(\varphi)\Big\} \\ \\ \mathcal{A}_{\mathcal{U}}\varphi = \varphi' \ \text{ for } \ \varphi \in D(\mathcal{A}_{\mathcal{U}}). \end{cases}$$

Then $\mathcal{A}_{\mathcal{U}}$ is the infinitesimal generator of the semigroup $(\mathcal{U}(t))_{t\geq 0}$ on C_0.

Let $\langle X_0\rangle$ be the space defined by

$$\langle X_0\rangle = \{X_0 x : \ x \in X\}$$

where the function $X_0 x$ is defined by

$$(X_0 x)(\theta) = \begin{cases} 0 \quad \text{if} \quad \theta \in [-r,0[, \\ \\ x \quad \text{if} \quad \theta = 0. \end{cases}$$

The space $C_0 \oplus \langle X_0\rangle$ equipped with the norm $|\phi + X_0 c|_C = |\phi|_C + |c|$ for $(\phi, c) \in C_0 \times X$ is a Banach space and consider the extension $\mathcal{A}_{\mathcal{U}}$ defined on $C_0 \oplus \langle X_0\rangle$ by

$$\begin{cases} D(\widetilde{\mathcal{A}_{\mathcal{U}}}) = \Big\{\varphi \in C^1([-r,0];X) : \ \varphi \in D(A) \ \text{ and } \varphi' \in \overline{D(A)}\Big\} \\ \widetilde{\mathcal{A}_{\mathcal{U}}}\varphi = \varphi' + X_0(A\varphi + L(\varphi) - \varphi'). \end{cases}$$

Proposition 2.5. *[3] Assume that $(\boldsymbol{H_0})$ holds. Then, $\widetilde{\mathcal{A}_{\mathcal{U}}}$ satisfies the Hille-Yosida condition on $C_0 \oplus \langle X_0\rangle$ there exist $\widetilde{M} \geq 0$, $\widetilde{\omega} \in \mathbb{R}$ such that $]\widetilde{\omega}, +\infty[\subset \rho(\widetilde{\mathcal{A}_{\mathcal{U}}})$ and*

$$|(\lambda I - \widetilde{\mathcal{A}_{\mathcal{U}}})^{-n}| \leq \frac{\widetilde{M}}{(\lambda - \widetilde{\omega)^n}} \ \ for \ n \in \mathbb{N} \ \ and \ \lambda > \widetilde{\omega}$$

Moreover,the part of $\widetilde{\mathcal{A}_{\mathcal{U}}}$ on $D(\widetilde{\mathcal{A}_{\mathcal{U}}}) = C_0$ is exactly the operator $\widetilde{\mathcal{A}_{\mathcal{U}}}$.

Definition 2.6. The semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic if

$$\sigma(\mathcal{A}_{\mathcal{U}}) \cap i\mathbb{R} = \emptyset$$

For the sequel, we make the following assumption:
$(\mathbf{H_1})$ $T_0(t)$ is compact on $\overline{D(A)}$ for every $t > 0$.

Proposition 2.7. *Assume that $(\boldsymbol{H_0})$ and $(\boldsymbol{H_1})$. then the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is compact for $t > r$.*

Proposition 2.8. *Assume that $(\boldsymbol{H_1})$ holds. If the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic then the space C_0 is decomposed as a direct sum*

$$C_0 = S \oplus U$$

of two $\mathcal{U}(t)$ invariant closed subspaces S and U such that the restricted semigroup on $\mathcal{U}$ is a group and there exist positive constant $\overline{M}$ and ω such that

$$|\mathcal{U}(t)\varphi| \leq \overline{M}e^{-\omega t}|\varphi| \text{ for } t \geq 0 \text{ and } \varphi \in S$$

$$|\mathcal{U}(t)\varphi| \leq \overline{M}e^{\omega t}|\varphi| \text{ for } t \leq 0 \text{ and } \varphi \in U,$$

Where S and U are called respectively the stable and unstable space, Π^s and Π^u denote respectively the projection operator on S and U.

3. Square-Mean (μ, ν)-Ergodic process of class r

Let $\mathcal{N}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{N}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < \infty$, for all $a, b \in \mathbb{R}$ $(a \leq b)$. $L^2(P, H)$ is a Hilbert space with following norm

$$\|x\|_{L^2} = \left(\int_\Omega \|x\|^2 dP\right)^{\frac{1}{2}}$$

Definition 3.1. Let $x : \mathbb{R} \to L^2(P, H)$ be a stochastic process.

(1) x said to be stochastically bounded if there exists $C > 0$ such that

$$\mathbb{E}\|x(t)\|^2 \leq C \; \forall \; t \in \mathbb{R}.$$

(2) x is said to be stochastically continuous if

$$\lim_{t\to s} \mathbb{E}\|x(t) - x(s)\|^2 = 0 \; \forall \; s \in \mathbb{R}.$$

Denote by $SBC(\mathbb{R}, L^2(P, H))$, the space of all stochastically bounded and continuous process. Otherwise, this space endowed the following norm

$$\|x\|_\infty = \sup_{t\in\mathbb{R}} \left(\mathbb{E}\|x(t)\|^2\right)^{\frac{1}{2}}$$

is a Banach space.

Definition 3.2. Let $\mu, \nu \in \mathcal{M}$. A stochastic process f is said to be square-mean $(\mu, \nu)-$ ergodic if $f \in SBC(\mathbb{R}, L^2(P, H))$ and satisfied

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \mathbb{E}\|f(t)\|^2 d\mu(t) = 0.$$

We denote by $\mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu)$, the space of all such process.

Definition 3.3. Let $\mu,\nu\in\mathcal{M}$. A stochastic process f is said to be square-mean $(\mu,\nu)-$ ergodic of class r if $f\in SBC(\mathbb{R},L^2(P,H))$ and satisfied

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^2d\mu(t)=0.$$

We denote by $\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,r)$, the space of all such process.
For $\mu,\nu\in\mathcal{M}$ and $a\in\mathbb{R}$, we denote by μ_a the positive measure on $(\mathbb{R},\mathcal{N})$ defined by

$$\mu_a(A)=\mu(a+b:b\in A)\ \text{ for } A\in\mathcal{N}. \tag{3.1}$$

From $\mu,\nu\in\mathcal{M}$, we formulate the following hypothesis
$(\mathbf{H_2})$: For all $a\in\mathbb{R}$, there exists $\beta>0$ and a bounded intervall I such that $\mu_a(A)\le\beta\mu(A)$ when $A\in\mathcal{N}$ satisfies $A\cap I=\emptyset$.
$(\mathbf{H_3})$ For all a, b and $c\in\mathbb{R}$, such that $0\le a<b\le c$, there exist δ_0 and $\alpha_0>0$ such that

$$|\delta|\ge\delta_0 \implies \mu(a+\delta,b+\delta)\ge\alpha_0\mu(\delta,c+\delta).$$

$(\mathbf{H_4})$ Let $\mu,\nu\in\mathcal{M}$ be such that $\limsup_{\tau\to+\infty}\dfrac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}=\alpha<\infty$.

Proposition 3.4. *Assume that $(\boldsymbol{H_4})$ holds. Then $\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,r)$ is a Banach space with the norm $\|,\|_\infty$.*

Proof. We can see that $\mathcal{E}(\mathbb{R};L^2(P,H),\mu,\nu,r)$ is a vector subspace of $SBC(\mathbb{R},L^2(P,H)))$. To complete the proof, it is enough to prove that $\mathcal{E}(\mathbb{R};L^2(P,H),\mu,\nu,r)$ is closed in $SBC(\mathbb{R};L^2(P,H))$. Let $(f_n)_n$ be a sequence in $\mathcal{E}(\mathbb{R};L^2(P,H),\mu,\nu,r)$ such that $\lim_{n\to+\infty}f_n=f$ uniformly in $SBC(\mathbb{R},L^2(P,H))$. From $\nu(\mathbb{R})=+\infty$, it follows $\nu([-\tau,\tau])>0$ for τ sufficiently large. Let $n_0\in\mathbb{N}$ such that for all $n\ge n_0$, $\|f_n-f\|_\infty<\varepsilon$. Let $n\ge n_0$, then

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^2\Big)d\mu(t) \le\ & \frac{2}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_n(\theta)-f(\theta)\|^2\Big)d\mu(t)\\
&+\frac{2}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_n(\theta)\|^2\Big)d\mu(t)\\
\le\ & \frac{2}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{t\in\mathbb{R}}\mathbb{E}\|f_n(t)-f(t)\|^2\Big)d\mu(t)\\
&+\frac{2}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_n(\theta)\|^2\Big)d\mu(t)\\
\le\ & 2\|f_n-f\|_\infty^2\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}\\
&+\frac{2}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_n(\theta)\|\Big)d\mu(t).
\end{aligned}$$

Consequently

$$\limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^2\Big)d\mu(t)\le 2\alpha\varepsilon\ \text{ for any } \varepsilon>0.$$

□

The following theorem is a characterization of square-mean (μ,ν)-ergodic processes(eventually $I=\emptyset$).

