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FIXED POINT THEOREM FOR $(\phi, F)-$CONTRACTION ON C^*-ALGEBRA VALUED METRIC SPACES

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ABSTRACT. Recently, a new type of mapping called $(\phi, F)-$contraction was introduced in the literature as a generalization of the concepts of contractive mappings. This present article extends the new notion in C^*-algebra valued metric spaces and establishing the existence and uniqueness of fixed point for them. Non-trivial examples are further provided to support the hypotheses of our results.

1. INTRODUCTION

Banach's contraction principle is a fundamental result in fixed point theory. Due to its importance, several authors have obtained many interesting extensions and generalizations see [1, 4, 8, 19, 21]. This approach is particularly associated with the work of Picard, although it was Stefan Banach who in 1922 in [2] developed the ideas involved in an abstract setting. Many generalizations of the concept of metric spaces are defined and some fixed point theorems were proved in these spaces. In particular, C^*-algebra valued metric spaces were introduced by Ma et al [13] as a generalization of metric spaces they proved certain fixed point theorems, by giving the definition of C^*-algebra valued contractive mapping analogous to Banach contraction principle. many mathematicians worked on this interesting space.

Various fixed point results were established on such spaces, see [6, 9, 11, 12, 16, 17] and references therein.

Combining conditions used for definitions of C^*-algebra valued metric and generalized metric spaces, Piri et al [15] announced the notions of C^*-algebra valued metric space and establish nice results of fixed point on such space.

In this paper, inspired by the work done in [14, 18], we introduce the notion of $(\phi, F)-$contraction and establish some new fixed point theorems for mappings in the setting of complete C^*-algebra valued metric spaces. Moreover, an illustrative examples is presented to support the obtained results.

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2. PRELIMINARIES

Throughout this paper, we denote $\mathbb{A}$ by an unital (i.e ,unity element I) C^*-algebra with linear involution $*$, such that for all $x, y \in \mathbb{A}$,

$$(xy)^* = y^*x^*, \text{and } x^{**} = x.$$

We call an element $x \in \mathbb{A}$ a positive element, denote it by $x \succeq \theta$ if $x \in \mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$ and $\sigma(x) \subset \mathbb{R}_+$,where $\sigma(x)$ is the spectrum of x.Using positive element ,we can define a partial ordering $\preceq$ on $\mathbb{A}_h$ as follows :

$$x \preceq y \text{ if and only if } y - x \succeq \theta$$

where θ means the zero element in $\mathbb{A}$.

we denote the set $x \in \mathbb{A} : x \succeq \theta$ by $\mathbb{A}_+$ and $|x| = (x^*x)^{\dfrac{1}{2}}$

Remark 2.1. When $\mathbb{A}$ is a unital C^*-algebra,then for any $x \in \mathbb{A}_+$ we have

$$x \preceq I \iff \|x\| \leq 1$$

Definition 2.2. Let X be a non-empty set and $d : X \times X \to \mathbb{A}_+$ be a mapping such that for all $x, y \in X$ and for all distinct points $z \in X$, each of them different from x and y, on has

(i) $d(x, y) = \theta$ if and only if $x = y$; and $\theta \preceq d(x, y)$ for all $x, y \in X$
(ii) $d(x, y) = d(y, x)$ for all distinct points $x, y \in X$;
(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $(X, \mathbb{A}_+, d)$ is called a C^*-algebra valued metric space.

Definition 2.3. [15] Let $(X, \mathbb{A}_+, d)$ be a C^*-algebra valued metric space.

Suppose that $\{x_n\} \subset X$ and $x \in X$.

If for any $\varepsilon > 0$ there is N such that for all $n, m > N$,$\|d(x_n, x_m)\| \leq \varepsilon$,then $\{x_n\}_{n\in\mathbb{N}}$ is called a Cauchy sequence with respect to $\mathbb{A}$.

We say $(X, \mathbb{A}_+, d)$ is a complete C^*-algebra valued metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent.

It is obvious that if X is a Banach space ,then $(X, \mathbb{A}_+, d)$ is a complete C^*-algebra valued metric space if we set

$$d(x, y) = \|x - y\|I$$

Example 2.4. Consider $X = \mathbb{R}$ and $\mathbb{A} = \mathbb{M}_2(\mathbb{R})$

Let $d : X \times X \to \mathbb{M}_2(\mathbb{R})$ be mapping defined by

$$d(x, y) = diag(|x - y|, \alpha|x - y|)$$

where $x, y \in \mathbb{R}$ and $\alpha > 0$ is a constant.It is clearly that d is a C^*-algebra valued metric and $(X, \mathbb{M}_2(\mathbb{R}), d)$ is a complete C^*-algebra valued metric space by the completeness of $\mathbb{R}$.

