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INVERSE SUM INDEG INVARIANT OF SOME GRAPHS

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ABSTRACT. The inverse sum indeg invariant $ISI(\Omega)$ of a simple graph Ω is defined as the sum of the terms $\frac{\gamma_\Omega(u)\gamma_\Omega(v)}{\gamma_\Omega(u)+\gamma_\Omega(v)}$ over all edges uv of Ω, where $\gamma_\Omega(u)$ denotes the degree of a vertex u of Ω. In this paper, we present several lower and upper bounds for inverse sum indeg invariant of some standard graphs.

1. INTRODUCTION

Molecular descriptors, results of functions mapping molecule's chemical information into a number [24], have found applications in modeling many physicochemical properties in QSAR and QSPR studies [3, 11]. A particularly common type of molecular descriptors are those that are defined as functions of the structure of the underlying molecular graph, such as the Wiener invariabt [26], the Zagreb indices [9], the Randić invariabt [19] or the Balaban J-invariabt [1]. Damir Vukicević and Marija Gasperov [25] observed that many of these descritors are defined simply as the sum of individual bond contributions.

Among the 148 discrete Adriatic indices studied in [25], whose predictive properties were evaluated against the benchmark datasets of the Internation Academy of Mathematical Chemistry [16], 20 invariants were selected as significant predictors of physicochemical properties. In this connection, Sedlar et al. [20] studied the properties of the inverse sum indeg invariant, the descriptor that was selected in [25] as a significant predictor of total surface area of octane isomers and for which the extremal graphs obtained with the help of Math. Chem. have a particularly simple and elegant structure. The inverse sum indeg invariant is defined as

$$ISI(\Omega) = \sum_{uv\in E(\Omega)} \frac{1}{\frac{1}{\gamma_\Omega(u)}+\frac{1}{\gamma_\Omega(v)}} = \sum_{uv\in E(\Omega)} \frac{\gamma_\Omega(u)\gamma_\Omega(v)}{\gamma_\Omega(u)+\gamma_\Omega(v)}.$$

Extremal values of inverse sum indeg invariant across several graph classes, including connected graphs, chemical graphs, trees and chemical trees were determined in [20]. The bounds of a descriptor are important information of a molecular graph in the sense that they establish the approximate range of the descriptor in terms of molecular structural parameters. In [6], some sharp bounds for the inverse sum indeg invariant of connected graphs are given. The inverse sum indeg invariant of some nanotubes is computed in [7]. In this connection, we present

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several upper and lower bounds on the inverse sum indeg invariant in terms of some molecular structural parameters and relate this invariant to various well-known molecular descriptors.

The Zagreb indices are amoung the oldest topological indices, and were introduced by Gutman and Trinajstić [9] in 1972. These indices have since been used to study molecular complexity, chirality, ZE-ismerism and hetero-systems. The first and second Zagreb indices of Ω are denoted by $M_1(\Omega)$ and $M_2(\Omega)$, respectively, and defined as $M_1(\Omega) = \sum\limits_{v\in V(\Omega)} (\gamma_\Omega(v))^2 = \sum\limits_{uv\in E(\Omega)} (\gamma_\Omega(u)+\gamma_\Omega(v))$ and $M_2(\Omega) = \sum\limits_{uv\in E(\Omega)} \gamma_\Omega(u)\gamma_\Omega(v)$.

Another variant of the Randić connectivity invariant named the harmonic invariant was introduced by Fajtlowicz [4] in 1987. The *harmonic invariant* $H(\Omega)$ of Ω is defined as $H(\Omega) = \sum\limits_{uv\in E(\Omega)} \frac{1}{\gamma_\Omega(u)+\gamma_\Omega(v)}$. The inverse degree of a connected graph Ω is defined as $ID(\Omega) = \sum\limits_{u\in E(\Omega)} \frac{1}{\gamma_\Omega(u)}$. In this paper, we present several lower and upper bounds for inverse sum indeg invariant of some derived graphs

2. Lower bounds for ISI

In this section, we obtain the lower bounds for inverse sum indeg invariant of a connected graph.

Theorem 2.1. *Let Ω be a graph with m edges. Then $ISI(\Omega) \geq m\Big(1+ln\Big(\frac{\delta(\Omega)^2}{2\Delta(\Omega)}\Big)\Big)$ with equality if and only if Ω is 2-regular graph.*

Proof. Assume that the function $f(x) = x - ln\ x - 1$. One can easily prove that $f(x) \geq 0$.