Theorem 3.5. *Assume that* $(\boldsymbol{H_4})$ *holds and let* $f \in SBC(\mathbb{R}, L^2(P,H))$. *Then the following assertions are equivalent:*

i) $\mathcal{E}(\mathbb{R}, L^2(P,H), \mu, \nu, r)$

ii) $\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau]\setminus I)} \int_{[-\tau,\tau]\setminus I} \sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|^2 d\mu(t) = 0$

iii) *For any* $\varepsilon > 0$, $\lim_{\tau\to+\infty} \frac{\mu\left\{t \in [-\tau,\tau]\setminus I : \sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|^2 > \varepsilon\right\}}{\nu([-\tau,\tau]\setminus I)} = 0$

Proof. The proof is made like the proof of Theorem(2.13) in [6].
First, we show that $i)$ is equivalent to $ii)$.
Denote by $A = \nu(I)$, $B = \int_I \Big(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|^2\Big) d\mu(t)$. A and B belong to $\mathbb{R}$, since the interval I is bounded and the process f is stochastically bounded and continuous. For $\tau > 0$ such that $I \subset [-\tau,\tau]$ and $\nu([-\tau,\tau]\setminus I) > 0$, it follows

$$\frac{1}{\nu([-\tau,\tau]\setminus I)} \int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|^2\Big) d\mu(t) = \frac{1}{\nu([-\tau,\tau]) - A}\Big[\int_{-\tau}^{\tau} \Big(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|^2\Big) d\mu(t) - B\Big]$$
$$= \frac{\nu([-\tau,\tau])}{\nu([-\tau,\tau]) - A}\Big[\frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau]} \Big(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|^2\Big) d\mu(t) - \frac{B}{\nu([-\tau,\tau])}\Big].$$

From above equalities and the fact that $\nu(\mathbb{R}) = +\infty$, $ii)$ is equivalent to

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|^2\Big) d\mu(t) = 0,$$

that is i).

Now, we show that $iii)$ implies $ii)$.
Denote by A_τ^ε and B_τ^ε the following sets

$$A_\tau^\varepsilon = \Big\{t \in [-\tau,\tau]\setminus I : \sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|^2 > \varepsilon\Big\}$$
$$B_\tau^\varepsilon = \Big\{t \in [-\tau,\tau]\setminus I) : \sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|^2 \le \varepsilon\Big\}.$$

Assume that $iii)$ holds, that is

$$\lim_{\tau\to+\infty} \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)} = 0. \tag{3.2}$$

From the equality

$$\int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|^2\Big) d\mu(t) = \int_{A_\tau^\varepsilon} \Big(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|^2\Big) d\mu(t) + \int_{B_\tau^\varepsilon} \Big(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|^2\Big) d\mu(t),$$

then for τ sufficiently large

$$\frac{1}{\nu([-\tau,\tau]\setminus I)} \int_{[-\tau,\tau]\setminus I} \Big(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|^2\Big) d\mu(t) \le \|f\|_\infty \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)} + \varepsilon \frac{\mu(B_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}.$$

By using ($\mathbf{H_4}$), it follows that

$$\limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^2\Big)d\mu(t)\le\alpha\varepsilon,\quad\text{for any }\ \varepsilon>0,$$

consequently $ii)$ holds.

Thus, we shall show that $ii)$ implies $iii)$.
Assume that $ii)$ holds. From the following inequality

$$\begin{aligned}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^2\Big)d\mu(t) &\ge \int_{A_\tau^\varepsilon}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^2\Big)d\mu(t)\\ \frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^2\Big)d\mu(t) &\ge \varepsilon\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}\\ \frac{1}{\varepsilon\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^2\Big)d\mu(t) &\ge \frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)},\end{aligned}$$

for τ sufficiently large, equation (3.2) is obtained, that is $iii)$. □

Definition 3.6. Let $f\in SBC(\mathbb{R},L^2(P,H))$ and $\tau\in\mathbb{R}$. We denote by f_τ the function defined by $f_\tau(t)=f(t+\tau)$ for $t\in\mathbb{R}$. A subset $\mathfrak{F}$ of $SBC(\mathbb{R},L^2(P,H))$ is said to translation invariant if for all $f\in\mathfrak{F}$ we have $f_\tau\in\mathfrak{F}$ for all $\tau\in\mathbb{R}$.

Definition 3.7. Let μ_1 and $\mu_2\in\mathcal{M}$. μ_1 is said to be equivalent to μ_2 ($\mu_1\sim\mu_2$) if there exist constants α and $\beta>0$ and a bounded interval I(eventually $I=\emptyset$) such that $\alpha\mu_1(A)\le\mu_2(A)\le\beta\mu_1(A)$ for $A\in\mathcal{N}$ satisfying $A\cap I=\emptyset$.

Remark 3.8. The relation $\sim$ is an equivalence relation on $\mathcal{M}$.

Theorem 3.9. *Let $\mu_1,\mu_2,\nu_1,\nu_2\in\mathcal{M}$. If $\mu_1\sim\mu_2$ and $\nu_1\sim\nu_2$, then $\mathcal{E}(\mathbb{R},L^2(P,H),\mu_1,\nu_1,r)=\mathcal{E}(\mathbb{R},L^2(P,H),\mu_2,\nu_2,r)$.*

Proof. Since $\mu_1\sim\mu_2$ and $\nu_1\sim\nu_2$ there exist some constants $\alpha_1,\alpha_2,\beta_1,\beta_2>0$ and a bounded interval I (eventually $I=\emptyset$) such that $\alpha_1\mu_1(A)\le\mu_2(A)\le\beta_1\mu_1(A)$ and $\alpha_2\nu_1(A)\le\nu_2(A)\le\beta_2\nu_1(A)$ for each $A\in\mathcal{N}$ satisfies $A\cap I=\emptyset$ i.e

$$\frac{1}{\beta_2\nu_1(A)}\le\frac{1}{\nu_2(A)}\le\frac{1}{\alpha_2\nu_1(A)}.$$

Since $\mu_1\sim\mu_2$ and $\mathcal{N}$ is the Lebesgue σ-field, then for τ sufficiently large, it follows that

$$\frac{\alpha_1\mu_1\Big(\Big\{t\in[-\tau,\tau]\setminus I:\ \sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^2>\varepsilon\Big\}\Big)}{\beta_2\nu_1([-\tau,\tau]\setminus I)}\le$$

$$\frac{\mu_2\Big(\Big\{t\in[-\tau,\tau]\setminus I:\ \sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^2>\varepsilon\Big\}\Big)}{\nu_2([-\tau,\tau]\setminus I)}\le$$

$$\frac{\beta_1\mu_1\Big(\Big\{t\in[-\tau,\tau]\setminus I:\ \sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^2>\varepsilon\Big\}\Big)}{\alpha_2\nu_1([-\tau,\tau]\setminus I)}$$

Consequently by Theorem 3.5, $\mathcal{E}(\mathbb{R},X,\mu_1,\nu_1,r)=\mathcal{E}(\mathbb{R},X,\mu_2,\nu_2,r)$. □

Let μ, $\nu \in \mathcal{M}$ denote by

$$cl(\mu,\nu) = \{\omega_1, \omega_2 : \mu \sim \omega_1 \text{ and } \nu \sim \omega_2\}.$$

Proposition 3.10. *[4] Let $\mu \in \mathcal{M}$. Then μ satisfies ($\boldsymbol{H_2}$) if and only if the measures μ and μ_τ are equivalent for all $\tau \in \mathbb{R}$.*

Proposition 3.11. *[6] ($\boldsymbol{H_3}$) implies for all σ, $\limsup_{\tau\to\infty} \dfrac{\mu([-\tau-\sigma,\tau+\sigma])}{\mu([-\tau,\tau])} < +\infty$.*

Theorem 3.12. *Assume that ($\boldsymbol{H_2}$) holds. Then $\mathcal{E}(\mathbb{R}, L^2(P,H),\mu,\nu,r)$ is translation invariant.*

Proof. The proof of this theorem is inspired by Theorem (3.5) in [4]. Let $f \in \mathcal{E}(\mathbb{R}, L^2(P,H),\mu,\nu,r)$ and $a \in \mathbb{R}$. Since $\nu(\mathbb{R}) = +\infty$, there exists $a_0 > 0$ such that $\nu([-\tau-|a|,\tau+|a|]) > 0$ for $|a| \geq a_0$. Denote by

$$M_a(\tau) = \frac{1}{\nu_a([-\tau,\tau])}\int_{[-\tau,\tau]}\left(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^2\right)d\mu_a(t) \ \ \forall\tau > 0 \text{ and } a \in \mathbb{R},$$

where ν_a is the positive measure defined by equation (3.1). By using Proposition (3.10), it follows that ν and ν_a are equivalent, μ and μ_a are equivalent by using Theorem (3.9) we have $\mathcal{E}(\mathbb{R}, L^2(P,H),\mu_a,\nu_a,r) = \mathcal{E}(\mathbb{R}, L^2(P,H),\mu,\nu,r)$ therefore $f \in \mathcal{E}(\mathbb{R}, L^2(P,H),\mu_a,\nu_a,r)$ that is $\lim_{\tau\to+\infty} M_a(\tau) = 0$ for all $a \in \mathbb{R}$.
For all $A \in \mathcal{N}$, we denote by $\mathcal{X}_A$ the characteristic function of A. By using definition of the measure μ_a, we obtain that

$$\int_{[-\tau,\tau]}\mathcal{X}_A(t)d\mu_a(t) = \int_{[-\tau,\tau]}\mathcal{X}_A(t)d\mu(t+a) = \int_{[-\tau+a,\tau+a]}d\mu(t) \text{ for all } A \in \mathcal{N}.$$

Since $t \mapsto \sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^2$ is the pointwise limit of an increasing sequence of linear combinations of functions, see([[13]; Theorem 1.17 p.15]), we deduce that

$$\int_{[-\tau,\tau]}\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^2 d\mu_a(t) = \int_{[-\tau+a,\tau+a]}\sup_{\theta\in[t-a-r,t-a]}\mathbb{E}\|f(\theta)\|^2 d\mu(t).$$

