SQUARE-MEAN PSEUDO ALMOST PERIODIC SOLUTIONS OF CLASS \( r \)
UNDER THE LIGHT OF MEASURE THEORY

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Abstract. The aim of this work is to present new concept of square-mean pseudo almost periodic of class \( r \) using the measure theory. We use the \((\mu, \nu)\)-ergodic process to define the spaces of \((\mu, \nu)\)-pseudo almost periodic processes of class \( r \) in the square-mean sense. We present many interesting results on those spaces like completeness and composition theorems and we study the existence and the uniqueness of the square-mean \((\mu, \nu)\)-pseudo almost periodic solutions of class \( r \) for the stochastic evolution equation.

1. Introduction

In this work, we study some properties of the square-mean \((\mu, \nu)\)-pseudo almost periodic process using the measure theory and we used those results to study the following stochastic evolution equations in a Hilbert space \( H \),

\[
dx(t) = [Ax(t) + L(x_t) + f(t)]dt + g(t)dW(t),
\]

where \( A : D(A) \subset H \) is the infinitesimal generator of a \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) on \( H \), \( f, g : \mathbb{R} \rightarrow L^2(P, H) \) are two stochastic processes and \( W(t) \) is a two-sided and standard one-dimensional Brownian motion defined on the filtered probability space \((\Omega, \mathcal{F}, P, \mathcal{F}_t)\) with \( \mathcal{F}_t = \sigma\{W(u) - W(v) \mid u, v \leq t\} \) and \( L \) is a bounded linear operator from \( C \) into \( L^2(P, H) \).

\( C = C([-r, 0]; L^2(P, H)) \) denotes the space of continuous functions from \([-r, 0]\) to \( L^2(P, H) \) endowed with the uniform topology norm. For every \( t \geq 0 \), \( x_t \) denotes the history function of \( x \) defined by \( x_t(\theta) = x(t + \theta) \) for \(-r \leq \theta \leq 0\).

We assume \((H, \| \cdot \|, \| \cdot \|)\) is a real separable Hilbert space and \( L^2(P, H) \) is the space of all \( H \)-valued random variables \( x \) such that

\[
\mathbb{E}\|x\|^2 = \int_{\Omega} \|x\|^2 dP < +\infty.
\]

This work is an extension of [14] whose authors have studied equation (1.1) in the deterministic case. Some recent contributions concerning square-mean pseudo almost periodic solutions for abstract differential equations similar to equation (1.1) have been made. For example in [10] the authors studied equation (1.1) without the operator \( L \). They showed that the equation has a unique square-mean \( \mu \)-pseudo almost periodic mild solution on \( \mathbb{R} \) when \( f \) and \( g \) are square
mean pseudo almost periodic functions.
In [5] the authors studied the square-mean almost periodic solutions to a class of nonautonomous stochastic differential equations without our operator \( L \) and without delay on a separable real Hilbert space. They established the existence and uniqueness of a square-mean almost periodic mild solution to those nonautonomous stochastic differential equations with the 'Acquistapace-Terreni' conditions.

In [9] The authors established the existence, uniqueness and stability of square-mean \( \mu \)-pseudo almost periodic (resp. automorphic) mild solution to a linear and semilinear case of the stochastic evolution equations in case when the functions forcing are both continuous and \( S^2 - \mu \)-pseudo almost periodic (resp. automorphic) and verify some suitable assumptions.

This work is organized as follow, in section 2, we give spectral decomposition of phase space in section 3 we study square-mean \((\mu, \nu)\)-ergodic process of class \( r \), in section 4 we study square-mean \((\mu, \nu)\)-pseudo almost process functions and properties and last section is devoted to an application.

### 2. Spectral decomposition

To equation (1.1), associate the following initial value problem

\[
\begin{align*}
\frac{du}{dt} &= [Au_t + L(u_t) + f(t)]dt + g(t)dW(t) \text{ for } t \geq 0 \\
\quad u_0 &= \varphi \in C = C([-r, 0], L^2(P, H)),
\end{align*}
\]

where \( f : \mathbb{R}^+ \rightarrow L^2(P, H) \) and \( g : \mathbb{R}^+ \rightarrow L^2(P, H) \) are stochastic processes continuous.

**Definition 2.1.** We say that a continuous function \( u \) from \([-r, +\infty[\) into \( L^2(P, H) \) is an integral solution of equation, if the following conditions hold:

1. \( \int_0^t u(s)ds \in D(A) \) for \( t \geq 0 \),
2. \( u(t) = \phi(0) + A\int_0^t u(s)ds + \int_0^t (L(u_s) + f(s))ds + \int_0^t g(s)dW(s), \) for \( t \geq 0 \),
3. \( u_0 = \varphi. \)

If \( \overline{D(A)} = L^2(P, H) \), the integral solution coincide with the known mild solutions. One can see that if \( u(t) \) is an integral of equation 2.1, then \( u(t) \in \overline{D(A)} \) for all \( t \geq 0 \), in particular \( \phi(0) \in \overline{D(A)} \)

Let us introduce the part \( A_0 \) of the operator \( A \) in \( \overline{D(A)} \) which defined by

\[
\begin{align*}
D(A_0) &= \{ x \in D(A) : Ax \in \overline{D(A)} \} \\
A_0x &= Ax \text{ for } x \in D(A_0)
\end{align*}
\]

The following assumption is supposed:

(H\(_0\)) \( A \) satisfies the Hille-Yosida condition.

**Proposition 2.2.** \([2]\) \( A_0 \) generates a strongly continuous semigroup \( (T_0(t))_{t \geq 0} \) on \( \overline{D(A)} \).

The phase space \( C_0 \) of equation (2.1) is defined by

\[
C_0 = \{ \varphi \in C : \varphi(0) \in \overline{D(A)} \}.
\]
For each $t \geq 0$, the linear operator $U(t)$ on $C_0$ is defined by

$$U(t) = v_t(.,\varphi)$$

where $v(.,\varphi)$ is the solution of the following homogeneous equation

$$\begin{cases}
\frac{d}{dt}v_t = Av_t + L(v_t) & \text{for } t \geq 0 \\
v_0 = \varphi \in C.
\end{cases}$$

**Proposition 2.3.** $[3]$ $(U(t))_{t \geq 0}$ is a strongly continuous semigroup of linear operators on $C_0$. Moreover, $(U(t))_{t \geq 0}$ satisfies, for $t \geq 0$ and $\theta \in [-r, 0]$, the following translation property

$$(U(t)\varphi)(\theta) = \begin{cases} (U(t+\theta)\varphi)(0) & \text{for } t+\theta \geq 0 \\ \varphi(t+\theta) & \text{for } t+\theta \leq 0. \end{cases}$$

**Theorem 2.4.** $[3]$ Let $A_U$ defined on $C_0$ by

$$\begin{cases}
D(A_U) = \{ \varphi \in C^1([-r, 0]; X) : \varphi(0) \in \overline{D(A)} \text{ and } \varphi'(0) = A \varphi(0) + L(\varphi) \} \\
A_U \varphi = \varphi' \text{ for } \varphi \in D(A_U).
\end{cases}$$

Then $A_U$ is the infinitesimal generator of the semigroup $(U(t))_{t \geq 0}$ on $C_0$.

Let $\langle X_0 \rangle$ be the space defined by

$$\langle X_0 \rangle = \{ X_0x : x \in X \}$$

where the function $X_0x$ is defined by

$$(X_0x)(\theta) = \begin{cases} 0 & \text{if } \theta \in [-r, 0], \\
x & \text{if } \theta = 0. \end{cases}$$

The space $C_0 \oplus \langle X_0 \rangle$ equipped with the norm $|\varphi + X_0c|_c = |\varphi|_c + |c|$ for $(\varphi, c) \in C_0 \times X$ is a Banach space and consider the extension $A_U$ defined on $C_0 \oplus \langle X_0 \rangle$ by

$$\begin{cases}
D(\tilde{A}_U) = \{ \varphi \in C^1([-r, 0]; X) : \varphi \in D(A) \text{ and } \varphi' \in \overline{D(A)} \} \\
\tilde{A}_U \varphi = \varphi' + X_0(A \varphi + L(\varphi) - \varphi').
\end{cases}$$

**Proposition 2.5.** $[3]$ Assume that $(H_0)$ holds. Then, $\tilde{A}_U$ satisfies the Hille-Yosida condition on $C_0 \oplus \langle X_0 \rangle$ there exist $\tilde{M} \geq 0, \tilde{\omega} \in \mathbb{R}$ such that $|\tilde{\omega}, +\infty| \subset \rho(\tilde{A}_U)$ and

$$|(\lambda I - \tilde{A}_U)^{-n}| \leq \frac{\tilde{M}}{(\lambda - \tilde{\omega})^n} \text{ for } n \in \mathbb{N} \text{ and } \lambda > \tilde{\omega}$$

Moreover, the part of $\tilde{A}_U$ on $D(\tilde{A}_U) = C_0$ is exactly the operator $\tilde{A}_U$.

**Definition 2.6.** The semigroup $(U(t))_{t \geq 0}$ is hyperbolic if

$$\sigma(\tilde{A}_U) \cap i\mathbb{R} = \emptyset$$
For the sequel, we make the following assumption:

\((H_1)\) \(T_0(t)\) is compact on \(D(A)\) for every \(t > 0\).

**Proposition 2.7.** Assume that \((H_0)\) and \((H_1)\). then the semigroup \((U(t))_{t \geq 0}\) is compact for \(t > r\).

**Proposition 2.8.** Assume that \((H_1)\) holds. If the semigroup \((U(t))_{t \geq 0}\) is hyperbolic then the space \(C_0\) is decomposed as a direct sum

\[ C_0 = S \oplus U \]

of two \(U(t)\) invariant closed subspaces \(S\) and \(U\) such that the restricted semigroup on \(U\) is a group and there exist positive constant \(\bar{M}\) and \(\omega\) such that

\[ |U(t)\varphi| \leq \frac{\bar{M}e^{-\omega t}}{\omega} |\varphi| \] for \(t \geq 0\) and \(\varphi \in S\)

\[ |U(t)\varphi| \leq \frac{\bar{M}e^{\omega t}}{\omega} |\varphi| \] for \(t \leq 0\) and \(\varphi \in U\).

Where \(S\) and \(U\) are called respectively the stable and unstable space, \(\Pi^s\) and \(\Pi^u\) denote respectively the projection operator on \(S\) and \(U\).