The following definition was given by D.Wardowski in [5].

Definition 2.5. [19] Let $\mathcal{F}$ be the family of all functions $F\colon\ \mathbb{R}_+ \to \mathbb{R}$ and Φ be the family of all functions $\phi :]0, +\infty[\to]0, +\infty[$ satisfying:

(i) F is strictly increasing ; ie for $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta$,$F(\alpha) < F(\beta)$.
(ii) For each sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive numbers

$$\lim_{n\to 0} x_n = 0, \quad \text{if and only if} \quad \lim_{n\to\infty} F(x_n) = -\infty;$$

(iii) $lim\ inf_{s\to\alpha^+}\phi(s) > 0$ for all $s > 0$

(iv) There exists $k \in]0,1[$ such that $\lim_{x\to 0} x^k F(x) = 0$.

Definition 2.6. [20] Let (X, d) be a complete metric space. A mapping $T : X \to X$ is called an $(\phi, F)-$ contraction on (X, d) if there exists $F \in \mathcal{F}$ and $\phi \in \Phi$ such that

$$(d(Tx, Ty) > 0 \Rightarrow F(d(Tx, Ty) + \phi(d(x, y)) \leq F(d(x, y))$$

for all $x, y \in X$ for which $Tx \neq Ty$

Theorem 2.7. *Let (X, d) be a complete metric space and $T : X \to X$ be an $(\phi, F)-$ contraction. Then T has a unique fixed point.*

Definition 2.8. [22] Let the function $\phi : A^+ \to A^+$ be positive if having the following constraints :

(i) ϕ is continous and nondecrasing

(ii) $\phi(a) = \theta$ if and only if $a = \theta$

(iii) $lim_{n\longrightarrow\infty}\phi^n(a) = \theta$

Definition 2.9. [22] Suppose that A and B are C^*-algebra .
A mapping $\phi : A \to B$ is said to be C^*- homomorphism if :

(i) $\phi(ax + by) = a\phi(x) + b\phi(y)$ for all $a, b \in \mathbb{C}$ and $x, y \in A$

(ii) $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A$

(iii) $\phi(x^*) = \phi(x)^*$ for all $x \in A$

(iv) ϕ maps the unit in A to the unit in B.

Definition 2.10. [22] Let A and B be C^*-algebra spaces and let $\phi : A \to B$ be a homomorphism
then ϕ is called an $*-$ homomorphism if it is one to one $*-$ homomorphism.
A C^*-algebra A is $*-$isomorphic to a C^*-algebra B if there exists $*-$ isomorphism of A onto B.

Lemma 2.11. *[7] Let A and B be C^*-algebra spaces and $\phi : A \to B$ is a C^*- homomorphism for all $x \in A$ we have*

$$\sigma(\phi(x)) \subset \sigma(x) \text{ and } \|\phi(x)\| \leq \|\phi\|.$$

Corollary 2.12. *[22] Every C^*- homomorphism is bounded.*

Corollary 2.13. *[22] Suppose that ϕ is C^*- isomorphism from A to B, then $\sigma(\phi(x)) = \sigma(x)$ and $\|\phi(x)\| = \|\phi\|$ for all $x \in A$.*

Lemma 2.14. *[22] Every $*-$ homomorphism is positive.*

3. Main result

Aspired by Wardowski in [10], we introduce the notion of (ϕ, F)-contraction on C^*-algebra valued metric space.

Definition 3.1. Let

$$F : \mathbb{A}_+ \to \mathbb{A}_+$$

a function satisfying:

(i) F is continuous and nondecreasing .

(ii) $F(t) = \theta$ if and only if $t = \theta$.

1. A mapping $T : X \to X$ is said to be a (ϕ, F) C^* valued contraction of type (I) if there exists $\phi : \mathbb{A}_+ \to \mathbb{A}_+$ an $*-$ homomorphism such that

$$\forall x, y \in X \ (d(Tx, Ty) \succeq \theta \Rightarrow F(d(Tx, Ty) + \phi(d(x, y)) \preceq F(d(x, y)) \ , \qquad (1)$$

2. A mapping $T : X \to X$ is said to be a (ϕ, F) C^* valued contraction of type (II) if there exists $\phi : \mathbb{A}_+ \to \mathbb{A}_+$ an $*-$ homomorphism satisfying:

(a) $\phi(a) \prec a$ for $a \in \mathbb{A}_+$

(b) Either $\phi(a) \preceq d(x, y)$ or $d(x, y) \preceq \phi(a)$, where $a \in \mathbb{A}_+$ and $x, y \in X$