If we denote by $a^+ := \max(a,0)$ and $a^- = \max(-a,0)$ we have $|a|+a = 2a^+$, $|a|-a = 2a^-$, and
$[-\tau+a-|a|,\tau+a+|a|] = [-\tau-2a^-,\tau+2a^+]$. Therefore we obtain

$$M_a(\tau+|a|) = \frac{1}{\nu([-\tau-2a^-,\tau+2a^+])}\int_{[-\tau-2a^-,\tau+2a^+]}\sup_{\theta\in[t-a-r,t-a]}\mathbb{E}\|f(\theta)\|^2 d\mu(t). \tag{3.3}$$

From equation (3.3) and the following inequality

$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in[t-a-r,t-a]}\mathbb{E}\|f(\theta)\|^2 d\mu(t) \leq$$
$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau-2a^-,\tau+2a^+]}\sup_{\theta\in[t-a-r,t-a]}\mathbb{E}\|f(\theta)\|^2 d\mu(t)$$

we obtain

$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in[t-a-r,t-a]}\mathbb{E}\|f(\theta)\|^2 d\mu(t) \leq \frac{\nu([-\tau-2a^-,\tau+2a^+]}{\nu([-\tau,\tau])}\times M_a(\tau+|a|).$$

This implies ,

$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in[t-a-r,t-a]}\mathbb{E}\|f(\theta)\|^2d\mu(t)\leq\frac{\nu([-\tau-2|a|,\tau+2|a|])}{\nu([-\tau,\tau])}\times M_a(\tau+|a|). \tag{3.4}$$

From equation (3.3) and equation (3.4) and using Proposition (3.11) we deduce that

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in[t-a-r,t-a]}\mathbb{E}\|f(\theta)\|^2d\mu(t)=0$$

which equivalent to

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta-a)\|^2d\mu(t)=0,$$

that is $f_{-a}\in\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,r)$. We have proved that $f\in\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,r)$ then $f_{-a}\in\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,r)$ for $a\in\mathbb{R}$. That is $\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,r)$ is translation invariant.

Proposition 3.13. *The space $SPAP(\mathbb{R},L^2(P,H),\mu,\nu,r)$ is translation invariant, that is for all $\alpha\in\mathbb{R}$ and $f\in SPAP(\mathbb{R},L^2(P,H),\mu,\nu,r)$, $f_\alpha\in SPAP(\mathbb{R},L^2(P,H),\mu,\nu,r)$.*

4. Square-Mean (μ,ν)-Pseudo Almost Periodic Process

In this section, we define square-mean (μ,ν)-pseudo almost periodic process and we study their basic properties.

Definition 4.1. Let $f:\mathbb{R}\to L^2(P,H)$ be a continuous stochastic process. f is said be square-mean almost periodic process if for all $\alpha\in\mathbb{R}$, there exists $\tau\in[\alpha,\alpha+l]$ such that

$$\sup_{t\in\mathbb{R}}\mathbb{E}\|f(t+\tau)-f(t)\|^2<\varepsilon \tag{4.1}$$

We denote the space of all such stochastic processes by $SAP(\mathbb{R},L^2(P,H))$.

Theorem 4.2. *[10] The space $SAP(\mathbb{R},L^2(P,H))$ endowed the norm $\|,\|_\infty$ is a Banach space.*

Definition 4.3. Let $\mu,\nu\in\mathcal{M}$ and $f:\mathbb{R}\to L^2(P,H)$ be a continuous stochastic process. f is said be $(\mu,\nu)-$ square-mean pseudo almost periodic process if it can be decomposed as follows

$$f=g+\phi$$

where $g\in SAP(\mathbb{R},L^2(P,H))$ and $\phi\in\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu)$.

We denote the space of such stochastic processes by $SPAP(\mathbb{R},L^2(P,H),\mu,\nu)$.

Proposition 4.4. *[7] Assume that ($\boldsymbol{H_3}$) holds. Then the decomposition of (μ,ν)-pseudo almost periodic function in the form $f=g+\phi$ where $g\in AP(\mathbb{R},X)$ and $\phi\in\mathcal{E}(\mathbb{R},X,\mu,\nu)$ is unique.*

Proposition 4.5. *[14] Let μ, $\nu\in\mathcal{M}$. Assume ($\boldsymbol{H_3}$) holds. Then the decomposition of a (μ,ν)-pseudo almost periodic function $\phi=\phi_1+\phi_2$, where $\phi_1\in AP(\mathbb{R},X)$ and $\phi_2\in\mathcal{E}(\mathbb{R},X,\mu,\nu)$ is unique.*

Remark 4.6. Let $X=L^2(P,H)$. Then the Proposition (4.4) always holds.

Definition 4.7. Let $\mu, \nu \in \mathcal{M}$ and $f : \mathbb{R} \to L^2(P, H)$ be a continuous stochastic process. f is said be $(\mu, \nu)-$ square-mean pseudo almost periodic process of class r if it can be decomposed as follows

$$f = g + \phi$$

where $g \in SAP(\mathbb{R}, L^2(P, H))$ and $\phi \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu, r)$.

We denote the space of such stochastic processes by $SPAP(\mathbb{R}, L^2(P, H), \mu, \nu, r)$.

Proposition 4.8. *$SPAP(\mathbb{R}, L^2(P, H), \mu, \nu, r)$ is a Banach space.*

Proof. This proposition is a consequence of Theorem(4.2) and Proposition(3.4). □

Proposition 4.9. *[14] Let $\mu, \nu \in \mathcal{M}$ and assume $(\boldsymbol{H_3})$ holds. Then the decomposition of (μ, ν)-pseudo almost periodic function $\phi = \phi_1 + \phi_2$, where $\phi \in AP(\mathbb{R}, L^2(P, H))$ and $\phi_2 \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu, r)$ is unique.*

Proposition 4.10. *Let μ_1, μ_2, ν_1 and ν_2 $\in \mathcal{M}$ if $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, then $SPAP(\mathbb{R}, L^2(P, H), \mu_1, \nu_1, r) = SPAP(\mathbb{R}; L^2(P, H), \mu_2, \nu_2, r)$.*

This Proposition is a consequence of Theorem(3.9).

Theorem 4.11. *Assume that $(\boldsymbol{H_3})$ holds. Let $\mu, \nu \in \mathcal{M}$ and $\phi \in SPAP(\mathbb{R}, L^2(P, H), \mu, \nu, r)$ then the function $t \to \phi_t$, belongs to $SPAP(C([-r, 0], L^2(P, H)), \mu, \nu, r)$.*

Proof. Assume that $\phi = g + h$, where $g \in SAP(\mathbb{R}, L^2(P, H))$ and $h \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu, r)$. Then we can see that, $\phi_t = g_t + h_t$ and g_t is square mean almost periodic process. Let us denote by

$$M_\alpha(\tau) = \frac{1}{\nu_\alpha([-\tau, \tau])} \int_{-\tau}^{\tau} (\sup_{\theta \in [t-r,t]} \mathbb{E}\|h(\theta)\|^2) d\mu_\alpha(t).$$

Where μ_α and ν_α are the positive measures defined by equation (3.1). By using Proposition (3.10), it follows that μ_α and μ are equivalent and ν_α and ν are also equivalent. Then by using Theorem (4.10) we have $\mathcal{E}(\mathbb{R}, L^2(P, H), \mu_\alpha, \nu_\alpha, r) = \mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu, r)$ therefore $h \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu_\alpha, \nu_\alpha, r)$ that is $\lim_{\tau \to +\infty} M_\alpha(\tau) = 0$ for all $\alpha \in \mathbb{R}$. On the other hand, for $r > 0$ we have

$$\begin{aligned}
\frac{1}{\nu([-\tau, \tau])} & \int_{-\tau}^{\tau} (\sup_{\theta \in [t-r,t]} \left(\sup_{\eta \in [-r,0]} \left(\mathbb{E}\|h(\theta + \eta)\|^2 \right) \right) d\mu(t) \leq \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-2r,t]} \left(\mathbb{E}\|h(\theta)\|^2 \right) d\mu(t) \\
\leq & \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \Big[\sup_{\theta \in [t-2r,t-r]} \left(\mathbb{E}\|h(\theta)\|^2 \right) + \sup_{\theta \in [t-r,t]} \left(\mathbb{E}\|h(\theta)\|^2 \right) \Big] d\mu(t) \\
\leq & \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [-\tau-r,\tau+r]} \left(\mathbb{E}\|h(\theta)\|^2 \right) d\mu(t + r) + \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \left(\mathbb{E}\|h(\theta)\|^2 \right) d\mu(t) \\
\leq & \frac{\nu([-\tau - r, \tau + r])}{\nu([-\tau, \tau])} \times \frac{1}{\nu([-\tau - r, \tau + r])} \int_{-\tau-r}^{\tau+r} \sup_{\theta \in [t-r,t]} \mathbb{E}\|h(\theta)\|^2 d\mu(t + r) \\
+ & \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \mathbb{E}\|h(\theta)\|^2 d\mu(t)
\end{aligned}$$

Consequently,

$$\begin{aligned}
\frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \left(\sup_{\eta \in [-r,0]} \left(\mathbb{E}\|h(\theta + \eta)\|^2 \right) \right) d\mu(t) \quad \leq & \quad \frac{\nu([-\tau - r, \tau + r])}{\nu([-\tau, \tau])} \times M_r(\tau + r) \\
+ & \quad \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \mathbb{E}\|h(\theta)\|^2 d\mu(t)
\end{aligned}$$

Using Proposition(3.11), and Proposition(3.10), it follows that,
$\phi_t \in SPAP(C[-r,0], L^2(P,H)), \mu, \nu, r)$. Thus, we obtain the desired result □

Next, we study the composition of the space square-mean (μ, ν) -pseudo almost periodic process.