### 3. Square-Mean \((\mu, \nu)\)-Ergodic process of class \(r\)

Let \(\mathcal{N}\) the Lebesgue \(\sigma\)-field of \(\mathbb{R}\) and by \(\mathcal{M}\) the set of all positive measures \(\mu\) on \(\mathcal{N}\) satisfying \(\mu(\mathbb{R}) = +\infty\) and \(\mu([a, b]) < \infty\), for all \(a, b \in \mathbb{R}\) (\(a \leq b\)). \(L^2(P, H)\) is a Hilbert space with following norm

\[ \|x\|_{L^2} = \left( \int_\Omega \|x\|^2 dP \right)^{\frac{1}{2}} \]

**Definition 3.1.** Let \(x : \mathbb{R} \rightarrow L^2(P, H)\) be a stochastic process.

1. \(x\) said to be stochastically bounded if there exists \(C > 0\) such that

\[ \mathbb{E}\|x(t)\|^2 \leq C \forall t \in \mathbb{R}. \]

2. \(x\) is said to be stochastically continuous if

\[ \lim_{t \rightarrow s} \mathbb{E}\|x(t) - x(s)\|^2 = 0 \forall s \in \mathbb{R}. \]

Denote by \(SBC(\mathbb{R}, L^2(P, H))\), the space of all stochastically bounded and continuous process. Otherwise, this space endowed the following norm

\[ \|x\| = \sup_{t \in \mathbb{R}} \left( \mathbb{E}\|x(t)\|^2 \right)^{\frac{1}{2}} \]

is a Banach space.

**Definition 3.2.** Let \(\mu, \nu \in \mathcal{M}\). A stochastic process \(f\) is said to be square-mean \((\mu, \nu)\)-ergodic if \(f \in SBC(\mathbb{R}, L^2(P, H))\) and satisfied

\[ \lim_{\tau \rightarrow +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \mathbb{E}\|f(t)\|^2 d\mu(t) = 0. \]

We denote by \(\mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu)\), the space of all such process.
Definition 3.3. Let $\mu, \nu \in \mathcal{M}$. A stochastic process $f$ is said to be square-mean $(\mu, \nu)$-ergodic of class $r$ if $f \in SBC(\mathbb{R}, L^2(P, H))$ and satisfied

$$\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-t, t]} \mathbb{E}\|f(\theta)\|^2 d\mu(t) = 0.$$  

We denote by $\mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu, r)$, the space of all such process.

For $\mu, \nu \in \mathcal{M}$ and $a \in \mathbb{R}$, we denote by $\mu_a$ the positive measure on $(\mathbb{R}, \mathcal{N})$ defined by

$$(3.1) \quad \mu_a(A) = \mu(a + b : b \in A) \quad \text{for } A \in \mathcal{N}.$$  

From $\mu, \nu \in \mathcal{M}$, we formulate the following hypothesis

$$(H_2): \text{For all } a \in \mathbb{R}, \text{ there exists } \beta > 0 \text{ and a bounded intervall } I \text{ such that } \mu_a(A) \leq \beta \mu(A) \quad \text{when } A \in \mathcal{N} \text{ satisfies } A \cap I = \emptyset.$$  

$$(H_3) \text{ For all } a, b \text{ and } c \in \mathbb{R}, \text{ such that } 0 \leq a < b \leq c, \text{ there exist } \delta_0 \text{ and } \alpha_0 > 0 \text{ such that}$$

$$|\delta| \geq \delta_0 \implies \mu(a + \delta, b + \delta) \geq \alpha_0 \mu(\delta, c + \delta).$$  

$$(H_4) \text{ Let } \mu, \nu \in \mathcal{M} \text{ be such that } \limsup_{\tau \to +\infty} \frac{\mu([-\tau, \tau])}{\nu([-\tau, \tau])} = \alpha < \infty.$$  

Proposition 3.4. Assume that $(H_4)$ holds. Then $\mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu, r)$ is a Banach space with the norm $\| \cdot \|_\infty$.

Proof. We can see that $\mathcal{E}(\mathbb{R}; L^2(P, H), \mu, \nu, r)$ is a vector subspace of $SBC(\mathbb{R}, L^2(P, H))$. To complete the proof, it is enough to prove that $\mathcal{E}(\mathbb{R}; L^2(P, H), \mu, \nu, r)$ is closed in $SBC(\mathbb{R}; L^2(P, H))$. Let $(f_n)_n$ be a sequence in $\mathcal{E}(\mathbb{R}; L^2(P, H), \mu, \nu, r)$ such that $\lim_{n \to +\infty} f_n = f$ uniformly in $SBC(\mathbb{R}, L^2(P, H))$. From $\nu(\mathbb{R}) = +\infty$, it follows $\nu([-\tau, \tau]) > 0$ for $\tau$ sufficiently large. Let $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\|f_n - f\|_\infty < \varepsilon$. Let $n \geq n_0$, then

$$\frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \left( \sup_{\theta \in [t-t, t]} \mathbb{E}\|f(\theta)\|^2 \right) d\mu(t) \leq \frac{2}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \left( \sup_{\theta \in [t-t, t]} \mathbb{E}\|f_n(\theta) - f(\theta)\|^2 \right) d\mu(t)$$

$$+ \frac{2}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \left( \sup_{\theta \in [t-t, t]} \mathbb{E}\|f_n(\theta)\|^2 \right) d\mu(t)$$

$$\leq \frac{2}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \left( \sup_{\theta \in [t-t, t]} \mathbb{E}\|f_n(t) - f(t)\|^2 \right) d\mu(t)$$

$$+ \frac{2}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \left( \sup_{\theta \in [t-t, t]} \mathbb{E}\|f_n(\theta)\|^2 \right) d\mu(t)$$

$$\leq 2 \|f_n - f\|_\infty^2 \frac{\mu([-\tau, \tau])}{\nu([-\tau, \tau])}$$

$$+ \frac{2}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \left( \sup_{\theta \in [t-t, t]} \mathbb{E}\|f_n(\theta)\| \right) d\mu(t).$$

Consequently

$$\limsup_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \left( \sup_{\theta \in [t-t, t]} \mathbb{E}\|f(\theta)\|^2 \right) d\mu(t) \leq 2\alpha\varepsilon \quad \text{for any } \varepsilon > 0.$$  

\[\square\]

The following theorem is a characterization of square-mean $(\mu, \nu)$-ergodic processes eventually $I = \emptyset$.
Theorem 3.5. Assume that \((H_4)\) holds and let \(f \in SBC(\mathbb{R}, L^2(P, H))\). Then the following assertions are equivalent:

i) \(\mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu, \tau)\)

\[
\lim_{\tau \to +\infty} \frac{1}{\nu([\tau, \tau] \setminus I)} \int_{[\tau, \tau] \setminus I} \frac{\sup_{\theta \in [\tau, \tau]} \mathbb{E}\|f(\theta)\|^2}{\mu([\tau, \tau] \setminus I)} d\mu(t) = 0
\]

ii) \(\lim_{\tau \to +\infty} \frac{1}{\nu([\tau, \tau] \setminus I)} \int_{[\tau, \tau] \setminus I} \sup_{\theta \in [\tau, \tau]} \mathbb{E}\|f(\theta)\|^2 d\mu(t) = 0\)

iii) For any \(\varepsilon > 0\), \(\lim_{\tau \to +\infty} \frac{1}{\nu([\tau, \tau] \setminus I)} \int_{[\tau, \tau] \setminus I} \sup_{\theta \in [\tau, \tau]} \mathbb{E}\|f(\theta)\|^2 > \varepsilon\)

Proof. The proof is made like the proof of Theorem(2.13) in [6]. First, we show that i) is equivalent to ii).

Denote by \(A = \nu(I), B = \int_I \left( \sup_{\theta \in [\tau, \tau]} \mathbb{E}\|f(\theta)\|^2 \right) d\mu(t)\). A and B belong to \(\mathbb{R}\), since the interval \(I\) is bounded and the process \(f\) is stochastically bounded and continuous. For \(\tau > 0\) such that \(I \subseteq [\tau, \tau] \setminus I\) and \(\nu([\tau, \tau] \setminus I) > 0\), it follows

\[
\frac{1}{\nu([\tau, \tau] \setminus I)} \int_{[\tau, \tau] \setminus I} \left( \sup_{\theta \in [\tau, \tau]} \mathbb{E}\|f(\theta)\|^2 \right) d\mu(t) = \frac{1}{\nu([\tau, \tau])} - A \left[ \int_{[\tau, \tau]} \sup_{\theta \in [\tau, \tau]} \mathbb{E}\|f(\theta)\|^2 d\mu(t) - B \right]
\]

\[
= \frac{1}{\nu([\tau, \tau])} - A \left[ \frac{1}{\nu([\tau, \tau])} \int_{[\tau, \tau]} \sup_{\theta \in [\tau, \tau]} \mathbb{E}\|f(\theta)\|^2 d\mu(t) - \frac{B}{\nu([\tau, \tau])} \right].
\]

From above equalities and the fact that \(\nu(\mathbb{R}) = +\infty\), ii) is equivalent to

\[
\lim_{\tau \to +\infty} \frac{1}{\nu([\tau, \tau] \setminus I)} \int_{[\tau, \tau] \setminus I} \sup_{\theta \in [\tau, \tau]} \mathbb{E}\|f(\theta)\|^2 d\mu(t) = 0,
\]

that is i).

Now, we show that iii) implies ii).

Denote by \(A_\tau^\varepsilon\) and \(B_\tau^\varepsilon\) the following sets

\[
A_\tau^\varepsilon = \left\{ t \in [\tau, \tau] \setminus I : \sup_{\theta \in [\tau, \tau]} \mathbb{E}\|f(\theta)\|^2 > \varepsilon \right\}
\]

\[
B_\tau^\varepsilon = \left\{ t \in [\tau, \tau] \setminus I : \sup_{\theta \in [\tau, \tau]} \mathbb{E}\|f(\theta)\|^2 \leq \varepsilon \right\}.
\]

Assume that iii) holds, that is

\[
\lim_{\tau \to +\infty} \frac{\mu(A_\tau^\varepsilon)}{\nu([\tau, \tau] \setminus I)} = 0.
\]

From the equality

\[
\int_{[\tau, \tau] \setminus I} \left( \sup_{\theta \in [\tau, \tau]} \mathbb{E}\|f(\theta)\|^2 \right) d\mu(t) = \int_{A_\tau^\varepsilon} \left( \sup_{\theta \in [\tau, \tau]} \mathbb{E}\|f(\theta)\|^2 \right) d\mu(t)
\]

\[
+ \int_{B_\tau^\varepsilon} \left( \sup_{\theta \in [\tau, \tau]} \mathbb{E}\|f(\theta)\|^2 \right) d\mu(t),
\]

then for \(\tau\) sufficiently large

\[
\frac{1}{\nu([\tau, \tau] \setminus I)} \int_{[\tau, \tau] \setminus I} \left( \sup_{\theta \in [\tau, \tau]} \mathbb{E}\|f(\theta)\|^2 \right) d\mu(t) \leq \|f\|_\infty \frac{\mu(A_\tau^\varepsilon)}{\nu([\tau, \tau] \setminus I)} + \varepsilon \frac{\mu(B_\tau^\varepsilon)}{\nu([\tau, \tau] \setminus I)}.
\]
By using \((H_4)\), it follows that
\[
\limsup_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \left( \sup_{\theta \in [t-\tau, t]} \mathbb{E}\|f(\theta)\|^2 \right) d\mu(t) \leq \alpha \varepsilon, \text{ for any } \varepsilon > 0,
\]
consequently \(ii)\) holds.

Thus, we shall show that \(ii)\) implies \(iii)\).