Hence for any edge $uv \in E(\Omega)$,

$$\frac{\gamma_\Omega(u)\gamma_\Omega(v)}{\gamma_\Omega(u)+\gamma_\Omega(v)} - ln\Big(\frac{\gamma_\Omega(u)\gamma_\Omega(v)}{\gamma_\Omega(u)+\gamma_\Omega(v)}\Big) - 1 \geq 0.$$

So,

$$\frac{\gamma_\Omega(u)\gamma_\Omega(v)}{\gamma_\Omega(u)+\gamma_\Omega(v)} \geq 1 + ln\Big(\frac{\gamma_\Omega(u)\gamma_\Omega(v)}{\gamma_\Omega(u)+\gamma_\Omega(v)}\Big).$$

By taking the summation over the edges of the graph, we get

$$\begin{aligned} ISI(\Omega) &\geq m + \sum_{uv\in E(\Omega)} ln\Big(\frac{\gamma_\Omega(u)\gamma_\Omega(v)}{\gamma_\Omega(u)+\gamma_\Omega(v)}\Big) \\ &= m + ln\Big(\prod_{uv\in E(\Omega)} \frac{\gamma_\Omega(u)\gamma_\Omega(v)}{\gamma_\Omega(u)+\gamma_\Omega(v)}\Big) \\ &= m + ln\Big(\prod_{uv\in E(\Omega)} \frac{\delta(\Omega)^2}{2\Delta(\Omega)}\Big) \\ &= m + ln\ \Big(\frac{\delta(\Omega)^2}{2\Delta(\Omega)}\Big)^m \\ &= m + m\ ln\Big(\frac{\delta(\Omega)^2}{2\Delta(\Omega)}\Big). \end{aligned}$$

Hence

$$ISI(\Omega) \geq m\Big(1 + ln\Big(\frac{\delta(\Omega)^2}{2\Delta(\Omega)}\Big)\Big).$$

To show the equality, let $f(x)=0$, then $x=1$. So, $\frac{\gamma_\Omega(u)\gamma_\Omega(v)}{\gamma_\Omega(u)+\gamma_\Omega(v)}=1$, for any edge $uv\in E(\Omega)$. Hence $\gamma_\Omega(u)\gamma_\Omega(v)=\gamma_\Omega(u)+\gamma_\Omega(v)$, which holds if and only if Ω is 2-regular graph. □

Theorem 2.2. *If Ω is a graph with n vertices and m edges. Then $ISI(\Omega)\geq 2m-n$ with equality if and only if Ω is 2-regular graph.*

Proof. Assume that the function $f(x)=x+\frac{1}{x}-2$. One can easily prove that $f(x)\geq 0$.

Hence for any edge $uv\in E(\Omega)$,

$$\frac{\gamma_\Omega(u)\gamma_\Omega(v)}{\gamma_\Omega(u)+\gamma_\Omega(v)}+\frac{\gamma_\Omega(u)+\gamma_\Omega(v)}{\gamma_\Omega(u)\gamma_\Omega(v)}-2\geq 0.$$

By taking the summation over the edges of the graph, we get

$$ISI(\Omega)+n-2m\geq 0.$$

Hence

$$ISI(\Omega)\geq 2m-n.$$

To show the equality, let $f(x)=0$, then $x=1$. The rest of the proof is similar to that in Theorem 2.1. □

3. Vertex- edge corona product graph

Let Ω_1 be a graph with vertex set $V(\Omega_1)=\{x_1,x_2,\ldots,x_{n_1}\}$ and edge set $E(\Omega_1)=\{e_1,e_2,\ldots,e_{m_1}\}$. Let Ω_2 be a graph with vertex set $V(\Omega_2)=\{y_1,y_2,\ldots,y_{n_2}\}$ and edge set $E(\Omega_1)=\{e'_1,e'_2,\ldots,e'_{m_2}\}$. The vertex-edge corona of Ω_1 and Ω_2, denoted by $\Omega_1\bullet\Omega_2$, is the graph obtained by taking one copy of Ω_1, $|V(\Omega_1)|$ copies of Ω_2 and $|E(\Omega_1)|$ copies of Ω_2, then joining i^{th} vertex of Ω_1 to every vertex in the i^{th} vertex copy of Ω_2 and also joining end vertices of j^{th} edge of Ω_1 to every vertex in j^{th} edge copy of Ω_2, where $1\leq i\leq n_1$ and $1\leq j\leq m_1$. One can see that $|V(\Omega_1\bullet\Omega_2)|=n_1+n_2(m_1+n_1)$ and $|E(\Omega_1\bullet\Omega_2)|=m_1+m_1(m_2+2n_2)+n_1(m_2+n_2)$. We denote the vertex set of the j^{th} edge copy of Ω_2 by $V_{je}(\Omega_2)=\{x_{j1},x_{j2},\ldots,x_{jn_2}\}$ and the vertes set of the i^{th} vertex copy of Ω_2 by $V_{iv}(\Omega_2)=\{w_{i1},w_{i2},\ldots,w_{in_2}\}$. Also, we denote by $E_{je}(\Omega_2)$ and $E_{iv}(\Omega_2)$, the set of the j^{th} edge i^{th} vertex copy of Ω_2, respectively.