Definition 4.12. [10] Let $f : \mathbb{R} \times L^2(P,H) \to L^2(P,H)$, $(t,x) \mapsto f(t,x)$ be continuous. f is said be square-mean almost periodic in $t \in \mathbb{R}$ uniformly in $x \in L^2(P,H)$ if for all compact K of $L^2(P,H)$ and for any $\varepsilon > 0$ there exists $l(\varepsilon, K)$ such that for all $\alpha \in \mathbb{R}$, there exists $\tau \in [\alpha, \alpha + l(\varepsilon, K)]$ with

$$x \in K, \ \sup_{t\in\mathbb{R}} \mathbb{E}\|f(t+\tau, x) - f(t,x)\|^2 < \varepsilon.$$

We denote the following space of stochastic processes by $SAP(\mathbb{R} \times L^2(P,H), L^2(P,H))$.

Theorem 4.13. *[10] Let $f : \mathbb{R} \times L^2(P,H) \to L^2(P,H)$, $(t,x) \mapsto f(t,x)$ be a square almost periodic process in t uniformly in $x \in L^2(P,H)$. Suppose that f is Lipschitz in the following sense: there exists a positive number L such that for any $x, y \in L^2(P,H)$,*

$$\mathbb{E}\|f(t,x) - f(t,y)\|^2 \le L \cdot \mathbb{E}\|x - y\|^2.$$

Then $t \mapsto f(t, x(t)) \in SAP(\mathbb{R}, L^2(P,H))$ for any $x \in SAP(\mathbb{R}, L^2(P,H))$.

Definition 4.14. Let $\mu, \nu \in \mathcal{M}$. A continuous functions $f(t,x) : \mathbb{R} \times L^2(P,H) \to L^2(P,H)$ is said to be square-mean (μ, ν)-pseudo almost periodic of class r in t for any $x \in L^2(P,H)$ if it can be decomposed as $f = g + \phi$, where $g \in SAP(\mathbb{R} \times L^2(P,H), L^2(P,H))$,$\phi \in \mathcal{E}(\mathbb{R} \times L^2(P,H), L^2(P,H), \mu, \nu, r)$. Denote the set of all such stochastically continuous processes by $SPAP(\mathbb{R} \times L^2(P,H), L^2(P,H), \mu, \nu, r)$.

Proposition 4.15. *Let $a_i \in \mathbb{R}$, $i \in \mathbb{N}$. Then $\left|\sum_{i=1}^{n} a_i\right|^2 \le n \sum_{i=1}^{n} |a_i|^2$.*

Theorem 4.16. *Let $\mu, \nu \in \mathcal{M}$ satisfy $(\boldsymbol{H_2})$. Suppose that $f \in SPAP(\mathbb{R} \times L^2(P,H), L^2(P,H), \mu, \nu, r)$ and that there exists a positive number L such that, for any x,$y \in L^2(P,H)$,*

$$\mathbb{E}\|f(t,x) - f(t,y)\|^2 \le L \cdot \mathbb{E}\|x - y\|^2$$

for $t \in \mathbb{R}$. Then $t \mapsto f(t, x(t)) \in SPAP(\mathbb{R}, L^2(P,H), \mu, \nu, r)$ for any $x \in SPAP(\mathbb{R}; L^2(P,H), \mu, \nu, r)$.

Proof. Since $x \in SPAP(\mathbb{R}; L^2(P,H), \mu, \nu, r)$, then we can decompose $x = x_1 + x_2$, where $x_1 \in SAP(\mathbb{R}, L^2(P,H))$ and $x_2 \in \mathcal{E}(\mathbb{R}, L^2(P,H), \mu, \nu, r)$.
Otherwise, since $f \in SPAP(\mathbb{R} \times L^2(P,H), L^2(P,H), \mu, \nu, r)$, then $f = f_1 + f_2$, where $f_1 \in SAP(\mathbb{R} \times L^2(P,H), L^2(P,H))$ and $f_2 \in \mathcal{E}(\mathbb{R} \times L^2(P,H), L^2(P,H), \mu, \nu, r)$.
The function f can be decomposed as

$$\begin{aligned} f(t,x(t)) &= f_1(t,x_1(t)) + [f(t,x(t)) - f(t,x_1(t))] + [f(t,x_1(t)) - f_1(t,x_1(t))] \\ &= f_1(t,x_1(t)) + [f(t,x(t)) - f(t,x_1(t))] + f_2(t,x_1(t)) \end{aligned}$$

Using Theorem (4.13), we have $(t \to f_1(t, x_1(t)) \in SAP(\mathbb{R} \times L^2(P,H), L^2(P,H))$.
It remains to show that the both functions $t \to [f(t,x(t)) - f(t,x_1(t))]$ and $t \to f_2(t, x_1(t))$

belong to $\mathcal{E}(\mathbb{R}\times L^2(P,H),L^2(P,H),\mu,\nu,r)$.
We have,

$$\begin{aligned}\mathbb{E}\|f(t,x(t))-f(t,x_1(t))\|^2 &\leq L.\mathbb{E}\|x(t)-x_1(t)\|^2\\ \sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta,x(\theta))-f(\theta,x_1(\theta)\|^2 &\leq L.\sup_{\theta\in[t-r,t]}\mathbb{E}\|x(\theta)-x_1(\theta)\|^2.\end{aligned}$$

It follows that

$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta,x(\theta))-f(\theta,x_1(\theta)\|^2d\mu(t)\leq$$
$$\frac{L}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in[t-r,t]}\mathbb{E}\|x(\theta)-x_1(\theta)\|^2d\mu(t)\leq$$
$$\frac{L}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in[t-r,t]}\mathbb{E}\|x_2(\theta)\|^2d\mu(t).$$

Since $x_2\in\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,r)$ then $t\to f(t,x(t))-f(t,x_1(t))$ is (μ,ν)-ergodic
Now to complete the proof, it is enough to prove $t\to f_2(t,x_1(t))$ is (μ,ν)-ergodic. Since f_2 is uniformly continuous on the compact set $K=\{x_1(t):t\in\mathbb{R}\}$ with respect to the second variable x, we deduce that for given ε, there exists $\delta>0$ such that for all $t\in\mathbb{R}$, ζ_1 and $\zeta_2\in K$, one has

$$\|\zeta_1-\zeta_2\|\leq\delta\implies\|f_2(t,\zeta_1)-f_2(t,\zeta_2)\|\leq\varepsilon.$$

Therefore, there exist $n(\varepsilon)\in\mathbb{N}$ and $\{x_i\}_{i=1}^{n(\varepsilon)}\subset K$, such that

$$K\subset\bigcup_{i=1}^{n(\varepsilon)}B(x_i,\delta),$$

then

$$\begin{aligned}||f_2(t,x_1(t))|| &\leq \varepsilon+\sum_{i=1}^{n(\varepsilon)}||f_2(t,x_i)||\\ \|f_2(t,x_1(t))\|^2 &\leq \left(\varepsilon+\sum_{i=1}^{n(\varepsilon)}\|f_2(t,x_i)\|\right)^2\\ &\leq 2\left(\varepsilon^2+\left(\sum_{i=1}^{n(\varepsilon)}\|f_2(t,x_i)\|\right)^2\right)\end{aligned}$$

By using the Proposition (4.15), we have

$$\|f_2(t,x_1(t))\|^2 \leq 2\left(\varepsilon+n(\varepsilon)\sum_{i=1}^{n(\varepsilon)}\|f_2(t,x_i)\|^2\right).$$

It follows that

$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_2(\theta,x_1(\theta))\|^2 d\mu(t)\leq$$

$$2\left(\frac{\varepsilon\mu([-\tau,\tau]}{\nu([-\tau,\tau])}+n(\varepsilon)\sum_{i=1}^{n(\varepsilon)}\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_2(\theta,x_i)\|^2 d\mu(t)\right).$$

By the fact that

$$\forall i\in\{1,...,n(\varepsilon)\},\quad \lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_2(\theta,x_i)\|^2\Big)d\mu(t)=0$$

we deduce that

$$\forall\varepsilon>0,\quad \limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_2(\theta,x_1(\theta))\|^2\Big)d\mu(t)\leq 2\alpha\varepsilon.$$

Therefore $t\to f_2(t,x_1(t))$ is ergodic and the theorem is proved. □

($\mathbf{H_5}$): g is a stochastically bounded process.

Theorem 4.17. *Assume that ($\boldsymbol{H_0}$), ($\boldsymbol{H_1}$), ($\boldsymbol{H_4}$) and ($\boldsymbol{H_5}$) hold and the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic. If f is bounded and continuous on $\mathbb{R}$, then there exists a unique bounded solution u of equation (1.1) on $\mathbb{R}$ given by*

$$\begin{aligned}u_t=&\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds\\&+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\end{aligned}$$

$\forall\ t\geq 0$, where $\widetilde{B}_\lambda=\lambda(\lambda I-\widetilde{\mathcal{A}}_{\mathcal{U}})^{-1},\Pi^s$ and Π^u are the projections of C_0 onto the stable and unstable subspaces.