Assume that \(ii)\) holds. From the following inequality
\[
\frac{1}{\nu([-\tau, \tau] \setminus I)} \int_{[-\tau, \tau] \setminus I} \left( \sup_{\theta \in [t-\tau, t]} \mathbb{E}\|f(\theta)\|^2 \right) d\mu(t) \geq \int_{\mathcal{A}_\tau^E} \left( \sup_{\theta \in [t-\tau, t]} \mathbb{E}\|f(\theta)\|^2 \right) d\mu(t)
\]
for \(\tau\) sufficiently large, equation (3.2) is obtained, that is \(iii)\). \(\square\)

**Definition 3.6.** Let \(f \in SBC(\mathbb{R}, L^2(P, H))\) and \(\tau \in \mathbb{R}\). We denote by \(f_{\tau}\) the function defined by \(f_{\tau}(t) = f(t + \tau)\) for \(t \in \mathbb{R}\). A subset \(\mathfrak{F}\) of \(SBC(\mathbb{R}, L^2(P, H))\) is said to translation invariant if for all \(f \in \mathfrak{F}\) we have \(f_{\tau} \in \mathfrak{F}\) for all \(\tau \in \mathbb{R}\).

**Definition 3.7.** Let \(\mu_1\) and \(\mu_2 \in \mathcal{M}\). \(\mu_1\) is said to be equivalent to \(\mu_2\) \((\mu_1 \sim \mu_2)\) if there exist constants \(\alpha\) and \(\beta > 0\) and a bounded interval \(I\) (eventually \(I = \emptyset\)) such that \(\alpha \mu_1(A) \leq \mu_2(A) \leq \beta \mu_1(A)\) for \(A \in \mathcal{N}\) satisfying \(A \cap I = \emptyset\).

**Remark 3.8.** The relation \(\sim\) is an equivalence relation on \(\mathcal{M}\).

**Theorem 3.9.** Let \(\mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{M}\). If \(\mu_1 \sim \mu_2\) and \(\nu_1 \sim \nu_2\), then \(\mathcal{E}(\mathbb{R}, L^2(P, H), \mu_1, \nu_1, r) = \mathcal{E}(\mathbb{R}, L^2(P, H), \mu_2, \nu_2, r)\).

**Proof.** Since \(\mu_1 \sim \mu_2\) and \(\nu_1 \sim \nu_2\) there exist some constants \(\alpha_1, \alpha_2, \beta_1, \beta_2 > 0\) and a bounded interval \(I\) (eventually \(I = \emptyset\)) such that \(\alpha_1 \mu_1(A) \leq \mu_2(A) \leq \beta_1 \mu_1(A)\) and \(\alpha_2 \nu_1(A) \leq \nu_2(A) \leq \beta_2 \nu_1(A)\) for each \(A \in \mathcal{N}\) satisfies \(A \cap I = \emptyset\) i.e
\[
\frac{1}{\beta_2 \nu_1(A)} \leq \frac{1}{\nu_2(A)} \leq \frac{1}{\alpha_2 \nu_1(A)}.
\]
Since \(\mu_1 \sim \mu_2\) and \(\mathcal{N}\) is the Lebesgue \(\sigma\)-field, then for \(\tau\) sufficiently large, it follows that
\[
\frac{\alpha_1 \mu_1(\left\{ t \in [-\tau, \tau] \setminus I : \sup_{\theta \in [t-\tau, t]} \mathbb{E}\|f(\theta)\|^2 > \varepsilon \right\})}{\beta_2 \nu_1([-\tau, \tau] \setminus I)} \leq \frac{\mu_2(\left\{ t \in [-\tau, \tau] \setminus I : \sup_{\theta \in [t-\tau, t]} \mathbb{E}\|f(\theta)\|^2 > \varepsilon \right\})}{\nu_2([-\tau, \tau] \setminus I)} \leq \frac{\beta_1 \mu_1(\left\{ t \in [-\tau, \tau] \setminus I : \sup_{\theta \in [t-\tau, t]} \mathbb{E}\|f(\theta)\|^2 > \varepsilon \right\})}{\alpha_2 \nu_1([-\tau, \tau] \setminus I)}
\]
Consequently by Theorem 3.5, \(\mathcal{E}(\mathbb{R}, X, \mu_1, \nu_1, r) = \mathcal{E}(\mathbb{R}, X, \mu_2, \nu_2, r)\). \(\square\)
Let \( \mu, \nu \in \mathcal{M} \) denote by
\[
cl(\mu, \nu) = \{ \omega_1, \omega_2 : \mu \sim \omega_1 \text{ and } \nu \sim \omega_2 \}.
\]

**Proposition 3.10.** \([4]\) Let \( \mu \in \mathcal{M} \). Then \( \mu \) satisfies \((H_2)\) if and only if the measures \( \mu \) and \( \mu_\tau \) are equivalent for all \( \tau \in \mathbb{R} \).

**Proposition 3.11.** \([6]\) \((H_3)\) implies for all \( \sigma \),
\[
\limsup_{\tau \to -\infty} \frac{\mu([-\tau - \sigma, \tau + \sigma])}{\mu([-\tau, \tau])} < +\infty.
\]

**Theorem 3.12.** Assume that \((H_2)\) holds. Then \( \mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu, r) \) is translation invariant.

**Proof.** The proof of this theorem is inspired by Theorem (3.5) in [4]. Let \( f \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu, r) \) and \( a \in \mathbb{R} \). Since \( \nu(\mathbb{R}) = +\infty \), there exists \( a_0 > 0 \) such that \( \nu([-\tau - |a|, \tau + |a|]) > 0 \) for \( |a| \geq a_0 \). Denote by
\[
M_a(\tau) = \frac{1}{\nu([-\tau, \tau], t)} \int_{[-\tau, \tau]} \left( \sup_{\theta \in [t-r, t]} \mathbb{E}\|f(\theta)\|^2 \right) d\mu_a(t) \quad \forall \tau > 0 \text{ and } a \in \mathbb{R},
\]
where \( \nu_a \) is the positive measure defined by equation (3.1). By using Proposition (3.10), it follows that \( \nu \) and \( \nu_a \) are equivalent, \( \mu \) and \( \mu_a \) are equivalent by using Theorem (3.9) we have \( \mathcal{E}(\mathbb{R}, L^2(P, H), \mu_a, \nu_a, r) = \mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu, r) \) therefore \( f \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu_a, \nu_a, r) \) that is
\[
\lim_{\tau \to +\infty} M_a(\tau) = 0 \text{ for all } a \in \mathbb{R}.
\]

For all \( A \in \mathcal{N} \), we denote by \( \chi_A \) the characteristic function of \( A \). By using definition of the measure \( \mu_a \), we obtain that
\[
\int_{[\tau, \tau]} \chi_A(t)d\mu_a(t) = \int_{[-\tau, \tau]} \chi_A(t)d\mu(t + a) = \int_{[-\tau + a, \tau + a]} d\mu(t) \text{ for all } A \in \mathcal{N}.
\]

Since \( t \mapsto \sup_{\theta \in [t-r, t]} \mathbb{E}\|f(\theta)\|^2 \) is the pointwise limit of an increasing sequence of linear combinations of functions, see([13]; Theorem 1.17 p.15)), we deduce that
\[
\int_{[-\tau, \tau]} \sup_{\theta \in [t-r, t]} \mathbb{E}\|f(\theta)\|^2 d\mu_a(t) = \int_{[-\tau + a, \tau + a]} \sup_{\theta \in [t-a-r, t-a]} \mathbb{E}\|f(\theta)\|^2 d\mu(t).
\]

If we denote by \( a^+ := \max(a, 0) \) and \( a^- := \max(-a, 0) \) we have \( |a| + a = 2a^+, \ |a| - a = 2a^- \), and \( [-\tau + a, \tau + a, \tau + a] = [-\tau - 2a^-, \tau + 2a^+] \). Therefore we obtain
\[
M_a(\tau + |a|) = \frac{1}{\nu([-\tau - 2a^-, \tau + 2a^+], t)} \int_{[-\tau - 2a^-, \tau + 2a^+]} \sup_{\theta \in [t-a-r, t-a]} \mathbb{E}\|f(\theta)\|^2 d\mu(t).
\]

From equation (3.3) and the following inequality
\[
\frac{1}{\nu([-\tau, \tau], t)} \int_{[-\tau, \tau]} \sup_{\theta \in [t-a-r, t-a]} \mathbb{E}\|f(\theta)\|^2 d\mu(t) \leq \frac{1}{\nu([-\tau, \tau], t)} \int_{[-\tau, \tau]} \sup_{\theta \in [t-a-r, t-a]} \mathbb{E}\|f(\theta)\|^2 d\mu(t)
\]
we obtain
\[
\frac{1}{\nu([-\tau, \tau], t)} \int_{[-\tau, \tau]} \sup_{\theta \in [t-a-r, t-a]} \mathbb{E}\|f(\theta)\|^2 d\mu(t) \leq \frac{\nu([-\tau - 2a^-, \tau + 2a^+])}{\nu([-\tau, \tau])} \times M_a(\tau + |a|).
\]
This implies,

\[\frac{1}{\nu([-\tau, \tau])} \int_{[-\tau, \tau]} \sup_{\theta \in [t-\tau, t+\tau]} \mathbb{E}\|f(\theta)\|^2 d\mu(t) \leq \frac{\nu([-\tau - 2|a|, \tau + 2|a|])}{\nu([-\tau, \tau])} \times M_a(\tau + |a|).\]

From equation (3.3) and equation (3.4) and using Proposition (3.11) we deduce that

\[\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{[-\tau, \tau]} \sup_{\theta \in [t-\tau, t+\tau]} \mathbb{E}\|f(\theta)\|^2 d\mu(t) = 0\]

which equivalent to

\[\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{[-\tau, \tau]} \sup_{\theta \in [t-\tau, t]} \mathbb{E}\|f(\theta - a)\|^2 d\mu(t) = 0,\]

that is \(f_{-a} \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu, r)\). We have proved that \(f \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu, r)\) then \(f_{-a} \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu, r)\) for \(a \in \mathbb{R}\). That is \(\mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu, r)\) is translation invariant.

**Proposition 3.13.** The space \(SPAP(\mathbb{R}, L^2(P, H), \mu, \nu, r)\) is translation invariant, that is for all \(\alpha \in \mathbb{R}\) and \(f \in SPAP(\mathbb{R}, L^2(P, H), \mu, \nu, r)\), \(f_{\alpha} \in SPAP(\mathbb{R}, L^2(P, H), \mu, \nu, r)\).

4. **Square-Mean \((\mu, \nu)\)-Pseudo Almost Periodic Process**

In this section, we define square-mean \((\mu, \nu)\)-pseudo almost periodic process and we study their basic properties.

**Definition 4.1.** Let \(f : \mathbb{R} \to L^2(P, H)\) be a continuous stochastic process. \(f\) is said be square-mean almost periodic process if for all \(\alpha \in \mathbb{R}\), there exists \(\tau \in [\alpha, \alpha + l]\) such that

\[(4.1) \sup_{t \in \mathbb{R}} \mathbb{E}\|f(t + \tau) - f(t)\|^2 < \varepsilon\]

We denote the space of all such stochastic processes by \(SAP(\mathbb{R}, L^2(P, H))\).