Lemma 3.1. *[28] Let f be a convex function on the interval I and $x_1,x_2,\ldots,x_n\in I$. Then $f\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)\leq\frac{f(x_1)+f(x_2)+\ldots f(x_n)}{n}$, with equality if and only if $x_1=x_2=\ldots=x_n$.* □

Theorem 3.2. *Let G_i be (n_i,m_i) graph, $i=\{1,2\}$. Then $ISI(\Omega_1\bullet\Omega_2)\leq\left(\frac{n_2+1}{4}\right)ISI(\Omega_1)+\left(\frac{n_1+m_1}{4}\right)ISI(\Omega_2)+\left(\frac{n_2(4m_2+5n_2)}{8(n_2+1)}\right)H(\Omega_1)+\left(\frac{4m_1+n_1}{8}\right)H(\Omega_2)+\left(\frac{(n_2+1)(2ID(\Omega_2)+3n_2+2m_2+2)}{16}\right)M_1(\Omega_1)+\left(\frac{n_1+m_1}{8}\right)M_1(\Omega_2)+\left(\frac{(n_2+1)^2}{8n_2}\right)M_2(\Omega_1)+\left(\frac{2n_1+m_1}{16}\right)M_2(\Omega_2)+\left(\frac{2n_2m_2+n_2^2}{16(n_2+1)}\right)ID(\Omega_1)+\left(\frac{n_1n_2+3n_2m_1+2m_1}{4}\right)ID(\Omega_2)+\frac{1}{16}\Big(2n_2m_1(3n_2+6m_2+8)+2n_1(m_1+m_2)+n_1n_2(2n_2+1)+32m_1m_2\Big)+\left(\frac{n_1n_2(2m_2+n_2)}{16(n_2+1)}\right)$.*

Proof. By using the definition of ISI of $\Omega_1\bullet\Omega_2$, we have

$$\begin{aligned}ISI(\Omega_1\bullet\Omega_2) &= \sum_{xy\in E(\Omega_1\bullet\Omega_2)}\frac{\gamma_{\Omega_1\bullet\Omega_2}(x)\gamma_{\Omega_1\bullet\Omega_2}(y)}{\gamma_{\Omega_1\bullet\Omega_2}(x)+\gamma_{\Omega_1\bullet\Omega_2}(y)}\\ &= I_1+I_2+I_3+I_4+I_5,\text{ where}\end{aligned}$$

$I_1 = \sum_{x_i x_j \in E(\Omega_1)} \frac{\gamma_{\Omega_1 \bullet \Omega_2}(x_i)\gamma_{\Omega_1 \bullet \Omega_2}(x_j)}{\gamma_{\Omega_1 \bullet \Omega_2}(x_i)+\gamma_{\Omega_1 \bullet \Omega_2}(x_j)}$,

$I_2 = \sum_{e_i \in E(\Omega_1)} \sum_{x_{ij}x_{ik} \in E_{ie}(\Omega_2)} \frac{\gamma_{\Omega_1 \bullet \Omega_2}(x_{ij})\gamma_{\Omega_1 \bullet \Omega_2}(x_{ik})}{\gamma_{\Omega_1 \bullet \Omega_2}(x_{ij})+\gamma_{\Omega_1 \bullet \Omega_2}(x_{ik})}$,

$I_3 = \sum_{e_i = x_l x_m \in E(\Omega_1)} \sum_{x_{ij} \in V_{ie}(\Omega_2)} \frac{(\gamma_{\Omega_1 \bullet \Omega_2}(x_l)+\gamma_{\Omega_1 \bullet \Omega_2}(x_m))\gamma_{\Omega_1 \bullet \Omega_2}(x_{ij})}{\gamma_{\Omega_1 \bullet \Omega_2}(x_l)+\gamma_{\Omega_1 \bullet \Omega_2}(x_m)+\gamma_{\Omega_1 \bullet \Omega_2}(x_{ij})}$,