Proof. Let

$$\begin{aligned}u_t\ =&\ v(t)+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\\&+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\forall\ t\geq 0,\end{aligned}$$

where

$$v(t)=\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds$$

Let us first prove that u_t exists. The existence of $v(t)$ have proved by [1]. Now, we show that the limit $\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s\widetilde{B}_\lambda(X_0 g(s))dW(s)$ exist.
For $t\in\mathbb{R}$ we have,

$$
\begin{aligned}
\mathbb{E}\left\|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\right\|^2 &\le \mathbb{E}\Big(\int_{-\infty}^{t}\overline{M}^2 e^{-2w(t-s)}|\Pi^s|^2\|\widetilde{B}_\lambda(X_0 g(s))\|^2 ds\Big)\\
&\le \overline{M}^2\mathbb{E}\Big(\int_{-\infty}^{t} e^{-2w(t-s)}|\Pi^s|^2\|\widetilde{B}_\lambda(X_0 g(s))\|^2 ds\Big)\\
&\le \overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\Big(\int_{-\infty}^{t} e^{-2w(t-s)}\|g(s)\|^2 ds\Big)\\
&\le \overline{M}^2\widetilde{M}^2|\Pi^s|^2\sum_{n=1}^{\infty}\mathbb{E}\left(\int_{t-n}^{t-n+1} e^{-2w(t-s)}\|g(s)\|^2 ds\right).
\end{aligned}
$$

Using the Hölder inequality, we obtain

$$
\begin{aligned}
&\mathbb{E}\left\|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s\ \widetilde{B}_\lambda(X_0 g(s))dW(s)\right\|^2 \le\\
&\overline{M}^2\widetilde{M}^2|\Pi^s|^2\sum_{n=1}^{+\infty}\left(\int_{t-n}^{t-n+1} e^{-4w(t-s)}ds\right)^{\frac{1}{2}}\mathbb{E}\left(\int_{t-n}^{t-n+1}\|g(s)\|^2 ds\right)^{\frac{1}{2}} \le\\
&\overline{M}^2\widetilde{M}^2|\Pi^s|^2\frac{1}{2\sqrt{w}}\sum_{n=1}^{\infty}\left(e^{-4w(n-1)}-e^{-4wn}\right)^{\frac{1}{2}}\mathbb{E}\left(\int_{t-n}^{t-n+1}\|g(s)\|^2 ds\right)^{\frac{1}{2}} \le\\
&\overline{M}^2\widetilde{M}^2|\Pi^s|^2\frac{1}{2\sqrt{w}}(e^{4wn}-1)^{\frac{1}{2}}\sum_{n=1}^{\infty}e^{-2wn}\times\mathbb{E}\left(\int_{t-n}^{t-n+1}\|g(s)\|^2 ds\right)^{\frac{1}{2}}.
\end{aligned}
$$

Since the serie $\sum_{n=1}^{\infty}e^{-2wn}$ is convergent, then it exists a constant $c>0$ such that $\sum_{n=1}^{\infty}e^{-2wn}\le c$, moreover it follows that

$$
\begin{aligned}
&\mathbb{E}\left\|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\right\|^2 \le\\
&\overline{M}\widetilde{M}^2|\Pi^s|^2\frac{1}{2\sqrt{w}}(e^{4w}-1)^{\frac{1}{2}}\mathbb{E}\|g(s)\|\sum_{n=1}^{\infty}e^{-2wn}\\
\le\ & \gamma\sum_{n=1}^{\infty}e^{-2wn}\\
\le\ & \gamma c,
\end{aligned}
$$

where, $\gamma=\overline{M}^2\widetilde{M}^2|\Pi^s|^2\frac{1}{2\sqrt{w}}(e^{4w}-1)^{\frac{1}{2}}\mathbb{E}\|g(s)\|$.

Let $F(n,s,t)=\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))$ for $n\in\mathbb{N}$ for $s\le t$.

For n is sufficiently large and $\sigma\le t$, we have

$$
\begin{aligned}
&\mathbb{E}\left\|\int_{-\infty}^{\sigma}F(n,s,t)dW(s)\right\|^2 \le\\
&\overline{M}^2\widetilde{M}^2|\Pi^s|^2\sum_{n=1}^{+\infty}\left(\int_{\sigma-n}^{\sigma-n+1} e^{-4w(t-s)}ds\right)^{\frac{1}{2}}\times\mathbb{E}\left(\int_{\sigma-n}^{\sigma-n+1}\|g(s)\|^2 ds\right)^{\frac{1}{2}} \le
\end{aligned}
$$

$$\overline{M}^2\widetilde{M}^2|\Pi^s|^2\frac{1}{2\sqrt{\omega}}\Big(\sum_{n=1}^{\infty}\big(e^{-4\omega(t-\sigma+n-1)}-e^{-4\omega(t-\sigma+n)}\big)^{\frac{1}{2}}\times\mathbb{E}\Big(\int_{\sigma-n}^{\sigma-n+1}\|g(s)\|^2ds)^{\frac{1}{2}}\Big)\Big)\leq$$

$$\overline{M}^2\widetilde{M}^2|\Pi^s|^2\frac{1}{2\sqrt{\omega}}e^{-2\omega(t-\sigma)}(e^{4\omega}-1)^{\frac{1}{2}}\sum_{n=1}^{\infty}e^{-2\omega n}\mathbb{E}\left(\int_{\sigma-n}^{\sigma-n+1}\|g(s)\|^2ds)^{\frac{1}{2}}\right)\leq\gamma ce^{-2w(t-\sigma)}$$

It follows that for n and m sufficiently large and $\sigma\leq t$, we have

$$\begin{aligned}&\mathbb{E}\left|\left|\int_{-\infty}^{t}F(n,s,t)dW(s)-\int_{\infty}^{t}F(m,s,t)dW(s)\right|\right|^2\leq\\&\mathbb{E}\Big|\Big|\int_{-\infty}^{\sigma}F(n,s,t)dW(s)+\int_{\sigma}^{t}F(n,s,t)dW(s)-\int_{-\infty}^{\sigma}F(m,s,t)dW(s)\\&-\int_{\sigma}^{t}F(m,s,t)dW(s)\Big|\Big|^2\leq\\&3\mathbb{E}\Big|\Big|\int_{-\infty}^{\sigma}F(n,s,t)dW(s)\Big|\Big|^2+3\mathbb{E}\Big|\Big|\int_{-\infty}^{\sigma}F(m,s,t)dW(s)\Big|\Big|^2\\&+3\mathbb{E}\Big|\Big|\int_{\sigma}^{t}F(n,s,t)dW(s)-\int_{\sigma}^{t}F(m,s,t)dW(s)\Big|\Big|^2\leq\\&6\gamma ce^{-2\omega(t-\sigma)}+3\mathbb{E}\Big|\Big|\int_{\sigma}^{t}F(n,s,t)dW(s)-\int_{\sigma}^{t}F(m,s,t)dW(s)\Big|\Big|^2\end{aligned}$$

Since $\lim\limits_{n\to+\infty}\mathbb{E}\Big|\Big|\int_{\sigma}^{t}F(n,s,t)dW(s)\Big|\Big|^2$ exists, then

$$\limsup_{n,m\to+\infty}\mathbb{E}\Big|\Big|\int_{-\infty}^{t}F(n,s,t)dW(s)-\int_{-\infty}^{t}F(m,s,t)dW(s)\Big|\Big|^2\leq 6\gamma ce^{-2\omega(t-\sigma)}$$

If $\sigma\to-\infty$, then

$$\limsup_{n,m\to+\infty}\mathbb{E}\Big|\Big|\int_{-\infty}^{t}F(n,s,t)dW(s)-\int_{-\infty}^{t}F(m,s,t)dW(s)\Big|\Big|^2=0.$$

We deduce that the limit

$$\lim_{n\to+\infty}\mathbb{E}\Big|\Big|\int_{-\infty}^{t}F(n,s,t)dW(s)\Big|\Big|^2=\lim_{n\to+\infty}\mathbb{E}\Big|\Big|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_nX_0g(s))dW(s)\Big|\Big|^2$$

exists. Therefore, $\lim\limits_{n\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_nX_0g(s))dW(s)$ exists. In addition, one can show that the function

$$t\to\lim_{n\to+\infty}\mathbb{E}\Big|\Big|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_nX_0g(s)ds\Big|\Big|^2$$

is bounded on $\mathbb{R}$. Similary, we can show that the function

$$t\to\lim_{n\to+\infty}\int_{t}^{+\infty}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_nX_0g(s))dW(s)$$

is well defined and bounded on $\mathbb{R}$. □

Proposition 4.18. *[14] A function $\phi\in C(\mathbb{R},X)$ is almost periodic if and only if the space of functions $\{\phi_\tau:\tau\in\mathbb{R}\}$, where $\phi_\tau(t)=\phi(t+\tau)$, is relatively compact in $BC(\mathbb{R};X)$*

Remark 4.19. As $L^2(P,H)$ is a space Banach then the Proposition(4.18) holds.

Theorem 4.20. *Assume that* $(\boldsymbol{H_5})$. *Let* $f, g \in SAP(\mathbb{R}; L^2(P,H))$ *and* Γ *be the mapping defined for* $t \in \mathbb{R}$ *by*

$$\begin{aligned}\Gamma(f,g)(t) = &\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds\\ &+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\\ &+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\end{aligned}$$

Then $\Gamma(f,g) \in SAP(\mathbb{R}; L^2(P,H))$.

Proof. $\Gamma(f,g)_\tau(t) = \Gamma(f,g)(t+\tau)$

$$\begin{aligned}
&= \lim_{\lambda\to+\infty}\int_{-\infty}^{t+\tau}\mathcal{U}^s(t+\tau-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t+\tau}\mathcal{U}^u(t+\tau-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds\\
&+ \lim_{\lambda\to+\infty}\int_{-\infty}^{t+\tau}\mathcal{U}^s(t+\tau-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s) + \lim_{\lambda\to+\infty}\int_{+\infty}^{t+\tau}\mathcal{U}^u(t+\tau-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\\
&= \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s+\tau))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s+\tau))ds\\
&+ \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s+\tau))dW(s) + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s+\tau))dW(s)\\
&= \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f_\tau(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f_\tau(s))ds\\
&+ \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g_\tau(s))dW(s) + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g_\tau(s))dW(s)\\
&= \Gamma(f_\tau, g_\tau)(t) \text{ for all } t \in \mathbb{R}.
\end{aligned}$$

Thus $\Gamma(f,g)_\tau = \Gamma(f_\tau, g_\tau)$ which implies $\{\Gamma(f,g)_\delta, \delta \in \mathbb{R}\}$ is relatively compact in $SBC(\mathbb{R}, L^2(P,H))$. Since Γ is continuous from $SBC(\mathbb{R}, L^2(P,H))$ into $SBC(\mathbb{R}, L^2(P,H))$ then $\Gamma(f,g) \in SAP(\mathbb{R}, L^2(P,H))$. □

Theorem 4.21. *Assume that* $(\boldsymbol{H_3})$ *and* $(\boldsymbol{H_5})$ *holds. Let* $f, g \in \mathcal{E}(\mathbb{R}, L^2(P,H), \mu, \nu, r)$ *then* $\Gamma(f,g) \in \mathcal{E}(\mathbb{R}, L^2(P,H), \mu, \nu, r)$.

