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ON THE SOLVABILITY OF FINITE GROUPS AND THE NUMBER OF SYLOW 2-SUBGROUPS

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Abstract. Let G be a finite group. Denoted by $n_2(G)$ the number of Sylow 2-subgroups of G. In this paper, we prove if G is non-solvable and $n_2(G)$ is a power of a prime p, then p is a Fermat prime.

1. Introduction

Let G be a finite group and p a prime. We denote by $n_p(G)$ the number of Sylow p-subgroups of G, which is called Sylow p-number of G (hereinafter referred to as Sylow number). The influence of the number of Sylow subgroups in finite groups on group structure is a very meaningful research topic. In 1967, M. Hall [1], studied the number of Sylow subgroups in finite groups, and proved that solvable group have solvable Sylow numbers, and 22 is never a Sylow 3-number and 21 a Sylow 5-number. In 1995, Zhang [2], proved that a finite group G is p-nilpotent if and only if p is prime to every sylow number of G. In 2003, G. Navarro [3] proved that if G is p-solvable, then $n_p(H)$ divides $n_p(G)$ for every $H \leq G$. In 2016 [4], Li and Liu classified finite non-abelian simple group with only solvable Sylow numbers. We say that a group G satisfies $DivSyl(p)$ if $n_p(H)$ divides $n_p(G)$ for every $H \leq G$. In 2018, Guo and E. P. Vdovin [5] generalized the results of G. Navarro, and proved that G satisfies $DivSyl(p)$ provided every non-abelian composition factor of G satisfies $DivSyl(p)$. Recently, Wu [6] proved that finite simple group does not satisfy $DivSyl(p)$. In this paper, we will study the relationship between the number of Sylow 2-subgroups and the solvability of groups. Obviously, the number of Sylow 2-subgroups is odd. By the famous Feit-Thompson odd order Theorem, if the number of Sylow 2-subgroups of G is 1, then G is solvable. A natural question is whether can we determine the solvability of G if the number of Sylow 2-subgroups of G is given? In this paper, we study the case that Sylow 2-numbers is a prime power and obtain the following main result.

Theorem. *If G is non-solvable and the number of Sylow 2-subgroups of G is a power of a prime p, then p is a Fermat prime.*

Note that a Fermat prime above means a prime of the type $2^a + 1$. Also when $n_2(G) = 3$, G is solvable. If $k \geq 2$, then there exists a non-solvable group $G = PSL(2, 8) \times S_3^{k-2}$ such that

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$n_2(G) = 3^k$. Suppose that $p = 2^a + 1 > 3$ is a Fermat prime, then there exists a non-solvable group $G = PSL(2, 2^a)^k$ such that $n_2(G) = p^k$.

2. Some Lemmas

Lemma 2.1 *Let G be a finite group and N a normal subgroup of G, then both $n_2(N)$ and $n_2(G/N)$ divide $n_2(G)$.*

Proof. Let P_2 be a Sylow 2-subgroup of G. By Theorem 2.1 in [1], we have $n_2(G) = a_2b_2c_2$, where a_2 is the number of Sylow 2-subgroups in G/N, b_2 is the number of Sylow 2-subgroups in N and c_2 is the number of Sylow 2-subgroups in $N_{P_2N}(P_2 \cap N)/P_2 \cap N$. Thus we get both $n_2(N)$ and $n_2(G/N)$ divide $n_2(G)$, as required. □

Lemma 2.2 *Let P_2 be a Sylow 2-group of $PSL(2, q)$, where q is power of odd prime, then*
(1) *if $3 < q \equiv \pm 3(mod\ 8)$, then $N_{PSL(2,q)}(P_2) \cong A_4$,*
(2) *if $3 < q \equiv \pm 1(mod\ 8)$, then $N_{PSL(2,q)}(P_2) \cong P_2$.*

Proof. Let G be a finite non-abelian simple group and P_2 a Sylow 2-subgroup of G. By Corollary in [7], we get that $N_G(P_2) \cong P_2$, except in the following case: $G \cong PSL(2, q)$, where $3 < q \equiv \pm 3(mod\ 8)$ and $N_G(P_2) \cong A_4$. Therefore, for $3 < q \equiv \pm 3(mod\ 8)$, we have $N_{PSL(2,q)}(P_2) \cong A_4$. For $3 < q \equiv \pm 1(mod\ 8)$, we know that $N_{PSL(2,q)}(P_2) \cong P_2$, as required. □

The following Lemma gives the formula for calculating the number of Sylow 2-subgroups of $PSL(2, q)$. We denoted by $2'$ and $n_{2'}$ the set of all odd primes and the $2'$-part of n (i.e. the largest odd factor of n), respectively.