**Theorem 4.2.** [10] The space \(SAP(\mathbb{R}, L^2(P, H))\) endowed the norm \(\|, \|_\infty\) is a Banach space.

**Definition 4.3.** Let \(\mu, \nu \in \mathcal{M}\) and \(f : \mathbb{R} \to L^2(P, H)\) be a continuous stochastic process. \(f\) is said be \((\mu, \nu)\)- square-mean pseudo almost periodic process if it can be decomposed as follows

\[f = g + \phi\]

where \(g \in SAP(\mathbb{R}, L^2(P, H))\) and \(\phi \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu)\).

We denote the space of such stochastic processes by \(SPAP(\mathbb{R}, L^2(P, H), \mu, \nu)\).

**Proposition 4.4.** [7] Assume that \((H_3)\) holds. Then the decomposition of \((\mu, \nu)\)-pseudo almost periodic function in the form \(f = g + \phi\) where \(g \in AP(\mathbb{R}, X)\) and \(\phi \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)\) is unique.

**Proposition 4.5.** [14] Let \(\mu, \nu \in \mathcal{M}\). Assume \((H_3)\) holds. Then the decomposition of a \((\mu, \nu)\)-pseudo almost periodic function \(\phi = \phi_1 + \phi_2\), where \(\phi_1 \in AP(\mathbb{R}, X)\) and \(\phi_2 \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)\) is unique.

**Remark 4.6.** Let \(X = L^2(P, H)\). Then the Proposition (4.4) always holds.
\textbf{Definition 4.7.} Let $\mu, \nu \in \mathcal{M}$ and $f : \mathbb{R} \to L^2(P, H)$ be a continuous stochastic process. $f$ is said to be $(\mu, \nu)$–square-mean pseudo almost periodic process of class $r$ if it can be decomposed as follows
\[
f = g + \phi
\]
where $g \in SAP(\mathbb{R}, L^2(P, H))$ and $\phi \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu, r)$.

We denote the space of such stochastic processes by $SPAP(\mathbb{R}, L^2(P, H), \mu, \nu, r)$.

\textbf{Proposition 4.8.} \(SPAP(\mathbb{R}, L^2(P, H), \mu, \nu, r)\) is a Banach space.

\textbf{Proof.} This proposition is a consequence of Theorem(4.2) and Proposition(3.4). \hfill \Box

\textbf{Proposition 4.9.} \cite{14} Let $\mu, \nu \in \mathcal{M}$ and assume $(H_9)$ holds. Then the decomposition of $(\mu, \nu)$-pseudo almost periodic function $\phi = \phi_1 + \phi_2$, where $\phi \in AP(\mathbb{R}, L^2(P, H))$ and $\phi_2 \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu, r)$ is unique.

\textbf{Proposition 4.10.} Let $\mu_1, \mu_2, \nu_1$ and $\nu_2 \in \mathcal{M}$ if $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, then $SPAP(\mathbb{R}, L^2(P, H), \mu_1, \nu_1, r) = SPAP(\mathbb{R}; L^2(P, H), \mu_2, \nu_2, r)$.

This Proposition is a consequence of Theorem(3.9).

\textbf{Theorem 4.11.} Assume that $(H_9)$ holds. Let $\mu, \nu \in \mathcal{M}$ and $\phi \in SPAP(\mathbb{R}, L^2(P, H), \mu, \nu, r)$ then the function $t \to \phi_t$, belongs to $SPAP(C([-r, 0], L^2(P, H)), \mu, \nu, r)$.

\textbf{Proof.} Assume that $\phi = g + h$, where $g \in SAP(\mathbb{R}, L^2(P, H))$ and $h \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu, r)$. Then we can see that, $\phi_t = gt + ht$ and $gt$ is square mean almost periodic process. Let us denote by
\[
M_\alpha(\tau) = \frac{1}{\nu([\tau - \tau, \tau])} \int_{\tau - \tau}^{\tau} \sup_{\theta \in [t - \tau, t]} (\mathbb{E} \|h(\theta)\|^2) d\mu_\alpha(t).
\]
Where $\mu_\alpha$ and $\nu_\alpha$ are the positive measures defined by equation (3.1). By using Proposition (3.10), it follows that $\mu_\alpha$ and $\mu$ are equivalent and $\nu_\alpha$ and $\nu$ are also equivalent. Then by using Theorem (4.10) we have $\mathcal{E}(\mathbb{R}, L^2(P, H), \mu_\alpha, \nu_\alpha, r) = \mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu, r)$ therefore $h \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu_\alpha, \nu_\alpha, r)$ that is $\lim_{\tau \to \infty} M_\alpha(\tau) = 0$ for all $\alpha \in \mathcal{M}$. On the other hand, for $r > 0$ we have
\[
\frac{1}{\nu([\tau - \tau, \tau])} \int_{\tau - \tau}^{\tau} \sup_{\theta \in [t - \tau, t]} \left(\sup_{\eta \in [-\tau, 0]} (\mathbb{E} \|h(\theta + \eta)\|^2)\right) d\mu(t) \leq \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t - 2\tau, t]} (\mathbb{E} \|h(\theta)\|^2) d\mu(t)
\]
\[
\leq \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t - 2\tau, t - \tau]} (\mathbb{E} \|h(\theta)\|^2) + \sup_{\theta \in [t - \tau, t]} (\mathbb{E} \|h(\theta)\|^2) d\mu(t)
\]
\[
\leq \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [-\tau, r] + [t - \tau, t + r]} (\mathbb{E} \|h(\theta)\|^2) \mu(t + r) + \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t - \tau, t]} (\mathbb{E} \|h(\theta)\|^2) d\mu(t)
\]
\[
\leq \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t - \tau, t + r]} (\mathbb{E} \|h(\theta)\|^2) d\mu(t + r)
\]
\[
+ \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t - \tau, t]} (\mathbb{E} \|h(\theta + \eta)\|^2) d\mu(t)
\]
Consequently,
\[
\frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t - \tau, t]} \left(\sup_{\eta \in [-\tau, 0]} (\mathbb{E} \|h(\theta + \eta)\|^2)\right) d\mu(t) \leq \frac{\nu([-\tau - r, \tau + r])}{\nu([-\tau, \tau])} \times M_r(\tau + r)
\]
\[
+ \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t - \tau, t]} (\mathbb{E} \|h(\theta)\|^2) d\mu(t)
\]
Using Proposition (3.11), and Proposition (3.10), it follows that, 
\( \phi \in \text{SPAP}(C[-r,0], L^2(P,H)), \mu,\nu, \tau ) \). Thus, we obtain the desired result \( \Box \).

Next, we study the composition of the space square-mean \((\mu,\nu)\) -pseudo almost periodic process.

**Definition 4.12.** [10] Let \( f: \mathbb{R} \times L^2(P,H) \to L^2(P,H), (t,x) \to f(t,x) \) be continuous. \( f \) is said be square-mean almost periodic in \( t \) uniformly in \( x \) if for all compact \( K \) of \( L^2(P,H) \) and for any \( \varepsilon > 0 \) there exists \( l(\varepsilon,K) \) such that for all \( \alpha \in \mathbb{R} \), there exists \( \tau \in [\alpha,\alpha + l(\varepsilon,K)] \) with

\[
\| \sup_{t \in \mathbb{R}} \mathbb{E} \| f(t+\tau,x) - f(t,x) \|^2 < \varepsilon. 
\]

We denote the following space of stochastic processes by \( \text{SAP}(\mathbb{R} \times L^2(P,H), L^2(P,H)) \).

**Theorem 4.13.** [10] Let \( f: \mathbb{R} \times L^2(P,H) \to L^2(P,H), (t,x) \to f(t,x) \) be a square almost periodic process in \( t \) uniformly in \( x \in L^2(P,H) \). Suppose that \( f \) is Lipschitz in the following sense: there exists a positive number \( L \) such that for any \( x,y \in L^2(P,H) \),

\[
\mathbb{E} \| f(t,x) - f(t,y) \|^2 \leq L \cdot \mathbb{E} \| x - y \|^2.
\]

Then \( t \to f(t,x(t)) \in \text{SAP}(\mathbb{R}, L^2(P,H)) \) for any \( x \in \text{SAP}(\mathbb{R}, L^2(P,H)) \).

**Definition 4.14.** Let \( \mu,\nu \in \mathcal{M} \). A continuous functions \( f(t,x): \mathbb{R} \times L^2(P,H) \to L^2(P,H) \) is said to be square-mean \((\mu,\nu)\)-pseudo almost periodic of class \( r \) in \( t \) for any \( x \in L^2(P,H) \) if it can be decomposed as \( f = g + \phi \), where \( g \in \text{SAP}(\mathbb{R} \times L^2(P,H), L^2(P,H)), \phi \in \mathcal{E}(\mathbb{R} \times L^2(P,H), L^2(P,H), \mu,\nu,r) \). Denote the set of all such stochastically continuous processes by \( \text{SAP}(\mathbb{R} \times L^2(P,H), L^2(P,H), \mu,\nu,r) \).

**Proposition 4.15.** Let \( a_i \in \mathbb{R}, i \in \mathbb{N} \). Then 
\[
\left| \sum_{i=1}^{n} a_i \right|^2 \leq n \sum_{i=1}^{n} |a_i|^2.
\]

**Theorem 4.16.** Let \( \mu,\nu \in \mathcal{M} \) satisfy \((H_2)\). Suppose that \( f \in \text{SAP}(\mathbb{R} \times L^2(P,H), L^2(P,H), \mu,\nu,r) \) and that there exists a positive number \( L \) such that, for any \( x,y \in L^2(P,H) \),

\[
\mathbb{E} \| f(t,x) - f(t,y) \|^2 \leq L \cdot \mathbb{E} \| x - y \|^2
\]

for \( t \in \mathbb{R} \). Then \( t \to f(t,x(t)) \in \text{SAP}(\mathbb{R}, L^2(P,H), \mu,\nu,r) \) for any \( x \in \text{SAP}(\mathbb{R}; L^2(P,H), \mu,\nu,r) \).

**Proof.** Since \( x \in \text{SAP}(\mathbb{R}; L^2(P,H), \mu,\nu,r) \), then we can decompose \( x = x_1 + x_2 \), where \( x_1 \in \text{SAP}(\mathbb{R}, L^2(P,H)) \) and \( x_2 \in \mathcal{E}(\mathbb{R}, L^2(P,H), \mu,\nu,r) \). Otherwise, since \( f \in \text{SAP}(\mathbb{R} \times L^2(P,H), L^2(P,H), \mu,\nu,r) \), then \( f = f_1 + f_2 \), where \( f_1 \in \text{SAP}(\mathbb{R} \times L^2(P,H), L^2(P,H), \mu,\nu,r) \) and \( f_2 \in \mathcal{E}(\mathbb{R} \times L^2(P,H), L^2(P,H), \mu,\nu,r) \).