(c) $F(a) \prec \phi(a)$ Such that

$$(d(Tx, Ty) \succeq \theta \Rightarrow F(d(Tx, Ty) + \phi(d(x, y)) \preceq F(M(x, y))$$

Where $M(x, y) = a_1 d(x, y) + a_2[d(Tx, y) + d(Ty, x)] + a_3[d(Tx, x) + d(Ty, y)]$, with $a_1, a_2, a_3 \geq 0$,$a_1 + 2a_2 + 2a_3 \leq 1$

3. T is said to be $(\phi, F)-$ Kannan-type C^*- valued contraction if there exist ϕ satisfy (a) , (b) and (c) such that $(d(Tx, Ty) \succeq \theta$ we have

$$F(d(Tx, Ty) + \phi(d(x, y)) \preceq F(\frac{d(x, Tx) + d(y, Ty)}{2}).$$

4. T is said to be $(\phi, F)-$ Reich-type C^*- valued contraction if there exist ϕ satisfy (a) , (b) and (c) such that $(d(Tx, Ty) \succeq \theta$ we have

$$F(d(Tx, Ty) + \phi(d(x, y)) \preceq F(\frac{d(x, y) + d(x, Tx) + d(y, Ty)}{3}).$$

Example 3.2. Let $X = [0, 1]$ and $\mathbb{A} = \mathbb{R}^2$ Then $\mathbb{A}$ is a C^*- algebra with norm $\|.\| : \mathbb{A} \to \mathbb{R}$ defined by

$$\|(x, y)\| = (x^2 + y^2)^{\frac{1}{2}}.$$

Define a C^*- algebra valued metric $d : X \times X \to \mathbb{A}$ on X by

$$d(x, y) = (|x - y|, 0)$$

With ordering on $\mathbb{A}$ by

$$(a, b) \preceq (c, d) \ \Leftrightarrow a \leq c \text{ and } b \leq d$$

A mapping $T : X \to X$ given by $Tx = \dfrac{x}{3}$ is continuous with respect to $\mathbb{A}$.

Let $F : \mathbb{A}_+ \to \mathbb{A}_+$. Defined by

$$F(x, y) = ((x - y)^2, 0)$$

It is clear that F satisfies (i) and (ii)

We have $F(d(Tx, Ty)) = F(d(\frac{x}{3}, \frac{y}{3})) = F((\frac{x}{3} - \frac{y}{3}))^2, 0)$.

And $(\frac{x}{3} - \frac{y}{3}))^2 - (x - y)^2 \leq -\frac{1}{3}(x - y)^2$. Therefore T is a valued (ϕ, F) C^* -valued contraction of type (I) with $\phi(d(x, y)) = (\frac{1}{3}(x - y)^2, 0)$.

Example 3.3. Let $X = [0, 1] \cup \{2, 3, 4, ...\}$ and $\mathbb{A} = \mathbb{C}$ with a norm $\| \ z \ \| = | \ z \ |$ be a C^*- algebra.We define $\mathbb{C}^+ = \{z = (x, y) \in \mathbb{C}; x = Re(z) \geq 0, y = Im(z) \geq 0\}$.

The partial order $\leq$ with respect to the C^*- algebra $\mathbb{C}$ is the partial order in $\mathbb{C}$, $z_1 \leq z_2$ if

$Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$ for any two elements z_1, z_2 in $\mathbb{C}$.
Let $d : X \times X \to \mathbb{C}$

$$d(x,y) = \{ \begin{array}{l} (|x-y|\,, |x-y|)\, if\ x,y \in [0,1], x \neq y \\ (x+y, x+y)\ \ if\, at\, least\, one\, of\, x\, or\, y \notin [0,1]\ \ and\, x \neq y \\ (0,0) if\ x = y \end{array}$$

Then $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space.
Let $F : \mathbb{C}^+ \to \mathbb{C}$ be defined as

$$F(t) = \{ \begin{array}{l} t\ \ if\ t \in [0,1], \\ t^2,\ \ if\ t > 1 \end{array}$$

It is clear that F satisfies(i) and (ii) Let $T : X \to X$ be defined as

$$T(x) = \{ \begin{array}{l} x - \frac{1}{2}x^2\ \ if\ x \in [0,1], \\ x-1,\ \ if\ x \in \{2,3,4,...\} \end{array}$$

Without loss of generality, we assume that $x > y$ and discuss the following cases.
Case $1(x \in [0;1])$.
Then

$$F(d(Tx,Ty)) = ((x - \frac{1}{2}x^2) - (y - \frac{1}{2}y^2), (x - \frac{1}{2}x^2) - (y - \frac{1}{2}y^2))$$
$$= ((x-y) - \frac{1}{2}(x-y)(x+y), (x-y) - \frac{1}{2}(x-y)(x+y))$$
$$\leq ((x-y) - \frac{1}{2}((x-y))^2, (x-y) - \frac{1}{2}((x-y))^2)$$
$$= d(x,y) - \frac{1}{2}(d(x,y))^2$$
$$= F(d(x,y)) - \frac{1}{2}(d(x,y))^2$$