Proof. We have,

$$\begin{aligned}\Gamma(f,g)(t) \;=\; &\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds\\ &+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds\\ &+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\\ &+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\end{aligned}$$

$$\begin{aligned}\mathbb{E}\Big|\Big|\Gamma(f,g)(\theta)\Big|\Big|^2 &= \mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{\theta}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds\\ &+\lim_{\lambda\to+\infty}\int_{+\infty}^{\theta}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds\\ &+\lim_{\lambda\to+\infty}\int_{-\infty}^{\theta}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\\ &+\lim_{\lambda\to+\infty}\int_{+\infty}^{\theta}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\Big|\Big|^2.\end{aligned}$$

$$\begin{aligned}&\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\mathbb{E}\|\Gamma(f,g)(\theta)\|^2d\mu(t)\le\\ &\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}4\mathbb{E}\Big(\widetilde{M}^2\overline{M}^2\int_{-\infty}^{\theta}e^{-2\omega(t-s)}|\Pi^s|^2\|f(s)\|^2ds\\ +\;&\widetilde{M}^2\overline{M}^2\int_{\theta}^{+\infty}e^{2\omega(t-s)}|\Pi^u|^2\|f(s)\|^2ds+\widetilde{M}^2\overline{M}^2\int_{-\infty}^{\theta}e^{-2\omega(t-s)}|\Pi^s|^2\|g(s)\|^2ds\\ +\;&\widetilde{M}^2\overline{M}^2\int_{\theta}^{+\infty}e^{2\omega(t-s)}|\Pi^u|^2\|g(s)\|^2ds\Big)d\mu(t)\\ \le\;&4\widetilde{M}^2\overline{M}^2\Big[\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{-\infty}^{\theta}e^{-2\omega(t-s)}|\Pi^s|^2\mathbb{E}\|f(s)\|^2ds\Big)d\mu(t)\\ +\;&\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{+\infty}e^{2\omega(t-s)}|\Pi^u|^2\mathbb{E}\|f(s)\|^2ds\Big)d\mu(t)\\ +\;&\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{-\infty}^{\theta}e^{-2\omega(t-s)}|\Pi^s|^2\mathbb{E}\|g(s)\|^2ds+\int_{\theta}^{+\infty}e^{2\omega(t-s)}|\Pi^u|^2\mathbb{E}\|g(s)\|^2ds\Big)d\mu(t)\Big]\\ \le\;&4\widetilde{M}^2\overline{M}^2\Big[|\Pi^s|^2\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{-\infty}^{\theta}e^{-2\omega(t-s)}(\mathbb{E}\|f(s)\|^2+\mathbb{E}\|g(s)\|^2)ds\Big)d\mu(t)\\ +\;&|\Pi^u|^2\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{+\infty}e^{2\omega(t-s)}(\mathbb{E}\|f(s)\|^2+\mathbb{E}\|g(s)\|^2)ds\Big)d\mu(t)\Big]\end{aligned}$$

one the one hand using Fubini's theorem, we have

$$\begin{aligned}&|\Pi^s|^2\int_{-\tau}^{\tau}\Big[\sup_{\theta\in[t-r,t]}\int_{-\infty}^{\theta}e^{-2\omega(t-s)}(\mathbb{E}\|f(s)\|^2+\mathbb{E}\|g(s)\|^2)ds\Big]d\mu(t)\\ \le\;&e^{\omega r}|\Pi^s|^2\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{-\infty}^{\theta}e^{-2\omega(t-s)}(\mathbb{E}\|f(s)\|^2+\mathbb{E}\|g(s)\|^2)ds\Big)d\mu(t)\\ \le\;&e^{\omega r}|\Pi^s|^2\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{-\infty}^{t}e^{-2\omega(t-s)}(\mathbb{E}\|f(s)\|^2+\mathbb{E}\|g(s)\|^2)ds\Big)d\mu(t)\\ \le\;&e^{\omega r}|\Pi^s|^2\int_{-\tau}^{\tau}\Big(\int_{-\infty}^{t}e^{-2\omega(t-s)}(\mathbb{E}\|f(s)\|^2+\mathbb{E}\|g(s)\|^2)ds\Big)d\mu(t)\\ \le\;&e^{\omega r}|\Pi^s|^2\int_{-\tau}^{\tau}\Big(\int_{0}^{+\infty}e^{-2\omega s}(\mathbb{E}\|f(t-s)\|^2+\mathbb{E}\|g(t-s)\|^2)ds\Big)d\mu(t)\\ \le\;&e^{\omega r}|\Pi^s|^2\int_{0}^{+\infty}e^{-2\omega s}\int_{-\tau}^{\tau}\Big(\mathbb{E}\|f(t-s)\|^2+\mathbb{E}\|g(t-s)\|^2\Big)d\mu(t)ds\end{aligned}$$

By using Proposition(3.13) we deduce that

$$\lim_{\tau\to+\infty}\frac{e^{-2\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\mathbb{E}\|f(t-s)\|^2+\mathbb{E}\|g(t-s)\|^2\Big)d\mu(t)\to 0$$

for all $s\in\mathbb{R}^+$ and

$$\frac{e^{-2\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\mathbb{E}\|f(t-s)\|^2+\mathbb{E}\|g(t-s)\|^2\Big)d\mu(t)\le\frac{e^{-2\omega s}\mu([-\tau,\tau])}{\nu([-\tau,\tau])}\Big(\|f\|_\infty^2+\|g\|_\infty^2\Big)$$

Since f and g are bounded functions, then the function $s\mapsto\dfrac{e^{-2\omega s}\mu([-\tau,\tau])}{\nu([-\tau,\tau])}\Big(\|f\|_\infty^2+\|g\|_\infty^2\Big)$ belongs to $L^1([0,+\infty[)$ in view of the Lebesgue dominated convergence Theorem, it follows that

$$e^{\omega r}\lim_{\tau\to+\infty}\int_0^{+\infty}\frac{e^{-2\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\mathbb{E}\|f(t-s)\|^2+\mathbb{E}\|g(t-s)\|^2\Big)d\mu(t)ds\to 0.$$

On the other hand by Fubini's theorem, we also have

$$\begin{aligned}
&|\Pi^u|^2\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{+\infty}e^{2\omega(t-s)}(\mathbb{E}\|f(s)\|^2+\mathbb{E}\|g(s)\|^2)ds\Big)d\mu(t)\\
\le\quad&|\Pi^u|^2\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{t-r}^{+\infty}e^{2\omega(t-s)}(\mathbb{E}\|f(s)\|^2+\mathbb{E}\|g(s)\|^2)ds\Big)d\mu(t)\\
\le\quad&|\Pi^u|^2\int_{-\tau}^{\tau}\Big(\int_{t-r}^{+\infty}e^{2\omega(t-s)}(\mathbb{E}\|f(s)\|^2+\mathbb{E}\|g(s)\|^2)ds\Big)d\mu(t)\\
\le\quad&|\Pi^u|^2\int_{-\tau}^{\tau}\Big(\int_{-\infty}^{r}e^{2\omega s}(\mathbb{E}\|f(s)\|^2+\mathbb{E}\|g(s)\|^2)ds\Big)d\mu(t)\\
\le\quad&|\Pi^u|^2\int_{-\infty}^{r}\Big(\int_{-\tau}^{\tau}e^{2\omega s}(\mathbb{E}\|f(s)\|^2+\mathbb{E}\|g(s)\|^2)d\mu(t)\Big)ds
\end{aligned}$$

Since the function $s\mapsto\dfrac{e^{2\omega s}}{\nu([-\tau,\tau])}\Big(\|f\|_\infty^2+\|g\|_\infty^2\Big)$ belongs to $L^1(]-\infty,r])$ resoning like above, it follows that

$$\lim_{\tau\to+\infty}\int_{-\infty}^{r}e^{\omega s}\times\frac{1}{\nu([-\tau,\tau])}\Big(\int_{-\tau}^{\tau}e^{2\omega s}(\mathbb{E}\|f(s)\|^2+\mathbb{E}\|g(s)\|^2)d\mu(t)\Big)ds=0$$

Consequently

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\mathbb{E}\|\Gamma(f,g)(\theta)\|^2d\mu(t)=0$$

Thus, we obtain the desired result. □

Theorem 4.22. *Assume ($\boldsymbol{H_0}$) , ($\boldsymbol{H_1}$) , ($\boldsymbol{H_3}$) and ($\boldsymbol{H_5}$) hold. Then equation (1.1) has a unique square mean $cl(\mu,\nu)$-pseudo almost periodic solution of class r.*

Proof. Since f and g are square mean (μ,ν)-pseudo almost periodic function, f, g has a decomposition $f=f_1+f_2$ and $g=g_1+g_2$ where f_1, $g_1\in SAP(\mathbb{R};L^2(P,H))$ and f_2, $g_2\in\mathcal{E}(\mathbb{R};L^2(P,H),\mu,\nu,r)$. Using Theorem(4.20), Theorem(4.2) and Theorem(4.17), we get the desired result. □