The function \( f \) can be decomposed as

\[
f(t,x(t)) = f_1(t,x_1(t)) + [f(t,x(t)) - f(t,x_1(t))] + [f(t,x_1(t)) - f_1(t,x_1(t))] \\
= f_1(t,x_1(t)) + [f(t,x(t)) - f(t,x_1(t))] + f_2(t,x_1(t))
\]

Using Theorem (4.13), we have \( t \to f_1(t,x_1(t)) \in \text{SAP}(\mathbb{R} \times L^2(P,H), L^2(P,H)) \).

It remains to show that the both functions \( t \to [f(t,x(t)) - f(t,x_1(t))] \) and \( t \to f_2(t,x_1(t)) \)
belong to $E(\mathbb{R} \times L^2(P, H), L^2(P, H), \mu, \nu, r)$.

We have,

$$
\mathbb{E}\|f(t, x(t)) - f(t, x_1(t))\|^2 \leq L \mathbb{E}\|x(t) - x_1(t)\|^2 
$$

$$
\sup_{\theta \in [t-r, t]} \mathbb{E}\|f(\theta, x(\theta)) - f(\theta, x_1(\theta))\|^2 \leq L \sup_{\theta \in [t-r, t]} \mathbb{E}\|x(\theta) - x_1(\theta)\|^2.
$$

It follows that

$$
\frac{1}{\nu([-\tau, \tau])} \int_{[-\tau, \tau]} \sup_{\theta \in [t-r, t]} \mathbb{E}\|f(\theta, x(\theta)) - f(\theta, x_1(\theta))\|^2 d\mu(t) \leq
$$

$$
\frac{L}{\nu([-\tau, \tau])} \int_{[-\tau, \tau]} \sup_{\theta \in [t-r, t]} \mathbb{E}\|x(\theta) - x_1(\theta)\|^2 d\mu(t) \leq
$$

$$
\frac{L}{\nu([-\tau, \tau])} \int_{[-\tau, \tau]} \sup_{\theta \in [t-r, t]} \mathbb{E}\|x_2(\theta)\|^2 d\mu(t).
$$

Since $x_2 \in E(\mathbb{R}, L^2(P, H), \mu, \nu, r)$ then $t \to f(t, x(t)) - f(t, x_1(t))$ is $(\mu, \nu)$-ergodic.

Now to complete the proof, it is enough to prove $t \to f_2(t, x_1(t))$ is $(\mu, \nu)$-ergodic. Since $f_2$ is uniformly continuous on the compact set $K = \{x_1(t) : t \in \mathbb{R}\}$ with respect to the second variable $x$, we deduce that for given $\varepsilon$, there exists $\delta > 0$ such that for all $t \in \mathbb{R}$, $\zeta_1$ and $\zeta_2 \in K$, one has

$$
\|\zeta_1 - \zeta_2\| \leq \delta \implies \|f_2(t, \zeta_1) - f_2(t, \zeta_2)\| \leq \varepsilon.
$$

Therefore, there exist $n(\varepsilon) \in \mathbb{N}$ and $\{x_i\}_{i=1}^{n(\varepsilon)} \subset K$, such that

$$
K \subset \bigcup_{i=1}^{n(\varepsilon)} B(x_i, \delta),
$$

then

$$
\|f_2(t, x_1(t))\| \leq \varepsilon + \sum_{i=1}^{n(\varepsilon)} \|f_2(t, x_i)\|
$$

$$
\|f_2(t, x_1(t))\|^2 \leq \left(\varepsilon + \sum_{i=1}^{n(\varepsilon)} \|f_2(t, x_i)\|\right)^2
$$

$$
\leq 2 \left(\varepsilon^2 + \left(\sum_{i=1}^{n(\varepsilon)} \|f_2(t, x_i)\|^2\right)\right)
$$

By using the Proposition (4.15), we have

$$
\|f_2(t, x_1(t))\|^2 \leq 2 \left(\varepsilon + n(\varepsilon) \sum_{i=1}^{n(\varepsilon)} \|f_2(t, x_i)\|^2\right).
$$
It follows that
\[
\frac{1}{\nu([-\tau, \tau])} \int_{[-\tau, \tau]} \sup_{\theta \in [t-\tau, t]} \mathbb{E} \| f_2(\theta, x_1(\theta)) \|^2 d\mu(t) \leq
\]
\[
2 \left( \frac{\varepsilon \mu([-\tau, \tau]) + n(\varepsilon)}{\nu([-\tau, \tau])} \int_{[-\tau, \tau]} \sup_{\theta \in [t-\tau, t]} \mathbb{E} \| f_2(\theta, x_i) \|^2 d\mu(t) \right).
\]

By the fact that
\[
\forall i \in \{1, ..., n(\varepsilon)\}, \quad \lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \left( \sup_{\theta \in [t-\tau, t]} \mathbb{E} \| f_2(\theta, x_i) \|^2 \right) d\mu(t) = 0
\]
we deduce that
\[
\forall \varepsilon > 0, \quad \limsup_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \left( \sup_{\theta \in [t-\tau, t]} \mathbb{E} \| f_2(\theta, x_1(\theta)) \|^2 \right) d\mu(t) \leq 2\alpha \varepsilon.
\]
Therefore $t \to f_2(t, x_1(t))$ is ergodic and the theorem is proved. \hfill \Box

\[(H_5): \text{g is a stochastically bounded process.}\]

**Theorem 4.17.** Assume that $(H_0)$, $(H_1)$, $(H_4)$ and $(H_5)$ hold and the semigroup $(\mathcal{U}(t))_{t \geq 0}$ is hyperbolic. If $f$ is bounded and continuous on $\mathbb{R}$, then there exists a unique bounded solution $u$ of equation (1.1) on $\mathbb{R}$ given by

\[
u(t) = \lim_{\lambda \to +\infty} \int_{-\infty}^{+\infty} \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 f(s)) ds + \lim_{\lambda \to +\infty} \int_{-\infty}^{+\infty} \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 f(s)) ds
\]
\[
+ \lim_{\lambda \to +\infty} \int_{-\infty}^{+\infty} \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 g(s)) dW(s) + \lim_{\lambda \to +\infty} \int_{-\infty}^{+\infty} \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 g(s)) dW(s)
\]
\[
\forall \ t \geq 0, \text{ where } \tilde{B}_\lambda = \lambda(\lambda I - \tilde{A}_d)^{-1}, \Pi^s \text{ and } \Pi^u \text{ are the projections of } C_0 \text{ onto the stable and unstable subspaces.}
\]

**Proof.** Let

\[
u(t) = v(t) + \lim_{\lambda \to +\infty} \int_{-\infty}^{+\infty} \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 g(s)) dW(s)
\]
\[
+ \lim_{\lambda \to +\infty} \int_{+\infty}^{+\infty} \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 g(s)) dW(s) \forall \ t \geq 0,
\]

where

\[
v(t) = \lim_{\lambda \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 f(s)) ds + \lim_{\lambda \to +\infty} \int_{+\infty}^{t} \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 f(s)) ds
\]

Let us first prove that $u_t$ exists. The existence of $v(t)$ have proved by [1]. Now, we show that the limit \( \lim_{\lambda \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 g(s)) dW(s) \) exist.

For $t \in \mathbb{R}$ we have,
\[
\mathbb{E} \left\| \int_{-\infty}^{t} \mathcal{U}^s(t-s) \Pi^s(\widetilde{B}_{\lambda}X_0g(s))dW(s) \right\|^2 \leq \mathbb{E} \left( \int_{-\infty}^{t} M^2 e^{-2w(t-s)} \| \Pi^s \| \| \widetilde{B}_{\lambda}(X_0g(s)) \|^2 ds \right) \\
\leq M^2 \mathbb{E} \left( \int_{-\infty}^{t} e^{-2w(t-s)} \| \Pi^s \| \| \widetilde{B}_{\lambda}(X_0g(s)) \|^2 ds \right) \\
\leq M^2 \widetilde{M}^2 \| \Pi^s \|^2 \mathbb{E} \left( \int_{-\infty}^{t} e^{-2w(t-s)} \| g(s) \|^2 ds \right) \\
\leq M^2 \widetilde{M}^2 \| \Pi^s \|^2 \sum_{n=1}^{+\infty} \mathbb{E} \left( \int_{\sigma-n}^{\sigma-n+1} e^{-2w(t-s)} \| g(s) \|^2 ds \right). 
\]

Using the Hölder inequality, we obtain

\[
\mathbb{E} \left\| \int_{-\infty}^{t} \mathcal{U}^s(t-s) \Pi^s(\widetilde{B}_{\lambda}X_0g(s))dW(s) \right\|^2 \leq \\
M^2 \widetilde{M}^2 \| \Pi^s \|^2 \sum_{n=1}^{+\infty} \left( \int_{\sigma-n}^{\sigma-n+1} e^{-4w(t-s)} ds \right) \frac{1}{2} \mathbb{E} \left( \int_{\sigma-n}^{\sigma-n+1} \| g(s) \|^2 ds \right) \frac{1}{2} \\
\leq M^2 \widetilde{M}^2 \| \Pi^s \|^2 \frac{1}{2 \sqrt{w}} \sum_{n=1}^{+\infty} \left( e^{-4w(n-1)} - e^{-4wn} \right) \frac{1}{2} \mathbb{E} \left( \int_{\sigma-n}^{\sigma-n+1} \| g(s) \|^2 ds \right) \frac{1}{2} \\
\leq M^2 \widetilde{M}^2 \| \Pi^s \|^2 \frac{1}{2 \sqrt{w}} \left( e^{4wn} - 1 \right) \frac{1}{2} \sum_{n=1}^{+\infty} e^{-2wn} \times \mathbb{E} \left( \int_{\sigma-n}^{\sigma-n+1} \| g(s) \|^2 ds \right) \frac{1}{2}. 
\]

Since the serie \( \sum_{n=1}^{+\infty} e^{-2wn} \) is convergent, then it exists a constant \( c > 0 \) such that

\[
\sum_{n=1}^{+\infty} e^{-2wn} \leq c, 
\]

moreover it follows that

\[
\mathbb{E} \left\| \int_{-\infty}^{t} \mathcal{U}^s(t-s) \Pi^s(\widetilde{B}_{\lambda}X_0g(s))dW(s) \right\|^2 \leq \\
\frac{\gamma}{2} \sum_{n=1}^{+\infty} e^{-2wn} \\
\leq \gamma c, 
\]

where \( \gamma = M^2 \widetilde{M}^2 \| \Pi^s \|^2 \frac{1}{2 \sqrt{w}} \left( e^{4wn} - 1 \right) \frac{1}{2} \mathbb{E} \| g(s) \|. \)

Let \( F(n, s, t) = \mathcal{U}^s(t-s) \Pi^s(\widetilde{B}_{\lambda}X_0g(s)) \) for \( n \in \mathbb{N} \) for \( s \leq t \).