Lemma 2.3

$$n_2(PSL(2,q)) = \begin{cases} q+1, \ \textit{where } q = 2^f \textit{ for } f \textit{ is a positive integer}, \\ \dfrac{q(q^2-1)}{24}, \ 3 < q \equiv \pm 3(mod\ 8), \\ (\dfrac{q(q^2-1)}{2})_{2'}, \ 3 < q \equiv \pm 1(mod\ 8). \end{cases}$$

Proof. First we denote by P_2 and n_2 the Sylow 2-subgroups of $PSL(2, q)$ and the number of Sylow 2-subgroups of $PSL(2, q)$, respectively. Next we get, by the second Sylow theorem in [8], that $n_2 = |G : N_G(P_2)|$. If $q = 2^f$, then $\left|N_{PSL(2,q)}(P_2)\right| = q(q-1)$ by [9], and so $n_2 = \left|PSL(2, q) : N_{PSL(2,q)}(P_2)\right| = \frac{q(q^2-1)}{q(q-1)} = q + 1$. If $3 < q \equiv \pm 3(mod\ 8)$, by Lemma 1.2, we have $N_{PSL(2,q)}(P_2) \cong A_4$, and then $n_2 = |PSL(2, q) : A_4| = \frac{q(q^2-1)}{24}$. Also by Lemma 1.2, if $3 < q \equiv \pm 1(mod\ 8)$, then $N_{PSL(2,q)}(P_2) \cong P_2$, and so $n_2 = |PSL(2, q) : P_2| = (\frac{q(q^2-1)}{2})_{2'}$, as required. □

Lemma 2.4 *Let p and r be primes, and m and n be positive integers. Then there exists a prime s such that $s \mid p^n - 1$ and $s \nmid p^m - 1$, where $m < n$, except $(p, n) = (2, 6)$ or $p = 2^r - 1$ is a Mersenne prime and $n = 2$.*

Proof. The Lemma follows from [10] and [11]. □

Note that the above s is called the n-th primitive prime factors of p, also known as the Zsigmondy primes. The following Lemma gives a complete classification of simple groups whose index of maximal subgroups are prime powers.

Lemma 2.5 *Let G be a finite non-abelian simple group with $H < G$ and $|G : H| = p^n$, p prime. One of the following holds.*

(1) $G = A_m$ and $H \cong A_{m-1}$ with $m = p^n$,

(2) $G = PSL(m, q)$ and H is the stabilizer of a line or hyperplane. Then $|G : H| = \frac{q^m-1}{q-1} = p^n$ (Note m must be prime),

(3) $G = PSL(2, 11)$ and $H \cong A_5$,

(4) $G = M_{23}$ and $H \cong M_{22}$ or $G = M_{11}$ and $H \cong M_{10}$,

(5) $G = PSU(4, 2) \cong PSp(4, 3)$ and H is the parabolic subgroup of index 27.

Proof. The Lemma follows immediately from Theorem 1 in [12]. □

Lemma 2.6 *Let G be a finite non-abelian simple group and P_2 a Sylow 2-group of G. If $|G : N_G(P_2)|$ is a prime power, then $G \cong PSL(2, q)$.*

Proof. Let H be a maximal subgroup of G. Suppose that $|G : N_G(P_2)|$ is a power of a prime p, then we set $|G : N_G(P_2)| = p^k$, where k is a positive integer. Now $|G : H|$ is also a prime power since $N_G(P_2) \le H$. Furthermore, by Lemma 1.5, we get that G is isomorphic to one of the following groups: A_m with $m = p^n$ and $k \ge n$, $PSL(m, q)$ for m prime, $PSL(2, 11)$, M_{23}, M_{11}, $PSU(4, 2)$.