Then there exists ϕ such $\phi(d(x,y)) = \frac{1}{2}(d(x,y))^2$ and

$$\forall x,y \in X \quad (d(Tx,Ty) \geq 0 \Rightarrow F(d(Tx,Ty) + \phi(d(x,y)) \leq F(d(x,y)).$$

Case $2(x \in \{3,4,...\})$.
Then

$$d(Tx,Ty) = d(x-1, y - \frac{1}{2}y^2) \text{ if } y \in [0,1]$$

or

$$d(Tx,Ty) = (x - 1 + y - \frac{1}{2}y^2, x - 1 + y - \frac{1}{2}y^2) \leq (x+y-1, x+y-1)$$
$$d(Tx,Ty) = d(x-1, y-1) \text{ if } y \in \{2,3,4,...\}$$

or

$$d(Tx,Ty) = (x+y-2, x+y-2) < (x+y-1, x+y-1)$$

.

Consequently

$$F(d(Tx,Ty)) = (d(Tx,Ty))^2 \leq ((x+y-1)^2, (x+y-1)^2)$$
$$< ((x+y-1)(x+y+1), (x+y-1)(x+y+1))$$

$$= ((x+y)^2-1,(x+y)^2-1) < ((x+y)^2-\frac{1}{2},(x+y)^2-\frac{1}{2})$$
$$= F(d(x,y)) - \frac{1}{2}$$

Case $3(x=2)$.
Then $y \in [0,1]$,$Tx = 1$,
and

$$d(Tx,Ty) = (1-(y-\frac{1}{2}y^2), 1-(y-\frac{1}{2}y^2))$$

So, we have $F(d(Tx,Ty)) \leq F(1) = 1$.
Again $d(x,y) = (2+y, 2+y)$.
So,

$$1 = F(d(Tx,Ty)) \leq F(d(x,y)) - \frac{1}{2}$$

Example 3.4. Let $X = \{\frac{1}{2},\frac{1}{3},\frac{1}{4},1\}$. Let $\mathbb{A}_+ = \{(x,y) \in \mathbb{R}^2 : x,y \geq 0\}$.
Define $d : X \times X \to \mathbb{A}_+$ as follows:

$$\{\begin{array}{l} d(x,y) = d(y,x) \;\; for\, x,y \in X, \\ d(x,y) = (0,0) \Leftrightarrow x = y \\ d(\frac{1}{2},1) = (0.5,0.5) \\ d(\frac{1}{2},\frac{1}{4}) = (2,3) \\ d(\frac{1}{2},\frac{1}{3}) = (2,2.5) \\ d(1,\frac{1}{3}) = (2,2.5) \\ d(1,\frac{1}{4}) = (2.3) \\ d(\frac{1}{3},\frac{1}{4}) = (2,2.6) \end{array}$$

Let $F,\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that they can defined as follows:
for $t = (x,y) \in \mathbb{R}^2$,

$$F(t) = \{\begin{array}{l} (x,y) \;\; if x \leq 1 \;\; and\, y \leq 1 \\ (x^2,y) \;\; if x > 1, y \leq 1 \\ (x,y^2)\; if x \leq 1\; and\, y > 1 \\ (x^2,y^2) \;\; if x > 1 \; and\, y > 1 \end{array}$$

and for $s = (s_1,s_2) \in \mathbb{R}^2$ with $v = min\{s_1,s_2\}$,

$$\phi = \{\begin{array}{l} (\frac{v^2}{2},\frac{v^2}{2}) \;\; if v \leq 1 \\ (\frac{1}{2},\frac{1}{2}) \;\; if v > 1 \end{array}$$

Define mapping $T : X \to X$ by $T(\frac{1}{2}) = 1$,$T(1) = 1$,$T(\frac{1}{4}) = \frac{1}{2}$ and $T(\frac{1}{3}) = 1$.
Then T can verified that

$$F(d(Tx,Ty) + \phi(d(x,y)) \preceq F(M(x,y))$$

for $a_1 = \frac{1}{2}$,$a_2 = \frac{1}{8}$ and $a_3 = \frac{1}{8}$

Theorem 3.5. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space and let $T : X \to X$ be a (ϕ, F)-contraction mapping of type (I).*
Then T has a unique fixed point $x* \in X$ and for every $x_0 \in X$ a sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to $x*$.