For \( n \) is sufficiently large and \( \sigma \leq t \), we have

\[
\mathbb{E} \left\| \int_{-\infty}^{\sigma} F(n, s, t)dW(s) \right\|^2 \leq \\
\frac{\gamma}{2} \sum_{n=1}^{+\infty} \left( \int_{\sigma-n}^{\sigma-n+1} e^{-4w(t-s)} ds \right) \frac{1}{2} \times \mathbb{E} \left( \int_{\sigma-n}^{\sigma-n+1} \| g(s) \|^2 ds \right) \frac{1}{2} \leq 
\]
It follows that for $n$ and $m$ sufficiently large and $\sigma \leq t$, we have

$$\mathbb{E} \left| \int_{-\infty}^{t} F(n, s, t) dW(s) - \int_{\sigma}^{t} F(m, s, t) dW(s) \right|^2 \leq$$

$$\mathbb{E} \left| \int_{-\infty}^{\sigma} F(n, s, t) dW(s) + \int_{\sigma}^{t} F(n, s, t) dW(s) - \int_{-\infty}^{\sigma} F(m, s, t) dW(s) - \int_{\sigma}^{t} F(m, s, t) dW(s) \right|^2 \leq$$

$$3\mathbb{E} \left| \int_{-\infty}^{\sigma} F(n, s, t) dW(s) \right|^2 + 3\mathbb{E} \left| \int_{-\infty}^{\sigma} F(m, s, t) dW(s) \right|^2$$

$$+ 3\mathbb{E} \left| \int_{\sigma}^{t} F(n, s, t) dW(s) - \int_{\sigma}^{t} F(m, s, t) dW(s) \right|^2 \leq$$

$$6\gamma ce^{-2\omega(t-\sigma)} + 3\mathbb{E} \left| \int_{\sigma}^{t} F(n, s, t) dW(s) - \int_{\sigma}^{t} F(m, s, t) dW(s) \right|^2$$

Since $\lim_{n \to +\infty} \mathbb{E} \left| \int_{\sigma}^{t} F(n, s, t) dW(s) \right|^2$ exists, then

$$\lim_{n \to +\infty} \limsup_{m \to +\infty} \mathbb{E} \left| \int_{-\infty}^{t} F(n, s, t) dW(s) - \int_{-\infty}^{t} F(m, s, t) dW(s) \right|^2 \leq 6\gamma ce^{-2\omega(t-\sigma)}$$

If $\sigma \to -\infty$, then

$$\lim_{n \to +\infty} \limsup_{m \to +\infty} \mathbb{E} \left| \int_{-\infty}^{t} F(n, s, t) dW(s) - \int_{-\infty}^{t} F(m, s, t) dW(s) \right|^2 = 0.$$
Theorem 4.20. Assume that \((H_5)\). Let \(f, g \in \text{SAP}(\mathbb{R}; L^2(P, H))\) and \(\Gamma\) be the mapping defined for \(t \in \mathbb{R}\) by

\[
\Gamma(f, g)(t) = \lim_{\lambda \to +\infty} \int_{-\infty}^{t} U^\lambda(t-s)\Pi^\lambda(B_\lambda X_0f(s))ds + \lim_{\lambda \to +\infty} \int_{+\infty}^{t} U^\lambda(t-s)\Pi^\lambda(B_\lambda X_0f(s))ds \\
+ \lim_{\lambda \to +\infty} \int_{-\infty}^{t} U^\lambda(t-s)\Pi^\lambda(B_\lambda X_0g(s))dW(s) \\
+ \lim_{\lambda \to +\infty} \int_{+\infty}^{t} U^\lambda(t-s)\Pi^\lambda(B_\lambda X_0g(s))dW(s)
\]

Then \(\Gamma(f, g) \in \text{SAP}(\mathbb{R}; L^2(P, H))\).

Proof. \(\Gamma(f, g)_\tau(t) = \Gamma(f, g)(t + \tau)\)

\[
\begin{align*}
\Gamma(f, g)(t) &= \lim_{\lambda \to +\infty} \int_{-\infty}^{t} U^\lambda(t-s)\Pi^\lambda(B_\lambda X_0f(s))ds + \lim_{\lambda \to +\infty} \int_{+\infty}^{t} U^\lambda(t-s)\Pi^\lambda(B_\lambda X_0f(s))ds \\
&+ \lim_{\lambda \to +\infty} \int_{-\infty}^{t} U^\lambda(t-s)\Pi^\lambda(B_\lambda X_0g(s))dW(s) + \lim_{\lambda \to +\infty} \int_{+\infty}^{t} U^\lambda(t-s)\Pi^\lambda(B_\lambda X_0g(s))dW(s) \\
&= \lim_{\lambda \to +\infty} \int_{-\infty}^{t} U^\lambda(t-s)\Pi^\lambda(B_\lambda X_0f(s))ds + \lim_{\lambda \to +\infty} \int_{+\infty}^{t} U^\lambda(t-s)\Pi^\lambda(B_\lambda X_0f(s))ds \\
&+ \lim_{\lambda \to +\infty} \int_{-\infty}^{t} U^\lambda(t-s)\Pi^\lambda(B_\lambda X_0g(s))dW(s) + \lim_{\lambda \to +\infty} \int_{+\infty}^{t} U^\lambda(t-s)\Pi^\lambda(B_\lambda X_0g(s))dW(s) \\
&= \Gamma(f_\tau, g_\tau)(t) \text{ for all } t \in \mathbb{R}.
\end{align*}
\]

Thus \(\Gamma(f, g)_\tau = \Gamma(f_\tau, g_\tau)\) which implies \(\{\Gamma(f, g)_\delta, \delta \in \mathbb{R}\}\) is relatively compact in \(\text{SBC}(\mathbb{R}, L^2(P, H))\). Since \(\Gamma\) is continuous from \(\text{SBC}(\mathbb{R}, L^2(P, H))\) into \(\text{SBC}(\mathbb{R}, L^2(P, H))\) then \(\Gamma(f, g) \in \text{SAP}(\mathbb{R}, L^2(P, H))\). \(\square\)

Theorem 4.21. Assume that \((H_3)\) and \((H_5)\) holds. Let \(f, g \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu, r)\) then \(\Gamma(f, g) \in \mathcal{E}(\mathbb{R}, L^2(P, H), \mu, \nu, r)\).

Proof. We have,

\[
\begin{align*}
\Gamma(f, g)(t) &= \lim_{\lambda \to +\infty} \int_{-\infty}^{t} U^\lambda(t-s)\Pi^\lambda(B_\lambda X_0f(s))ds \\
&+ \lim_{\lambda \to +\infty} \int_{+\infty}^{t} U^\lambda(t-s)\Pi^\lambda(B_\lambda X_0f(s))ds \\
&+ \lim_{\lambda \to +\infty} \int_{-\infty}^{t} U^\lambda(t-s)\Pi^\lambda(B_\lambda X_0g(s))dW(s) \\
&+ \lim_{\lambda \to +\infty} \int_{+\infty}^{t} U^\lambda(t-s)\Pi^\lambda(B_\lambda X_0g(s))dW(s)
\end{align*}
\]
\[
\begin{align*}
\mathbb{E}\left|\Gamma(f,g)(\theta)\right|^2 &= \mathbb{E}\left| \lim_{\lambda \to +\infty} \int_{-\infty}^{\theta} \mathcal{U}^\theta(t-s)\Pi^\theta(\tilde{B}_\lambda X_0 f(s))ds \right. \\
&\hspace{4em} + \lim_{\lambda \to +\infty} \int_{+\infty}^{\theta} \mathcal{U}^\theta(t-s)\Pi^\theta(\tilde{B}_\lambda X_0 f(s))ds \\
&\hspace{4em} + \lim_{\lambda \to +\infty} \int_{-\infty}^{\theta} \mathcal{U}^\theta(t-s)\Pi^\theta(\tilde{B}_\lambda X_0 g(s))dW(s) \\
&\hspace{4em} + \lim_{\lambda \to +\infty} \int_{+\infty}^{\theta} \mathcal{U}^\theta(t-s)\Pi^\theta(\tilde{B}_\lambda X_0 g(s))dW(s)\right|^2.
\end{align*}
\]

\[
\int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \mathbb{E}\left|\Gamma(f,g)(\theta)\right|^2 d\mu(t) \leq \\
\int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \left( \int_{-\infty}^{\theta} e^{-2\omega(t-s)}\Pi^\theta \|f(s)\|^2 ds + \int_{-\infty}^{\theta} e^{-2\omega(t-s)}\Pi^\theta \|g(s)\|^2 ds \right) d\mu(t)
\]

\[
\int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \left( \int_{-\infty}^{\theta} e^{-2\omega(t-s)}\Pi^\theta \|f(s)\|^2 ds + \int_{-\infty}^{\theta} e^{-2\omega(t-s)}\Pi^\theta \|g(s)\|^2 ds \right) d\mu(t)
\]

\[
\leq 4M^2 \mathcal{M}^2 \left[ \int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \left( \int_{-\infty}^{\theta} e^{-2\omega(t-s)}\Pi^\theta \|f(s)\|^2 ds + \int_{-\infty}^{\theta} e^{-2\omega(t-s)}\Pi^\theta \|g(s)\|^2 ds \right) d\mu(t) \right]
\]

one the one hand using Fubini’s theorem, we have

\[
\left|\Pi^\theta\right|^2 \int_{-\tau}^{\tau} \left[ \sup_{\theta \in [t-r,t]} \left( \int_{-\infty}^{\theta} e^{-2\omega(t-s)}(\mathbb{E}\|f(s)\|^2 + \mathbb{E}\|g(s)\|^2)ds \right) d\mu(t) \right]
\]

\[
\leq e^{\omega r} \left|\Pi^\theta\right|^2 \int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \left( \int_{-\infty}^{\theta} e^{-2\omega(t-s)}(\mathbb{E}\|f(s)\|^2 + \mathbb{E}\|g(s)\|^2)ds \right) d\mu(t)
\]

\[
\leq e^{\omega r} \left|\Pi^\theta\right|^2 \int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \left( \int_{-\infty}^{t} e^{-2\omega(t-s)}(\mathbb{E}\|f(s)\|^2 + \mathbb{E}\|g(s)\|^2)ds \right) d\mu(t)
\]

\[
\leq e^{\omega r} \left|\Pi^\theta\right|^2 \int_{-\tau}^{\tau} \left( \int_{-\infty}^{t} e^{-2\omega(t-s)}(\mathbb{E}\|f(t-s)\|^2 + \mathbb{E}\|g(t-s)\|^2)ds \right) d\mu(t)
\]

\[
\leq e^{\omega r} \left|\Pi^\theta\right|^2 \int_{0}^{+\infty} e^{-2\omega s} \left( \mathbb{E}\|f(t-s)\|^2 + \mathbb{E}\|g(t-s)\|^2 \right) ds d\mu(t)
\]
By using Proposition(3.13) we deduce that

\[
\lim_{\tau \to +\infty} \frac{e^{-2\omega s}}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \left( \mathbb{E} \| f(t-s) \|^2 + \mathbb{E} \| g(t-s) \|^2 \right) d\mu(t) \to 0
\]

for all \( s \in \mathbb{R}^+ \) and

\[
\frac{e^{-2\omega s}}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \left( \mathbb{E} \| f(t-s) \|^2 + \mathbb{E} \| g(t-s) \|^2 \right) d\mu(t) \leq \frac{e^{-2\omega s} \mu([-\tau, \tau])}{\nu([-\tau, \tau])} \left( \| f \|_\infty^2 + \| g \|_\infty^2 \right)
\]