If $G \cong A_m$ with $m = p^n$, then $|G| = \frac{m!}{2}$. By Corollary in [7], we know that $N_G(P_2) = P_2$, thus $n_2(A_m) = |A_m : N_{A_m}(P_2)| = (\frac{p^n \cdot (p^n-1)\cdot(p^n-2)\cdot\dots\cdot 2\cdot 1}{2})_{2'}$, which contradicts $n_2(A_m) = p^k$ since $p^n \ge 5$.

If $G \cong PSL(m, q)$ for m prime, then $|G| = \frac{1}{(m,q-1)} q^{\frac{m(m-1)}{2}} \prod_{i=1}^{m-1}(q^{i+1} - 1)$. Suppose first that the characteristic of G is 2 and $m \ge 3$, we see that $N_G(P_2)$ is a Borel subgroup B of G which differs from P_2 by Corollary in [7]. Moreover, by [13], we get that B is the subgroup of all lower-triangular matrices, and then $B \cong P_2 : D$, where D of $PSL(m, q)$ consisting of all diagonal matrices is easily seen to be a subgroup of order $\frac{(q-1)^{m-1}}{(m,q-1)}$. Hence

$$\begin{aligned} n_2(PSL(m, q)) &= \left|PSL(m, q) : N_{PSL(m,q)}(P_2)\right| \\ &= |PSL(m, q) : B| = \frac{q^{\frac{m(m-1)}{2}} \cdot \prod_{i=1}^{m-1}(q^{i+1} - 1) \cdot (m, q - 1)}{(m, q - 1) \cdot (q - 1)^{m-1} \cdot |P_2|} \\ &= (\frac{q^{\frac{m(m-1)}{2}} \cdot \prod_{i=1}^{m-1}(q^{i+1} - 1)}{(q - 1)^{m-1}})_{2'} = \frac{(q^2 - 1)(q^3 - 1)\cdot\dots\cdot(q^m - 1)}{(q - 1)^{m-1}}. \end{aligned}$$

By Lemma 2.4 the existence of primitive prime factor, there must exists 2-th and 3-th primitive prime factor of q in $n_2(PSL(m, q))$, so $n_2(PSL(m, q))$ has at least two different prime factors, and then $n_2(PSL(m, q)) = p^k$ is impossible. Next suppose that q is odd and $m \ge 3$. By Corollary in [7], we get that $P_2 \ne N_G(P_2) = P_2 \times C_1 \times \dots \times C_{t-1}$, where the number $t \ge 2$ can be found from the 2-adic expansion $m = 2^{s_1} + \dots + 2^{s_t}$, $s_1 > \dots > s_t \ge 0$, and $C_1, \dots, C_{t-2}, C_{t-1}$ are cyclic groups of orders $(q + 1)_{2'}, \dots, (q + 1)_{2'}, \frac{(q+1)_{2'}}{(q+1,m)_{2'}}$, respectively. Thus $|N_G(P_2)| = \frac{|P_2|\cdot((q+1)_{2'})^{t-1}}{(q+1,m)_{2'}}$, and then $n_2(PSL(m, q)) = \left|PSL(m, q) : N_{PSL(m,q)}(P_2)\right| = (\frac{q^{\frac{m(m-1)}{2}} \cdot \prod_{i=1}^{m-1}(q^{i+1}-1)\cdot(q+1,m)_{2'}}{(m,q-1)\cdot((q+1)_{2'})^{t-1}})_{2'}$. We set $i + 1 = n$. For $n \ne 2$ or q is not a Mersenne prime, we know that $n_2(PSL(m, q))$ has at least two different primitive prime factors by Lemma 1.4

the existence of primitive prime factor, and so $n_2(PSL(m,q)) = p^k$ is impossible. Next we consider the case $n = 2$ or q is a Mersenne prime of the type $2^r - 1$. Since $m \geq 3$ is a prime, we conclude that $n_2(PSL(m,q)) = \left|PSL(m,q) : N_{PSL(m,q)}(P_2)\right| = (\frac{q^{\frac{m(m-1)}{2}} \cdot \prod_{i=1}^{m-1}(q^{i+1}-1)}{(m,q-1)})_{2'}$ and $q^n - 1 = (2^r - 1)^2 - 1 = 2^{r+1}(2^{r-1} - 1)$. Assume first that $r \neq 7$, we see, by Lemma 1.4, that $n_2(PSL(m,q))$ has at least two different prime factors, and so $n_2(PSL(m,q)) = p^k$ is impossible. Next assume that $r = 7$. If $m = 3$, then $n_2(PSL(3,127)) = (\frac{127^3 \cdot (127^2-1) \cdot (127^3-1)}{3})_{2'} = (2^9 \cdot 3^5 \cdot 7^2 \cdot 127^3 \cdot 5419)_{2'} = 3^5 \cdot 7^2 \cdot 127^3 \cdot 5419$, which contradicts $n_2(PSL(m,q)) = p^k$. If $m \geq 5$ then, by Lemma 1.4 the existence of primitive prime factor, there must exists 3-th, 4-th and 5-th primitive prime factor of q in $n_2(PSL(m,q))$, and so $n_2(PSL(m,q))$ has at least three different prime factors, contrary to $n_2(PSL(m,q)) = p^k$. From the above, we get $m = 2$, and then $G \cong PSL(2,q)$.