$I_4 = \sum_{x_i \in V(\Omega_1)} \sum_{w_{ij}w_{ik} \in E_{iv}(\Omega_2)} \frac{\gamma(w_{ij})\gamma(w_{ik})}{\gamma(w_{ij})+\gamma(w_{ik})}$ and

$I_5 = \sum_{x_i \in V(\Omega_1)} \sum_{w_{ij} \in V_{iv}(\Omega_2)} \frac{\gamma(x_i)\gamma(w_{ij})}{\gamma(x_i)+\gamma(w_{ij})}$.

From the structure of the graph $\Omega_1 \bullet \Omega_2$, we have the following;

◦ If $x_i \in V(\Omega_1)$, then $\gamma_{\Omega_1 \bullet \Omega_2}(x_i) = (n_2+1)\gamma_{\Omega_1}(x_i) + n_2$.

◦ If $x_{ij} \in V_{iv}(\Omega_2)$, then $\gamma_{\Omega_1 \bullet \Omega_2}(x_{ij}) = \gamma_{\Omega_2}(y_j) + 2$.

◦ If $w_{ij} \in V_{iv}(\Omega_2)$, then $\gamma_{\Omega_1 \bullet \Omega_2}(w_{ij}) = \gamma_{\Omega_2}(w_j) + 1$. Hence

$$\begin{aligned} I_1 &= \sum_{x_i x_j \in E(\Omega_1)} \frac{\gamma_{\Omega_1 \bullet \Omega_2}(x_i)\gamma_{\Omega_1 \bullet \Omega_2}(x_j)}{\gamma_{\Omega_1 \bullet \Omega_2}(x_i) + \gamma_{\Omega_1 \bullet \Omega_2}(x_j)} \\ &= \sum_{x_i x_j \in E(\Omega_1)} \frac{((n_2+1)\gamma_{\Omega_1}(x_i)+n_2)((n_2+1)\gamma_{\Omega_1}(x_j)+n_2)}{(n_2+1)\gamma_{\Omega_1}(x_i)+n_2+(n_2+1)\gamma_{\Omega_1}(x_j)+n_2} \\ &= \sum_{x_i x_j \in E(\Omega_1)} \frac{(n_2+1)^2(\gamma_{\Omega_1}(x_i)\gamma_{\Omega_1}(x_j)) + n_2^2 + n_2(n_2+1)(\gamma_{\Omega_1}(x_i)+\gamma_{\Omega_1}(x_j))}{(n_2+1)(\gamma_{\Omega_1}(x_i)+\gamma_{\Omega_1}(x_j)+2n_2)}. \end{aligned}$$

Jensen's inequality is used for the convex function $f : \mathcal{R}_+ \to \mathcal{R}_+, f(x) = \frac{1}{x}$ according to Lemma 3.1, we have $\frac{1}{(\gamma_{\Omega_1}(x_i)+\gamma_{\Omega_1}(x_j)+2n_2)} = \frac{1}{4(\gamma_{\Omega_1}(x_i)+\gamma_{\Omega_1}(x_j))} + \frac{1}{8n_2}$ with equality if and only if $\gamma_{\Omega_1}(x_i)+\gamma_{\Omega_1}(x_j) = 2n_2$. Thus

$$\begin{aligned} I_1 &\leq \sum_{x_i x_j \in E(\Omega_1)} \frac{(n_2+1)^2(\gamma_{\Omega_1}(x_i)\gamma_{\Omega_1}(x_j)) + n_2(n_2+1)(\gamma_{\Omega_1}(x_i)+\gamma_{\Omega_1}(x_j)+n_2^2)}{4(n_2+1)(\gamma_{\Omega_1}(x_i)+\gamma_{\Omega_1}(x_j))} \\ &+ \frac{(n_2+1)^2(\gamma_{\Omega_1}(x_i)\gamma_{\Omega_1}(x_j)) + n_2(n_2+1)(\gamma_{\Omega_1}(x_i)+\gamma_{\Omega_1}(x_j)+n_2^2)}{8n_2} \\ &= \frac{1}{4}ISI(\Omega_1)(n_2+1) + \frac{n_2 m_1}{4} + \frac{n_2^2}{8(n_2+1)}H(\Omega_1) \\ &+ \frac{(n_2+1)^2}{8n_2}M_2(\Omega_1) + \frac{(n_2+1)}{8}M_1(\Omega_1) + \frac{n_2 m_1}{8} \\ &= \frac{(n_2+1)}{4}ISI(\Omega_1) + \frac{n_2^2}{8(n_2+1)}H(\Omega_1) + \frac{(n_2+1)^2}{8n_2}M_2(\Omega_1) \\ &+ \frac{(n_2+1)}{8}M_1(\Omega_1) + \frac{3n_2 m_1}{8}. \end{aligned}$$