Since \( f \) and \( g \) are bounded functions, then the function \( s \mapsto \frac{e^{-2\omega s} \mu([-\tau, \tau])}{\nu([-\tau, \tau])} \left( \| f \|_\infty^2 + \| g \|_\infty^2 \right) \) belongs to \( L^1([0, +\infty[) \) in view of the Lebesgue dominated convergence theorem, it follows that

\[
\lim_{\tau \to +\infty} \frac{e^{2\omega \tau}}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \left( \mathbb{E} \| f(t-s) \|^2 + \mathbb{E} \| g(t-s) \|^2 \right) d\mu(t) ds \to 0.
\]

On the other hand by Fubini’s theorem, we also have

\[
|\Pi^u|^2 \int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \left( \int_{-\tau}^{+\infty} e^{2\omega(t-s)} (\mathbb{E} \| f(s) \|^2 + \mathbb{E} \| g(s) \|^2) ds \right) d\mu(t)
\]

\[
\leq \ |\Pi^u|^2 \int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \left( \int_{-\tau}^{+\infty} e^{2\omega(t-s)} (\mathbb{E} \| f(s) \|^2 + \mathbb{E} \| g(s) \|^2) ds \right) d\mu(t)
\]

\[
\leq \ |\Pi^u|^2 \int_{-\tau}^{\tau} \left( \int_{-\tau}^{+\infty} e^{2\omega(t-s)} (\mathbb{E} \| f(s) \|^2 + \mathbb{E} \| g(s) \|^2) ds \right) d\mu(t)
\]

\[
\leq \ |\Pi^u|^2 \int_{-\tau}^{\tau} \left( \int_{-\tau}^{+\infty} e^{2\omega(t-s)} (\mathbb{E} \| f(s) \|^2 + \mathbb{E} \| g(s) \|^2) ds \right) d\mu(t)
\]

Since the function \( s \mapsto \frac{e^{2\omega s}}{\nu([-\tau, \tau])} \left( \| f \|_\infty^2 + \| g \|_\infty^2 \right) \) belongs to \( L^1([- \infty, r]) \) resoning like above, it follows that

\[
\lim_{\tau \to +\infty} \int_{-\infty}^{\infty} e^{2\omega s} \times \frac{1}{\nu([-\tau, \tau])} \left( \int_{-\tau}^{\tau} e^{2\omega s} (\mathbb{E} \| f(s) \|^2 + \mathbb{E} \| g(s) \|^2) d\mu(t) \right) ds = 0
\]

Consequently

\[
\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \mathbb{E} \| \Gamma(f, g)(\theta) \|^2 d\mu(t) = 0
\]

Thus, we obtain the desired result. \( \square \)

**Theorem 4.22.** Assume \((H_0), (H_1), (H_3)\) and \((H_5)\) hold. Then equation \((1.1)\) has a unique square mean \(cl(\mu, \nu)\)-pseudo almost periodic solution of class \(r\).

**Proof.** Since \( f \) and \( g \) are square mean \((\mu, \nu)\)-pseudo almost periodic function, \( f, g \) has a decomposition \( f = f_1 + f_2 \) and \( g = g_1 + g_2 \) where \( f_1, g_1 \in SAP(\mathbb{R}; L^2(P, H)) \) and \( f_2, g_2 \in \mathcal{E}(\mathbb{R}; L^2(P, H), \mu, \nu, r) \). Using Theorem(4.20), Theorem(4.2) and Theorem(4.17), we get the desired result. \( \square \)

Our next objective is to show the existence of square mean \((\mu, \nu)\)-pseudo almost periodic solutions of class \(r\) for the following problem

\[
(4.2) \quad du(t) = [Au(t) + L(u_t) + f(t, u_t)] dt + g(t, u_t) dW(t) \quad t \in \mathbb{R}
\]
where \( f : \mathbb{R} \times C \to L^2(P, H) \) and \( g : \mathbb{R} \times C \to L^2(P, H) \) are two stochastic continuous processes. For the sequel, we formulate the following assumptions

(H\(_6\)) Let \( \mu, \nu \in \mathcal{M} \) and \( f : \mathbb{R} \times C([-r, 0], L^2(P, H)) \to L^2(P, H) \) square mean \( cl(\mu, \nu) \)-pseudo almost periodic of class \( r \) such that there exists a constant \( L_f \) such that \( E \| f(t, \phi_1) - f(t, \phi_2) \|^2 \leq L_f \times E \| \phi_1 - \phi_2 \|^2 \) for all \( t \in \mathbb{R} \) and \( \phi_1, \phi_2 \in C([-r, 0], L^2(P, H)). \)

(H\(_7\)) Let \( \mu, \nu \in \mathcal{M} \) and \( g : \mathbb{R} \times C([-r, 0], L^2(P, H)) \to L^2(P, H) \) square mean \( cl(\mu, \nu) \)-pseudo almost periodic of class \( r \) such that there exists a constant \( L_g \) such that \( E \| g(t, \phi_1) - g(t, \phi_2) \|^2 \leq L_g \times E \| \phi_1 - \phi_2 \|^2 \) for all \( t \in \mathbb{R} \) and \( \phi_1, \phi_2 \in C([-r, 0], L^2(P, H)). \)

**Theorem 4.23.** Assume (H\(_9\)), (H\(_4\)), (H\(_2\)), (H\(_4\)), (H\(_6\)) and (H\(_7\)) hold. If

\[
\tilde{M}^2 \tilde{M}^2 \sup_{t \in \mathbb{R}} \left( \| \Pi^v \|^2 \int_{-\infty}^{t} e^{-2\omega(t-s)}(L_f^2 + L_g^2)ds + \| \Pi^u \|^2 \int_{t}^{+\infty} e^{2\omega(t-s)}(L_f^2 + L_g^2)ds \right) < \frac{1}{4},
\]

then equation (4.2) has a unique square mean \( cl(\mu, \nu) \)-pseudo almost periodic solution of class \( r \).

**Proof.** Let \( x \) be a function in \( S P A P(\mathbb{R}, L^2(P, H), \mu, \nu, r) \) from Theorem(4.11) the function \( t \to x_t \) belongs to \( S P A P(C([-r, 0]; L^2(P, H), \mu, \nu, r) \). Hence Theorem(4.16) implies that the function \( g(\cdot) := f(\cdot, x) \) is in \( S P A P(\mathbb{R}; L^2(P, H), \mu, \nu, r) \). Consider the following mapping:

\[ \mathcal{H} : S P A P(\mathbb{R}; L^2(P, H), \mu, \nu, r) \to S P A P(\mathbb{R}; L^2(P, H), \mu, \nu, r) \]

defined for \( t \in \mathbb{R} \) by

\[
(\mathcal{H}x)(t) = \lim_{\lambda \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^v \tilde{B}_\lambda(X_0 f(s, x_s))ds
+ \lim_{\lambda \to +\infty} \int_{t}^{+\infty} \mathcal{U}^s(t-s)\Pi^v \tilde{B}_\lambda(X_0 f(s, x_s))ds
+ \lim_{\lambda \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^v \tilde{B}_\lambda(X_0 g(s, x_s))dW(s)
+ \lim_{\lambda \to +\infty} \int_{t}^{+\infty} \mathcal{U}^s(t-s)\Pi^v \tilde{B}_\lambda(X_0 g(s, x_s))dW(s)
\]

From Theorem(4.20), Theorem(4.22) and Theorem(4.17), it suffices now to show that the operator \( \mathcal{H} \) has a unique fixed point in \( S P A P(\mathbb{R}; L^2(P, H), \mu, \nu, r) \). Let \( x_1, x_2 \in S P A P(\mathbb{R}; L^2(P, H), \mu, \nu, r) \). Then we have

\[
E\| \mathcal{H}x_1(t) - \mathcal{H}x_2(t) \|^2 \leq 4E \left( \lim_{\lambda \to +\infty} \int_{-\infty}^{t} \| \mathcal{U}^s(t-s)\Pi^v \tilde{B}_\lambda(X_0(f(s, x_{1s}) - f(s, x_{2s})))ds \|^2 \right)
+ 4E \left( \lim_{\lambda \to +\infty} \int_{t}^{+\infty} \| \mathcal{U}^s(t-s)\Pi^v \tilde{B}_\lambda(X_0(f(s, x_{2s}) - f(s, x_{1s})))ds \|^2 \right)
+ 4E \left( \lim_{\lambda \to +\infty} \int_{-\infty}^{t} \| \mathcal{U}^s(t-s)\Pi^v \tilde{B}_\lambda(X_0(g(s, x_{1s}) - g(s, x_{2s})))ds \|^2 \right)
+ 4E \left( \lim_{\lambda \to +\infty} \int_{t}^{+\infty} \| \mathcal{U}^s(t-s)\Pi^v \tilde{B}_\lambda(X_0(g(s, x_{2s}) - g(s, x_{1s})))ds \|^2 \right)
\]
and only one square mean unique fixed point (\(\Omega\)-dimensional Brownian notion defined on the filtered probability space 
Lipschitzian with respect to the second argument.

Let us pose

\[
\text{Proof.}
\]

\[
\text{Consequently}
\]

\[
\text{This means that } H \text{ is a strict contraction. Thus by Banach’s fixed point theorem, } H \text{ has a unique fixed point } u \text{ in } SPAP(\mathbb{R}, L^2(P, H)), \mu, \nu, r). \text{ We conclude that equation (4.2), has one and only one square mean cl}(\mu, \nu)\text{-pseudo almost periodic solution of class } r. \]

**Proposition 4.24.** Assume \((H_0), (H_1), (H_2), (H_4)\) and \(f, g\) are lipschitz continuous with respect the second argument. If

\[
\text{Lip}(f) = \text{Lip}(g) < \left(\frac{\omega}{4\tilde{M}^2\tilde{M}^2(\|\Pi^s\|^2 + \|\Pi^u\|^2)}\right)^{\frac{1}{2}}
\]

then equation (5.1) has a unique cl}(\mu, \nu)\text{-pseudo almost periodic solution of class } r, \text{ where Lip}(f), \text{Lip}(g) \text{ are respectively the Lipschitz constant of } f \text{ and } g.