Proof. : First ,let us observe that T has at most one fixed point.
Indeed if

$$x_1^* \ ;x_2^* \in X \ Tx_1^* = x_1^* \neq x_2^* = Tx_2^*$$

then we get

$$\phi(d(x,y)) \preceq F(d(x_1^*;x_2^*) - F(d(Tx_1^*;Tx_2^*)) = \theta$$

wich is a contradiction.
In order to show that thas a fixed point let $x_0 \in X$ be arbitrary and fixed we define a sequence

$$\{x_n\}_{n\in\mathbb{N}} \subset X \ ;x_{n+1} = Tx_n, \ n = 0;1;2....$$

denote

$$d_n = d(x_{n+1};x_n); n = 0;1;2;...$$

if there exists $n_0 \in \mathbb{N}$ for which $x_{n_0+1} = x_{n_0}$ then $Tx_{n_0} = x_{n_0}$ and the proof is finished.

Suppose now that $x_{n+1} \neq x_n$ for every $n \in X$ then $d_n \succ \theta$ for all $n \in \mathbb{N}$ and using (1) the following holds for every $n \in \mathbb{N}$

$$F(d_n) \preceq F(d_{n-1}) - \phi(d_{n-1}) \prec F(d_{n-1}) \qquad (2)$$

Hence F is non decreasing and so the sequence (d_n) is monotonically decreasing in $\mathbb{A}_+$. So there exists $\theta \preceq t \in \mathbb{A}_+$ such that

$$d(x_n, x_{n+1}) \to t \ as \ \ n \to \infty$$

From (2) we obtain $lim_{n\to\infty}F(d_n) = \theta$ that together with (ii) gives

$$lim_{n\to\infty}d_n = \theta \qquad (3)$$

Now we shall show that $\{x_n\}$ is a Cauchy sequence in $(X, \mathbb{A}, d)$.To prove it ,we shall that

$$lim_{n\to\infty}d_n = \theta.$$

Assume that $\{x_n\}$ is not a Cauchy sequence in $(X, \mathbb{A}, d)$.
Then exist $\varepsilon > 0$ and subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ with $n_k > m_k > k$ such that

$$\|d(x_{m_k}, x_{n_k})\| \geq \varepsilon$$

Now , corresponding to m_k , wecan choose n_k such that it isthe smallest integr with $n_k > m_k$ and satisfing above inequality. Hence

$$\|d(x_{m_k}, x_{n_{k-1}})\| < \varepsilon$$

So we have

$$\varepsilon \leq \|d(x_{m_k}, x_{n_k})\| \leq \|d(x_{m_k}, x_{n_{k-1}})\| + \|d(x_{n_{k-1}}, x_{n_k})\| \leq \varepsilon + \|d(x_{n_{k-1}}, x_{n_k})\|$$

Using (3) we have

$$\varepsilon \leq lim_{k\to\infty}\|d(x_{m_k}, x_{n_k})\| < \varepsilon + \theta.$$

This implies

$$lim_{k\to\infty}\|d(x_{m_k}, x_{n_k})\| = \varepsilon. \qquad (4)$$

Again,

$$\begin{aligned}\|d(x_{n_k}, x_{m_k})\| &\leq \|d(x_{n_k}, x_{n_{k-1}})\| + \|d(x_{n_{k-1}}, x_{m_k})\| \\ &\leq \|d(x_{n_k}, x_{m_{k-1}})\| + \|d(x_{n_{k-1}}, x_{m_{k-1}})\| + \|d(x_{m_{k-1}}, x_{m_k})\| \qquad (5)\end{aligned}$$

Also,

$$\begin{aligned}\|d(x_{n_{k-1}}, x_{m_{k-1}})\| \leq \|d(x_{n_{k-1}}, x_{n_k})\| + \|d(x_{n_k}, x_{m_{k-1}})\|\|d(x_{n_{k-1}}, x_{n_k})\| \\ + \|d(x_{n_k}, x_{m_k})\| + \|d(x_{m_k}, x_{m_{k-1}})\|. \qquad (6)\end{aligned}$$

Letting $k \to \infty$ in (5) and (6) and using (4) we have

$$lim_{k\to\infty}\|d(x_{n_{k-1}}, x_{m_{k-1}})\| = \varepsilon.$$

Since $d(x_{n_{k-1}}, x_{m_{k-1}})$, $d(x_{n_k}, x_{m_k}) \in \mathbb{A}_+$ and

$$lim_{k\to\infty}\|d(x_{n_{k-1}}, x_{m_{k-1}})\| = lim_{k\to\infty}\|d(x_{n_k}, x_{m_k})\| = \varepsilon$$