Our next objective is to show the existence of square mean(μ,ν)-pseudo almost periodic solutions of class r for the following problem

$$du(t)=[Au(t)+L(u_t)+f(t,u_t)]dt+g(t,u_t)dW(t)\text{ for }t\in\mathbb{R}\tag{4.2}$$

where $f : \mathbb{R}\times C \to L^2(P,H)$ and $g : \mathbb{R}\times C \to L^2(P,H)$ are two stochastic continuous processes. For the sequel, we formulate the following assumptions

- ($\mathbf{H_6}$) Let μ, $\nu \in \mathcal{M}$ and $f : \mathbb{R}\times C([-r,0], L^2(P,H)) \to L^2(P,H)$ square mean $cl(\mu,\nu)$-pseudo almost periodic of class r such that there exists a constant L_f such that $\mathbb{E}\Big|\Big|f(t,\phi_1) - f(t,\phi_2)\Big|\Big|^2 \leq L_f \times \mathbb{E}\|\phi_1-\phi_2\|^2$ for all $t\in\mathbb{R}$ and ϕ_1, $\phi_2 \in C([-r,0], L^2(P,H))$.
- ($\mathbf{H_7}$) Let μ, $\nu \in \mathcal{M}$ and $g : \mathbb{R}\times C([-r,0], L^2(P,H)) \to L^2(P,H)$ square mean $cl(\mu,\nu)$-pseudo almost periodic of class r such that there exists a constant L_g such that $\mathbb{E}\Big|\Big|g(t,\phi_1)-g(t,\phi_2)\Big|\Big|^2 \leq L_g\times\mathbb{E}\|\phi_1-\phi_2\|^2$ for all $t\in\mathbb{R}$ and ϕ_1, $\phi_2 \in C([-r,0], L^2(P,H))$.

Theorem 4.23. *Assume ($\boldsymbol{H_0}$), ($\boldsymbol{H_1}$), ($\boldsymbol{H_2}$), ($\boldsymbol{H_4}$) ,($\boldsymbol{H_6}$) and ($\boldsymbol{H_7}$) hold. If*

$$\widetilde{M}^2\overline{M}^2\sup_{t\in\mathbb{R}}\left(|\Pi^s|^2\int_{-\infty}^{t} e^{-2\omega(t-s)}(L_f^2+L_g^2)ds + |\Pi^u|^2\int_{t}^{+\infty} e^{2\omega(t-s)}(L_f^2+L_g^2)ds\right) < \frac{1}{4},$$

then equation (4.2) has a unique square mean $cl(\mu,\nu)$-pseudo almost periodic solution of class r.

Proof. Let x be a function in $SPAP(\mathbb{R}, L^2(P,H),\mu,\nu,r)$ from Theorem(4.11) the function $t\to x_t$ belongs to $SPAP(C([-r,0]; L^2(P,H),\mu,\nu,r)$. Hence Theorem(4.16) implies that the function $g(.) := f(.,x_.)$ is in $SPAP(\mathbb{R}; L^2(P,H),\mu,\nu,r)$. Consider the following mapping: $\mathcal{H} : SPAP(\mathbb{R}; L^2(P,H),\mu,\nu,r) \to SPAP(\mathbb{R}; L^2(P,H),\mu,\nu,r)$ defined for $t\in\mathbb{R}$ by

$$\begin{aligned}(\mathcal{H}x)(t) &= \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s\widetilde{B}_\lambda(X_0 f(s,x_s))ds\\ &+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u\widetilde{B}_\lambda(X_0 f(s,x_s))ds\\ &+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s\widetilde{B}_\lambda(X_0 g(s,x_s))dW(s)\\ &+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u\widetilde{B}_\lambda(X_0 g(s,x_s))dW(s)\end{aligned}$$

From Theorem(4.20), Theorem(4.22) and Theorem(4.17), it suffices now to show that the operator $\mathcal{H}$ has a unique fixed point in $SPAP(\mathbb{R}; L^2(P,H),\mu,\nu,r)$. Let $x_1, x_2 \in SPAP(\mathbb{R}; L^2(P,H),\mu,\nu,r)$. Then we have

$$\begin{aligned}\mathbb{E}\|\mathcal{H}x_1(t)-\mathcal{H}x_2(t)\|^2 &\leq 4\mathbb{E}\left(\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\Big|\Big|\mathcal{U}^s(t-s)\Pi^s\widetilde{B}_\lambda(X_0(f(s,x_{1s})-f(s,x_{2s}))ds\Big|\Big|^2\right)\\ &+4\mathbb{E}\left(\lim_{\lambda\to+\infty}\int_{t}^{+\infty}\Big|\Big|\mathcal{U}^u(t-s)\Pi^u\widetilde{B}_\lambda(X_0(f(s,x_{2s})-f(s,x_{1s}))ds\Big|\Big|^2\right)\\ &+4\mathbb{E}\left(\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\Big|\Big|\mathcal{U}^s(t-s)\Pi^s\widetilde{B}_\lambda(X_0(g(s,x_{1s})-g(s,x_{2s}))ds\Big|\Big|^2\right)\\ &+4\mathbb{E}\left(\lim_{\lambda\to+\infty}\int_{t}^{+\infty}\Big|\Big|\mathcal{U}^u(t-s)\Pi^u\widetilde{B}_\lambda(X_0(g(s,x_{2s})-g(s,x_{1s}))ds\Big|\Big|^2\right)\end{aligned}$$

$$\begin{aligned}
&\leq 4\widetilde{M}^2\overline{M}^2\mathbb{E}\Big(|\Pi^s|^2\int_{-\infty}^{t} e^{-2\omega(t-s)}(L_f^2+L_g^2)\|x_{1s}-x_{2s}\|^2ds \\
&\quad +|\Pi^u|^2\int_{t}^{+\infty} e^{2\omega(t-s)}(L_f^2+L_g^2)\|x_{1s}-x_{2s}\|^2ds\Big) \\
&\leq 4\widetilde{M}^2\overline{M}^2\mathbb{E}(\|x_1-x_2\|^2)\sup_{t\in\mathbb{R}}\Big(|\Pi^s|^2\int_{-\infty}^{t} e^{-2\omega(t-s)}(L_f^2+L_g^2)ds \\
&\quad +|\Pi^u|^2\int_{t}^{+\infty} e^{2\omega(t-s)}(L_f^2+L_g^2)ds\Big).
\end{aligned}$$

This means that $\mathcal{H}$ is a strict contraction. Thus by Banach? fixed point theorem, $\mathcal{H}$ has a unique fixed point u in $SPAP(\mathbb{R};L^2(P,H),\mu,\nu,r)$. We conclude that equation (4.2), has one and only one square mean $cl(\mu,\nu)$-pseudo almost periodic solution of class r. □

Proposition 4.24. *Assume ($\boldsymbol{H_0}$), ($\boldsymbol{H_1}$), ($\boldsymbol{H_2}$), ($\boldsymbol{H_4}$) and f, g are lipschitz continuous with respect the second argument. If*

$$Lip(f)=Lip(g)<\Big(\frac{\omega}{4\overline{M}^2\widetilde{M}^2(|\Pi^s|^2+|\Pi^u|^2)}\Big)^{\frac{1}{2}}$$

then equation (5.1) has a unique $cl(\mu,\nu)$-pseudo almost periodic solution of class r, where $Lip(f)$, $Lip(g)$ are respectively the Lipschitz constant of f and g.

Proof. Let us pose $k=Lip(f)=Lip(g)$, we have

$$\begin{aligned}
\mathbb{E}\|\mathcal{H}x_1(t)-\mathcal{H}x_2(t)\|^2 &\leq 4\widetilde{M}^2\overline{M}^2\mathbb{E}(\|x_1-x_2\|^2)\sup_{t\in\mathbb{R}}\Big(|\Pi^s|^2\int_{-\infty}^{t} e^{-2\omega(t-s)}(L_f^2+L_g^2)ds \\
&\quad +|\Pi^u|^2\int_{t}^{+\infty} e^{2\omega(t-s)}(L_f^2+L_g^2)ds\Big) \\
&\leq 4\widetilde{M}^2\overline{M}^2\mathbb{E}(\|x_1-x_2\|^2)\sup_{t\in\mathbb{R}}\Big(|\Pi^s|^2\int_{-\infty}^{t} 2k^2e^{-2\omega(t-s)}ds+|\Pi^u|^2\int_{t}^{+\infty} 2k^2e^{2\omega(t-s)}ds\Big) \\
&\leq \frac{4k^2\widetilde{M}^2\overline{M}^2(|\Pi^s|^2+|\Pi^u|^2)}{\omega}\mathbb{E}(\|x_1-x_2\|^2).
\end{aligned}$$

Consequently $\mathcal{H}$ is a strict contraction if $k^2<\dfrac{\omega}{4\widetilde{M}^2\overline{M}^2(|\Pi^s|^2+|\Pi^s|^2)}$. □

5. Application

For illustration, we propose to study the existence of solutions for the following model

$$(5.1)\quad \begin{cases}
dz(t,x)=\dfrac{\partial^2}{\partial x^2}z(t,x)dt+\Big[\displaystyle\int_{-r}^{0}G(\theta)z(t+\theta,x)d\theta+\sin(t)+\sin(\sqrt{2}t)+\arctan(t) \\
\quad +\displaystyle\int_{-r}^{0}h(\theta,z(t+\theta,x)d\theta\Big]dt+\Big[\frac{\cos(t)}{2+\cos(\sqrt{2}t)}+\arctan(t) \\
\quad +\displaystyle\int_{-r}^{\theta}h(\theta,z(t+\theta,x))d\theta\Big]dW(t) \\
z(t,0)=z(t,\pi)=0 \text{ for } t\in\mathbb{R}
\end{cases}$$