**Proof.** Let us pose \(k = \text{Lip}(f) = \text{Lip}(g)\), we have

\[
\mathbb{E}[\|Hx_1(t) - Hx_2(t)\|^2] \leq 4\tilde{M}^2\tilde{M}^2\mathbb{E}[\|x_1 - x_2\|^2] \sup_{t \in \mathbb{R}} \left(\|\Pi^s\|^2 \int_{-\infty}^{t} e^{-2\omega(t-s)}(L_j^2 + L_o^2)ds + \|\Pi^u\|^2 \int_{t}^{+\infty} 2k^2 e^{-2\omega(t-s)}ds + \|\Pi^s\|^4 \int_{-\infty}^{t} 2k^2 e^{2\omega(t-s)}ds\right)
\]

\[
\leq \frac{4kL^2\tilde{M}^2(\|\Pi^s\|^2 + \|\Pi^u\|^2)}{\omega} \mathbb{E}[\|x_1 - x_2\|^2].
\]

Consequently \(H\) is a strict contraction if \(k^2 < \frac{\omega}{4\tilde{M}^2\tilde{M}^2(\|\Pi^s\|^2 + \|\Pi^u\|^2)}\). \(\square\)

5. APPLICATION

For illustration, we propose to study the existence of solutions for the following model

\[
\begin{cases}
\quad dz(t,x) = \frac{\partial^2}{\partial x^2}z(t,x)dt + \left[\int_{-r}^{0} G(\theta)z(t + \theta, x)d\theta + \sin(t) + \sin(\sqrt{2}t) + \arctan(t)\right] + \int_{-r}^{0} h(\theta, z(t + \theta, x))d\theta dt + \left[\frac{\cos(t)}{2 + \cos(\sqrt{2}t)} + \arctan(t)\right] + \int_{-r}^{\theta} h(\theta, z(t + \theta, x))d\theta dW(t)
\end{cases}
\]

\[
z(t,0) = z(t, \pi) = 0 \text{ for } t \in \mathbb{R}
\]

Where \(G : [-r, 0] \rightarrow \mathbb{R}\) is a continuous function and \(h : [-r, 0] \times \mathbb{R} \rightarrow \mathbb{R}\) is continuous, Lipschitzian with respect to the second argument. \(W(t)\) is a two-sided and standard one-dimensional Brownian motion defined on the filtered probability space \((\Omega, \mathcal{F}, P, \mathcal{F}_t)\) with
Consider the measures \( F_t = \sigma \{ W(u) - W(v) \mid u, v \leq t \} \). To rewrite equation (5.1) in the abstract form, we introduce the space \( H = L^2((0, \pi)) \). Let \( A : D(A) \to L^2((0, \pi)) \) defined by

\[
\begin{cases}
D(A) = H^1((0, \pi)) \cap H_0^1((0, 1)) \\
Ag(t) = y''(t) \text{ for } t \in (0, \pi) \text{ and } y \in D(A)
\end{cases}
\]

Then \( A \) generates a \( C_0 \)-semigroup \( (U(t))_{t \geq 0} \) on \( L^2((0, \pi)) \) given by

\[
(U(t)x)(r) = \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t} < x, e_n >_{L^2} e_n(r)
\]

Where \( e_n(r) = \sqrt{2} \sin(n \pi r) \) for \( n = 1, 2, \ldots \), and \( \|U(t)\| \leq e^{-\pi^2 t} \) for all \( t \geq 0 \). Thus \( \overline{M} = 1 \) and \( \omega = \pi^2 \). Then \( A \) satisfied the Hille-Yosida condition in \( L^2((0, \pi)) \). Moreover the part \( A_0 \) of \( A \) in \( \overline{D(A)} \). It follows that \( (H_0) \) and \( (H_1) \) are satisfied.

We define \( f : \mathbb{R} \times C \to L^2((0, \pi)) \) and \( L : C \to L^2((0, \pi)) \) as follows

\[
f(t, \phi)(x) = (\sin(t) + \sin(\sqrt{2}t)) + \arctan(t) + \int_{-\tau}^{\theta} h(\theta, \phi(\theta)(x))d\theta
\]
\[
g(t, \phi)(x) = \frac{\cos(t)}{2 + \cos(\sqrt{2}t)} + \arctan(t) + \int_{-\tau}^{\theta} h(\theta, \phi(\theta)(x))d\theta
\]
\[
L(\phi)(x) = \int_{-\tau}^{\theta} G(\theta, \phi(\theta)(x))d\theta \text{ for } -r \leq \theta \leq 0 \text{ and } x \in (0, \pi)
\]

let us pose \( v(t) = z(t, x) \). Then equation (5.1) takes the following abstract form

\[
dv(t) = [Av(t) + L(v_t) + f(t, v_t)]dt + g(t, v_t)dW(t) \text{ for } t \in \mathbb{R}
\]

Consider the measures \( \mu \) and \( \nu \) where its Radon-Nikodym derivative are respectively \( \rho_1, \rho_2 : \mathbb{R} \to \mathbb{R} \) defined by

\[
\rho_1(t) = \begin{cases} 1 \text{ for } t > 0 \\
e^t \text{ for } t \leq 0 \end{cases}
\]

and

\[
\rho_2(t) = |t| \text{ for } t \in \mathbb{R}
\]

i.e \( d\mu(t) = \rho_1(t)dt \) and \( dv(t) = \rho_2(t)dt \) where \( dt \) denotes the Lebesgue measure on \( \mathbb{R} \) and

\[
\mu(A) = \int_A \rho_1(t)dt \text{ for } \nu(A) = \int_A \rho_2(t)dt \text{ for } A \in \mathcal{B}.
\]

From [6] \( \mu, \nu \in \mathcal{M} \), \( \mu, \nu \) satisfy \( (H_4) \) and \( \sin(t) + \sin(\sqrt{2}t) + \frac{\pi}{2} \) is almost periodic. We have

\[
\limsup_{\tau \to +\infty} \frac{\mu([-\tau, \tau])}{\nu([-\tau, \tau])} = \limsup_{\tau \to +\infty} \frac{\int_{-\tau}^{0} e^t dt + \int_{0}^{\tau} dt}{2 \int_{-\tau}^{0} dt} = \limsup_{\tau \to +\infty} \frac{1 - e^{-\tau} + \tau}{\tau^2} = 0 < \infty,
\]

which implies that \( (H_2) \) is satisfied.

For all \( t \in \mathbb{R} \), \( \frac{\pi}{2} \leq \arctan t \leq \frac{\pi}{2} \) therefore, for all \( \theta \in [t - r, t] \), \( \arctan(t - r) \leq \arctan(\theta) \). It follows \( |\arctan(\theta - \pi) - \arctan(\theta) | = \frac{\pi}{2} \arctan(\theta) \leq |\arctan(t - r) - \arctan(\theta) | = \frac{\pi}{2} - \arctan(t - r) \), implies that
\[
\left| \arctan \theta - \frac{\pi}{2} \right|^2 \leq \left| \arctan(t-r) - \frac{\pi}{2} \right|^2 \quad \text{hence} \quad \sup_{\theta \in [t-r,t]} \mathbb{E} \left| \arctan \theta - \frac{\pi}{2} \right|^2 \leq \mathbb{E} \left| \arctan(t-r) - \frac{\pi}{2} \right|^2.
\]
One the one hand, we have the following:
\[
\frac{1}{\nu([-\tau, \tau])} \int_0^\tau \mathbb{E} \left| \arctan(t-r) - \frac{\pi}{2} \right|^2 dt = \frac{1}{\nu([-\tau, \tau])} \int_0^\tau \mathbb{E} \left( \frac{\pi}{2} - \arctan(t-r) \right)^2 dt \\
\leq \frac{1}{\nu([-\tau, \tau])} \int_0^\tau \frac{\pi^2}{4} dt \\
\leq \frac{\pi^2}{4\tau} \to 0 \quad \text{as} \quad \tau \to +\infty
\]
On the other hand we have
\[
\frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^0 \mathbb{E} \left| \arctan(t-r) - \frac{\pi}{2} \right|^2 e^t dt \leq \frac{1}{\nu([-\tau, \tau])} \int_0^\tau \frac{\pi^2}{4} e^t dt \\
\leq \frac{\pi^2(1-e^{-\tau})}{4\tau} \to 0 \quad \text{as} \quad \tau \to +\infty
\]
Consequently
\[
\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \sup_{\theta \in [t-r,t]} \mathbb{E} \left| \arctan \theta - \frac{\pi}{2} \right|^2 d\mu(t) = 0
\]
It follows that \( t \mapsto \arctan t - \frac{\pi}{2} \) is square mean \((\mu, \nu)\)-ergodic of class \( r \), consequently, \( f \) is uniformly square mean \((\mu, \nu)\)-pseudo almost periodic of class \( r \). Moreover, \( L \) is bounded linear operator from \( C \) to \( L^2(P, L^2((0, \pi)) \).
Let \( k \) be the lipschit constant of \( h \), then for every \( \phi_1, \phi_2 \in C \) and \( t \geq 0 \), we have
\[
\mathbb{E} \| f(t, \phi_1)(x) - f(t, \phi_2)(x) \|^2 = \mathbb{E} \left\| \int_{-\tau}^{0} \left[ h(\theta, \phi_1(\theta)(x)) - h(\theta, \phi_2(\theta)(x)) \right] \right\|^2 d\theta \\
\leq \int_{-\tau}^{0} \mathbb{E} \left\| h(\theta, \phi_1(\theta)(x)) - h(\theta, \phi_2(\theta)(x)) \right\|^2 d\theta \\
\leq \int_{-\tau}^{0} k \mathbb{E} \left\| \phi_1(\theta)(x) - \phi_2(\theta)(x) \right\|^2 d\theta
\]
\[
\mathbb{E} \| f(t, \phi_1)(x) - f(t, \phi_2)(x) \|^2 \leq kr \sup_{-\tau \leq \theta \leq 0} \mathbb{E} \left\| \phi_1(\theta)(x) - \phi_2(\theta)(x) \right\|^2 \\
\leq k \alpha \mathbb{E} \left\| \phi_1(\theta)(x) - \phi_2(\theta)(x) \right\|^2 \quad \text{for a certain} \quad \alpha \in \mathbb{R}_+
\]
Consequently, we conclude that \( f \) and \( g \) are Lipschitz continuous and \( cl(\mu, \nu)\)-pseudo almost periodic of class \( r \).
Moreover, since \( h \) is stochastically bounded, i.e \( \mathbb{E} \| h(t, \phi(t)(x)) \| \leq \beta \), \( t \in \mathbb{R} \), we have
\[
\mathbb{E} \| g(t, \phi)(x) \|^2 \leq \frac{4 + \pi}{2} + \int_{-\tau}^{0} \mathbb{E} \left\| h(\theta, \phi(\theta)(x)) \right\|^2 d\theta \\
\leq \frac{4 + \pi}{2} + r, \beta \\
\leq \beta_1 \quad \text{with} \quad \beta_1 = \frac{4 + \pi}{2} + r, \beta.
Which implies that $g$ satisfies $(H_5)$

For the hyperbolicity, we suppose that

$$(H_5) \int_{-r}^{0} |G(\theta)|d\theta < 1.$$ 

**Proposition 5.1.** [11] Assume that $(H_\theta)$ and $(H_\eta)$ holds. Then the semigroup $(U(t))_{t \geq 0}$ is hyperbolic.

Then by Proposition (4.24) we deduce the following result.

**Theorem 5.2.** Under the above assumptions, if $\text{Lip}(h)$ is small enough, then equation (5.1) has a unique $\text{cl}(\mu, \nu)$-pseudo almost periodic solution $\nu$ of class $r$.

**REFERENCES**