. there is exists $s \in \mathbb{A}_+$ with $\|s\| = \varepsilon$ such that

$$lim_{k\to\infty}\|d(x_{n_{k-1}}, x_{m_{k-1}})\| = lim_{k\to\infty}\|d(x_{n_k}, x_{m_k})\| = s \qquad (7)$$

by 7 we have

$$F(s) = lim_{k\to\infty}F(d(x_{n_k}, x_{m_k})) \preceq lim_{k\to\infty}F(d(x_{n_{k-1}}, x_{m_{k-1}}))$$

Therefore

$$F(s) \prec F(s)$$

Thus $F(s) = \theta$ and so $s = \theta$ which is a contradiction .Hence $\{x_n\}$ is a Cauchy sequence in $(X, \mathbb{A}, d)$. Hence there exist $z \in X$ such that

$$lim_{n\to\infty}d(x_n, z) = \theta$$

Now ,we shall show that z is fixed point of T .Using (7) ,we get

$$F(d(x_n, Tz)) \prec F(d(x_{n-1}, z))$$

Letting $n \to \infty$ and using the concept of continuity of the function of T.
We have $d(z, Tz) = \theta$.Hence by Definition 2.2 ,we have $Tz = z$.
wich completes the proof.

□

Example 3.6. Considering all cases in Example 3.3 , we conclude that inequality (1) remains valid for F and T constructed as above and consequently by an application of Theorem 3.4 ,T has a unique fixed point.
it is seen that 0 is the unique fixed point of T.

Theorem 3.7. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space.*
Let $T : X \to X$ be a (ϕ, F) of type (II), i.e, there exist F and ϕ two $*-$ homomorphisms such that for any $x, y \in X$ we have

$$(d(Tx, Ty) \succeq \theta \Rightarrow F(d(Tx, Ty) + \phi(d(x, y)) \preceq F(M(x, y))$$

Where $M(x, y) = a_1d(x, y) + a_2[d(Tx, y) + d(Ty, x)] + a_3[d(Tx, x) + d(Ty, y)]$, with $a_1, a_2, a_3 \geq 0$,$a_1 + 2a_2 + 2a_3 \leq 1$.
Then, T has a fixed point.

Proof. Let $x_0 \in X$ and define $x_1 = Tx_0, x_2 = Tx_1, ..., x_n = Tx_{n-1}$.
We have

$$F(d(x_{n+2}, x_{n+1})) = F(d(Tx_{n+1}, Tx_n)) \preceq F(M(x_{n+1}, x_n)) + \phi(d(x_{n+1}, x_n)) = F(a_1d(x_{n+1}, x_n) + a_2[d(x_{n+2}, x_n) + d(x_{n+1}, x_{n+1})] + a_3[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)]) - \phi(d(x_{n+1}, x_n)).$$

Then we have

$$F(d(x_{n+2}, x_{n+1})) \preceq F(a_1d(x_{n+1}, x_n) + a_2[d(x_{n+2}, x_n) + d(x_{n+1}, x_{n+1})] + a_3[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)])$$

Using the strongly monotone proprety of F, we have
$d(x_{n+2}, x_{n+1}) \preceq a_1d(x_{n+1}, x_n) + a_2[d(x_{n+2}, x_n) + d(x_{n+1}, x_{n+1})] + a_3[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)]$.
That is

$$(1 - a_2 - a_3)d(Tx_{n+1}, Tx_n) \preceq (a_1 + a_2 + a_3)d(x_{n+1}, x_n).$$

Therefore

$$d(x_{n+2}, x_{n+1}) \preceq \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} d(x_{n+1}, x_n).$$

Wich implies that

$$d(x_{n+2}, x_{n+1}) \preceq d(x_{n+1}, x_n).$$

Since

$$\frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} < 1$$

Therefore $\{d(x_{n+1}, x_n)\}$ is monotone decreasing sequence. There exists $u \in \mathbb{A}_+$ such that $d(x_{n+1}, x_n) \to u$ as $n \to \infty$.
Taking $n \to \infty$ in

$$F(d(x_{n+2}, x_{n+1})) \preceq F(a_1d(x_{n+1}, x_n) + a_2[d(x_{n+2}, x_n) + d(x_{n+1}, x_{n+1})] + a_3[d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)])$$

Using the continuities of F and ϕ, we have

$$F(u) \preceq F((a_1 + 2a_2 + 2a_3)u) - \phi(u)$$

wich implies that $F(u) \preceq F(u) - \phi(u)$ since $a_1 + 2a_2 + 2a_3 \leq 1$ and F is strongly monotonic increasing wich is a contradiction unless $u = \theta$. Hence $d(x_{n+1}, x_n) \to \theta$ as $n \to \infty$ (8).
Next
we show that $\{x_n\}$ is a Cauchy sequence.
If $\{x_n\}$ is not a Cauchy sequence then there exists $c \in \mathbb{A}$ such that $\forall n_0 \in \mathbb{N}$,$\exists n, m \in \mathbb{N}$ with $n > m \geq n_0$