Where $G:[-r,0]\to\mathbb{R}$ is a continuous function and $h:[-r,0]\times\mathbb{R}\to\mathbb{R}$ is continuous, Lipschitzian with respect to the second argument. $W(t)$ is a two-sided and standard one-dimensional Brownian notion defined on the filtered probability space $(\Omega,\mathcal{F},P,\mathcal{F}_t)$ with

$\mathcal{F}_t = \sigma\{W(u) - W(v) \mid u, v \le t\}$. To rewrite equation (5.1) in the abstract form , we introduce the space $H = L^2((0,\pi))$. Let $A : D(A) \to L^2((0,\pi))$ defined by

$$\begin{cases} D(A) = \mathbf{H}^1((0,\pi)) \cap \mathbf{H}^1_0((0,1)) \\ Ay(t) = y''(t) \ \text{ for } t \in (0,\pi) \text{ and } y \in D(A) \end{cases}$$

Then A generates a C_0-semigroup $(\mathcal{U}(t))_{t\ge 0}$ on $L^2((0,\pi))$ given by

$$(\mathcal{U}(t)x)(r) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} < x, e_n >_{L^2} e_n(r)$$

Where $e_n(r) = \sqrt{2}\sin(n\pi r)$ for $n = 1, 2, ...,$ and $\|\mathcal{U}(t)\| \le e^{-\pi^2 t}$ for all $t \ge 0$. Thus $\overline{M} = 1$ and $\omega = \pi^2$. Then A satisfied the Hille-Yosida condition in $L^2((0,\pi))$. Moreover the part A_0 of A in $\overline{D(A)}$. It follows that $(\mathbf{H_0})$ and $(\mathbf{H_1})$ are satisfied.

We define $f : \mathbb{R} \times C \to L^2((0,\pi))$ and $L : C \to L^2((0,\pi))$ as follows

$$\begin{aligned} f(t,\phi)(x) &= (\sin(t) + \sin(\sqrt{2}t)) + \arctan(t) + \int_{-r}^{\theta} h(\theta, \phi(\theta)(x))d\theta \\ g(t,\phi)(x) &= \frac{\cos(t)}{2+\cos(\sqrt{2}t)} + \arctan(t) + \int_{-r}^{\theta} h(\theta, \phi(\theta)(x))d\theta \\ L(\phi)(x) &= \int_{-r}^{\theta} G(\theta, \phi(\theta)(x))d\theta \text{ for } \ -r \le \theta \le 0 \text{ and } x \in (0,\pi) \end{aligned}$$

let us pose $v(t) = z(t,x)$. Then equation(5.1) takes the following abstract form

$$dv(t) = [Av(t) + L(v_t) + f(t, v_t)]dt + g(t, v_t)dW(t) \text{ for } t \in \mathbb{R}$$

Consider the measures μ and ν where its Radon-Nikodyn derivative are respectively ρ_1, ρ_2 : $\mathbb{R} \to \mathbb{R}$ defined by

$$\rho_1(t) = \begin{cases} 1 \text{ for } t > 0 \\ e^t \text{ for } t \le 0 \end{cases}$$

and

$$\rho_2(t) = |t| \ for \ t \in \mathbb{R}$$

i.e $d\mu(t) = \rho_1(t)dt$ and $d\nu(t) = \rho_2(t)dt$ where dt denotes the Lebesgue measure on $\mathbb{R}$ and

$$\mu(A) = \int_A \rho_1(t)dt \text{ for } \nu(A) = \int_A \rho_2(t)dt \text{ for } A \in \mathcal{B}.$$

From [6] $\mu, \nu \in \mathcal{M}$, μ, ν satisfy $(\mathbf{H_4})$ and $\sin(t) + \sin(\sqrt{2}t) + \dfrac{\pi}{2}$ is almost periodic.

We have

$$\limsup_{\tau\to+\infty} \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \limsup_{\tau\to+\infty} \frac{\displaystyle\int_{-\tau}^{0} e^t dt + \int_0^{\tau} dt}{\displaystyle 2\int_{-\tau}^{0} t dt} = \limsup_{\tau\to+\infty} \frac{1 - e^{-\tau} + \tau}{\tau^2} = 0 < \infty,$$

which implies that $(\mathbf{H_2})$ is satisfied.

For all $t \in \mathbb{R}$, $\dfrac{\pi}{2} \le \arctan t \le \dfrac{\pi}{2}$ therefore, for all $\theta \in [t-r, t]$, $\arctan(t-r) \le \arctan(\theta)$. It follows $\left|\arctan\theta - \dfrac{\pi}{2}\right| = \dfrac{\pi}{2} - \arctan\theta \le \left|\arctan(t-r) - \dfrac{\pi}{2}\right| = \dfrac{\pi}{2} - \arctan(t-r)$, implies that

$\Big|\arctan\theta-\frac{\pi}{2}\Big|^2\leq\Big|\arctan(t-r)-\frac{\pi}{2}\Big|^2$ hence $\sup_{\theta\in[t-r,t]}\mathbb{E}\Big|\arctan\theta-\frac{\pi}{2}\Big|^2\leq\mathbb{E}\Big|\arctan(t-r)-\frac{\pi}{2}\Big|^2$.
One the one hand, we have the following:

$$\begin{aligned}\frac{1}{\nu([-\tau,\tau])}\int_0^\tau\mathbb{E}\Big|\arctan(t-r)-\frac{\pi}{2}\Big|^2dt &= \frac{1}{\nu([-\tau,\tau])}\int_0^\tau\mathbb{E}\Big(\frac{\pi}{2}-\arctan(t-r)\Big)^2dt\\ &\leq \frac{1}{\nu([-\tau,\tau])}\int_0^\tau\frac{\pi^2}{4}dt\\ &\leq \frac{\pi^2}{4\tau}\to 0 \text{ as}\to+\infty\end{aligned}$$

On the other hand we have

$$\begin{aligned}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^0\mathbb{E}\Big|\arctan(t-r)-\frac{\pi}{2}\Big|^2e^tdt &\leq \frac{1}{\nu([-\tau,\tau])}\int_0^\tau\frac{\pi^2}{4}e^tdt\\ &\leq \frac{\pi^2(1-e^{-\tau})}{4\tau}\to 0 \text{ as}\to+\infty\end{aligned}$$

Consequently

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in[t-r,t]}\mathbb{E}\Big|\arctan\theta-\frac{\pi}{2}\Big|^2d\mu(t)=0$$

It follows that $t\mapsto\arctan t-\frac{\pi}{2}$ is square mean (μ,ν)-ergodic of class r, consequently, f is uniformly square mean (μ,ν)-pseudo almost periodic of class r. Moreover, L is bounded linear operator from C to $L^2(P,L^2((0,\pi))$.
Let k be the lipschiz constant of h, then for every ϕ_1, $\phi_2\in C$ and $t\geq 0$, we have

$$\begin{aligned}\mathbb{E}\|f(t,\phi_1)(x)-f(t,\phi_2)(x)\|^2 &= \mathbb{E}\Big|\Big|\int_{-r}^0\Big[h(\theta,\phi_1(\theta)(x))-h(\theta,\phi_2(\theta)(x))\Big]\Big|\Big|^2d\theta\\ &\leq \int_{-r}^0\mathbb{E}\Big|\Big|h(\theta,\phi_1(\theta)(x))-h(\theta,\phi_2(\theta)(x))\Big|\Big|^2d\theta\\ &\leq \int_{-r}^0k\mathbb{E}\Big|\Big|\phi_1(\theta)(x)-\phi_2(\theta)(x)\Big|\Big|^2d\theta\end{aligned}$$

$$\begin{aligned}\mathbb{E}\|f(t,\phi_1)(x)-f(t,\phi_2)(x)\|^2 &\leq kr\sup_{-r\leq\theta\leq 0}\mathbb{E}\Big|\Big|\phi_1(\theta)(x)-\phi_2(\theta)(x)\Big|\Big|^2\\ &\leq kr\alpha\mathbb{E}\Big|\Big|\phi_1(\theta)(x)-\phi_2(\theta)(x)\Big|\Big|^2\text{for a certain }\alpha\in\mathbb{R}_+\end{aligned}$$

Consequently, we conclude that f and g are Lipschitz continuous and $cl(\mu,\nu)$-pseudo almost periodic of class r.
Moreover, since h is stochastically bounded, i.e $\mathbb{E}||h(t,\phi(t)(x))||\leq\beta$, $t\in\mathbb{R}$, we have

$$\begin{aligned}\mathbb{E}||g(t,\phi)(x)||^2 &\leq \frac{4+\pi}{2}+\int_{-r}^0\mathbb{E}\Big|\Big|h(\theta,\phi(\theta)(x))\Big|\Big|^2d\theta\\ &\leq \frac{4+\pi}{2}+r.\beta\\ &\leq \beta_1\ \text{ with }\beta_1=\frac{4+\pi}{2}+r.\beta.\end{aligned}$$

Which implies that g satisfies $(\mathbf{H_5})$
For the hyberbolicity, we suppose that
$(\mathbf{H_8}) \int_{-r}^{0} |G(\theta)| d\theta < 1.$

Proposition 5.1. *[11] Assume that $(\boldsymbol{H_6})$ and $(\boldsymbol{H_7})$ holds. Then the semigroup $(U(t))_{t\geq 0}$ is hyperbolic.*

Then by Proposition (4.24) we deduce the following result.

Theorem 5.2. *Under the above assumptions, if Lip(h) is small enough, then equation (5.1) has a unique $cl(\mu,\nu)$-pseudo almost periodic solution ν of class r.*

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