If $G \cong M_{23}$, then $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. Moreover, by Corollary in [7], we see that $N_{M_{23}}(P_2) = P_2$, thus $n_2(M_{23}) = |M_{23} : N_{M_{23}}(P_2)| = |M_{23} : P_2| = 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$, which contradicts the fact that $n_2(M_{23}) = p^k$.

If $G \cong M_{11}$, then $|G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$. Also by Corollary in [7], we have $N_{M_{11}}(P_2) = P_2$, hence $n_2(M_{11}) = |M_{11} : N_{M_{11}}(P_2)| = |M_{11} : P_2| = 3^2 \cdot 5 \cdot 11$, contrary to $n_2(M_{11}) = p^k$.

If $G \cong PSU(4,2)$, then $|PSU(4,2)| = 25920$. On the other hand, by the GAP [14] Small-Groups package, we get $N_G(P_2) = 192$, so

$$n_2(PSU(4,2)) = \left|PSU(4,2) : N_{PSU(4,2)}(P_2)\right| = 135 = 3^3 \cdot 5,$$

which contradicts $n_2(PSU(4,2)) = p^k$, as required. □

Lemma 2.7 *If $n_2(PSL(2,q)) = p^k$, where p is a prime and k a positive integer, then p is Fermat.*

Proof. We set $n_2(PSL(2,q)) = p^k$. By Lemma 1.3, we divide three cases.

Case I. If $n_2(PSL(2,q)) = q + 1$, where $q = 2^f$ for f is a positive integer, then we have $2^f + 1 = p^k$, and so $p^k - 1 = 2^f$. Furthermore, we conclude that $p - 1 \mid 2^f$, then $p - 1 = 2^{f'}$, where $f' \leq f$ is a positive integer. Thus $p = 2^{f'} + 1$ is Fermat.

Case II. If $n_2(PSL(2,q)) = \frac{q(q^2-1)}{24}$, where $3 < q \equiv \pm 3 (mod\ 8)$, then $\frac{q(q^2-1)}{24} = p^k$, and so $\frac{q^2-1}{24} = 1$. Furthermore we get $q = p = 5$ is a Fermat prime.

Case III. If $n_2(PSL(2,q)) = (\frac{q(q^2-1)}{2})_{2'}$, where $3 < q \equiv \pm 1 (mod\ 8)$, then $(\frac{q(q^2-1)}{2})_{2'} = p^k$, and so $(q^2 - 1)_{2'} = 1$. Furthermore we set $q^2 - 1 = 2^l$, where l is a positive integer. Since $q > 3$, we have $q^2 - 1 \equiv 0 (mod\ 3)$, contrary to $q^2 - 1 = 2^l$.

Therefore, if $n_2(PSL(2,q)) = p^k$, then p is a Fermat prime, as required. □

3. Proof of Main Result

By Lemma 2.1, the Sylow number of the normal subgroups and quotient group of G is still a power of a prime p. So we need prove if p is not a Fermat prime and $n_2(G)$ is a power of a prime p, then G is solvable. Let G be a counterexample of a minimal order non-solvable group satisfying $n_2(G) = p^k$ and p is not a Fermat prime, and k is a positive integer. By Lemma 1.1, if the number of Sylow 2-subgroups of normal subgroup and factor subgroup of G is a power of p, then G must be a non-abelian simple group. Furthermore by lemma 1.6, $G \cong PSL(2,q)$. And by Lemma 1.7, we get p is a Fermat prime, a contradiction. As required. □

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