$F(c) \preceq d(x_n, x_m)$.Therefore there exists sequences $\{m_k\}$ and $\{n_k\}$ in $\mathbb{N}$ such that for all positive integers k, $n_k > m_k > k$ and
$d(x_{n_{(k)}}, x_{m_{(k)}}) \succeq \phi(c)$ and $d(x_{n_{(k)-1}}, x_{m_{(k)}} \preceq \phi(c)$
then

$$\phi(c) \preceq d(x_{n_{(k)}}, x_{m_{(k)}}) \preceq [d(x_{n_{(k)}}, x_{n_{(k)-1}}) + d(x_{n_{(k)-1}}, x_{m_{(k)}})$$

that is

$$\phi(c) \preceq d(x_{n_{(k)}}, x_{m_{(k)}}) \preceq [d(x_{n_{(k)}}, x_{n_{(k)-1}}) + \phi(c)]$$

letting $k \to \infty$ we have

$$lim_{k\to\infty} d(x_{n_{(k)}}, x_{m_{(k)}}) = \phi(c) \qquad (9)$$

again

$$d(x_{n_{(k)}}, x_{m_{(k)}}) \preceq [d(x_{n_{(k)}}, x_{n_{(k)+1}}) + d(x_{n_{(k)+1}}, x_{m_{(k)+1}}) + d(x_{m_{(k)+1}}, x_{m_{(k)}})]$$

and

$$d(x_{n_{(k)+1}}, x_{m_{(k)+1}}) \preceq [d(x_{n_{(k)+1}}, x_{n_{(k)}}) + d(x_{n_{(k)}}, x_{m_{(k)}}) + d(x_{m_{(k)}}, x_{m_{(k)+1}})]$$

letting $k \to \infty$ in above inequalities , we have

$$lim_{k\to\infty} d(x_{n_{(k)+1}}, x_{m_{(k)+1}}) = \phi(c) \qquad (10)$$

Again

$$d(x_{n_{(k)}}, x_{m_{(k)+1}}) \preceq [d(x_{n_{(k)}}, x_{m_{(k)}}) + d(x_{m_{(k)}}, x_{m_{(k)+1}})]$$

and

$$d(x_{n_{(k)+1}}, x_{m_{(k)}}) \preceq [d(x_{n_{(k)+1}}, x_{n_{(k)}}) + d(x_{n_{(k)}}, x_{m_{(k)+1}}) + d(x_{m_{(k)+1}}, x_{m_{(k)}})]$$

Further,

$$d(x_{n_{(k)+1}}, x_{m_{(k)}}) \preceq [d(x_{n_{(k)+1}}, x_{n_{(k)}}) + d(x_{n_{(k)}}, x_{m_{(k)}})]$$

and

$$d(x_{n_{(k)}}, x_{m_{(k)}}) \preceq [d(x_{n_{(k)}}, x_{n_{(k)+1}}) + d(x_{n_{(k)+1}}, x_{m_{(k)}})]$$

Letting $k \to \infty$ in the above four inequalities we have

$$lim_{k\to\infty} d(x_{n_{(k)}}, x_{m_{(k)+1}}) = \phi(c) \qquad (11)$$

$$lim_{k\to\infty} d(x_{n_{(k)+1}}, x_{m_{(k)}}) = \phi(c) \qquad (12)$$

Using (8), (9), (11), *and* (12) we have

$$lim_{k\to\infty} M(x_{n_{(k)}}, x_{m_{(k)}}) = lim_{k\to\infty} a_1 d(x_{n_{(k)}}, x_{m_{(k)}}) + a_2[d(x_{n_{(k)}}, x_{m_{(k)}}) + d(x_{m_{(k)}}, x_{m_{(k)+1}})] + a_3[d(x_{n_{(k)}}, x_{m_{(k)+1}}) + d(x_{m_{(k)}}, x_{n_{(k)+1}})]$$

$$= (a_1 + 2a_2)\phi(c) \qquad (13)$$

Clearly $x_{m_k} \preceq x_{n_k}$.Putting $x = x_{n_{(k)}}$,$y = x_{m_{(k)}}$

$$F(d(x_{n_{(k)+1}}, x_{m_{(k)+1}})) = F(d(Tx_{n_{(k)}}, Tx_{m_{(k)}})) \preceq F(M(x_{n_{(k)}}, x_{m_{(k)}})) - \phi(x_{n_{(k)}}, x_{m_{(k)}})$$

Letting $k \to \infty$ in the above inequality using (9), (10) *and*(13) and the continuities of F and ϕ we have

$$F(\phi(c)) \preceq F((a_1 + 2a_2)\phi(c)) - \phi(\phi(c))$$

that is
$F(\phi(c)) \preceq F(\phi(c)) - \phi(\phi(c))$,(since $(a_1 + 2a_2) < 1$) and F is strongly monotonic increasing .Which a contradiction by virtue of a proprety of ϕ. Hence $\{x_n\}$is a Cauchy sequence .From the completness of X, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Since T is continous and $Tx_n \to Tz$ as $n \to \infty$ that is $lim_{n\to\infty} x_{n+1} = Tz$, that is $z = Tz$. Hence z is a fixed point of T. □

Example 3.8. Let $X = [0, 1]$ and $\mathbb{A} = \mathbb{C}$ with a norm $\| z \| = | z |$ be a C^*- algebra.
We define $\mathbb{C}^+ = \{z = (x, y) \in \mathbb{C}; x = Re(z) \geq 0, y = Im(z) \geq 0\}$.
The partial order $\leq$ with respect to the C^*- algebra $\mathbb{C}$ is the partial order in $\mathbb{C}$, $z_1 \leq z_2$ if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$ for any two elements z_1, z_2 in $\mathbb{C}$.
Let $d : X \times X \to \mathbb{C}$
Suppose that $d(x, y) = (| x - y |, | x - y |)$ for $x, y \in X$.
Then ,$(X, \mathbb{C}, d)$ is a C^*- algebra valued metric space with the required propreties of theorem 3.8.
Let $F, \phi : \mathbb{C}^+ \to \mathbb{C}^+$ such that they can defined as follows:
for $t = (x, y) \in \mathbb{C}^+$,

$$F(t) = \begin{cases} (x, y) \;\; if x \leq 1 \;\; and\, y \leq 1 \\ (x^2, y) \;\; if x > 1, y \leq 1 \\ (x, y^2)\; if x \leq 1\; and\, y > 1 \\ (x^2, y^2) \;\; if x > 1\; and\, y > 1 \end{cases}$$

and for $s = (s_1, s_2) \in \mathbb{C}^+$ with $v = min\{s_1, s_2\}$,

$$\phi = \begin{cases} (\frac{v^2}{2}, \frac{v^2}{2}) \;\; if v \leq 1 \\ \\ (\frac{1}{2}, \frac{1}{2}) \;\; if v > 1 \end{cases}$$

Then F and ϕ have the propreties mentioned in definitions 2.8 and 2.9.

Let $T : X \to X$ be defined as follows : $T(x) = \begin{cases} 0 \;\; if 0 \leq x \leq \frac{1}{2} \\ \\ \frac{1}{16} \;\; if \frac{1}{2} < x \leq 1 \end{cases}$

Then ,T has the required properties montioned in theorem 3.8.

Let $a_1 = \frac{1}{2}$,$a_2 = \frac{1}{8}$ and $a_3 = \frac{1}{8}$. It can be verified that

$$F(d(Tx, Ty)) \preceq F(M(x, y)) - \phi(d(x, y)) \text{ for all } x, y \in X \text{ with } y \preceq x$$

the conditions of theorem 3.8 are satisfied .Here it is seen that 0 is a fixed point of T.

Theorem 3.9. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space. Let $T : X \to X$ be a $(\phi, F)-$ Kannan-type C^*- valued contraction.Then T has a unique fixed point.*

Proof. Since T is a $(\phi, F)-$ Kannan-type C^*- valued contraction, then exist F and ϕ such that $F(d(Tx, Ty) + \phi(d(x, y)) \preceq F(\frac{d(x, Tx) + d(y, Ty)}{2}) \preceq F(M(x, y))$. where $M(x, y) = a_1 d(x, y) + a_2[d(Tx, y) + d(Ty, x)] + a_3[d(Tx, x) + d(Ty, y)]$ with $a_1 = 0$,$a_2 = 0$ and $a_3 = \frac{1}{2}$.As in the proof of theorem 3.7 T has a fixed point.

□

Theorem 3.10. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space. Let $T : X \to X$ be a $(\phi, F)-$ Reich-type C^*- valued contraction.Then T has a unique fixed point.*

Proof. By taking $a_1 = \frac{1}{3}$,$a_2 = 0$ and $a_3 = \frac{1}{3}$ we have

$$F(d(Tx, Ty) + \phi(d(x, y)) \preceq F(M(x, y)) = F(\frac{d(x, y) + d(x, Tx) + d(y, Ty)}{3}).$$

As in the proof of Theorem 3.7 T has a fixed point. □

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