$$\begin{aligned} I_2 &= \sum_{e_i \in E(\Omega_1)} \sum_{x_{ij}x_{ik} \in E_{ie}(\Omega_2)} \frac{\gamma_{\Omega_1 \bullet \Omega_2}(x_{ij})\gamma_{\Omega_1 \bullet \Omega_2}(x_{ik})}{\gamma_{\Omega_1 \bullet \Omega_2}(x_{ij}) + \gamma_{\Omega_1 \bullet \Omega_2}(x_{ik})} \\ &= \sum_{e_i \in E(\Omega_1)} \sum_{y_j y_k \in E(\Omega_2)} \frac{(\gamma_{\Omega_2}(y_j)+2)(\gamma_{\Omega_2}(y_k)+2)}{\gamma_{\Omega_2}(y_j)+\gamma_{\Omega_2}(y_k)+4}. \end{aligned}$$

Apply Jensen's inequality, we have $\frac{1}{\gamma_{\Omega_2}(y_j)+\gamma_{\Omega_2}(y_k)+4} \leq \frac{1}{4(\gamma_{\Omega_2}(y_j)+\gamma_{\Omega_2}(y_k))} + \frac{1}{16}$ with equality if and only if $\gamma_{\Omega_2}(y_j) + \gamma_{\Omega_2}(y_k) = 4$. Thus

$$
\begin{aligned}
I_2 &\leq \frac{1}{4} \sum_{e_i \in E(\Omega_1)} \sum_{y_j y_k \in E(\Omega_2)} \Big(\frac{(\gamma_{\Omega_2}(y_j)+2)(\gamma_{\Omega_2}(y_k)+2)}{\gamma_{\Omega_2}(y_j)+\gamma_{\Omega_2}(y_k)} \\
&+ \frac{(\gamma_{\Omega_2}(y_j)+2)(\gamma_{\Omega_2}(y_k)+2)}{4} \Big) \\
&= \frac{1}{4} \sum_{e_i \in E(\Omega_1)} \sum_{y_j y_k \in E(\Omega_2)} \Big(\frac{\gamma_{\Omega_2}(y_j)\gamma_{\Omega_2}(y_k)}{\gamma_{\Omega_2}(y_j)+\gamma_{\Omega_2}(y_k)} + 2(\gamma_{\Omega_2}(y_j)+\gamma_{\Omega_2}(y_k)) \\
&+ \frac{4}{\gamma_{\Omega_2}(y_j)\gamma_{\Omega_2}(y_k)} \\
&+ \frac{\gamma_{\Omega_2}(y_j)\gamma_{\Omega_2}(y_k) + 2(\gamma_{\Omega_2}(y_j)+\gamma_{\Omega_2}(y_k)) + 4}{4} \Big) \\
&= \frac{1}{4} \sum_{e_i \in E(\Omega_1)} \Big(ISI(\Omega_2) + 3m_2 + 2H(\Omega_2) + \frac{M_2(\Omega_2)}{4} + \frac{M_1(\Omega_2)}{2} \Big) \\
&= \frac{m_1}{4} \Big(ISI(\Omega_2) + 2H(\Omega_2) + \frac{M_2(\Omega_2)}{4} + \frac{M_1(\Omega_2)}{2} + 3m_2 \Big).
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \sum_{e_i = x_l x_m \in E(\Omega_1)} \sum_{x_i \in V_{ie}(\Omega_2)} \frac{(\gamma_{\Omega_1 \bullet \Omega_2}(x_l) + \gamma_{\Omega_1 \bullet \Omega_2}(x_m))\gamma_{\Omega_1 \bullet \Omega_2}(x_{ij})}{\gamma_{\Omega_1 \bullet \Omega_2}(x_l) + \gamma_{\Omega_1 \bullet \Omega_2}(x_m) + \gamma_{\Omega_1 \bullet \Omega_2}(x_{ij})} \\
&= \sum_{e_i \in E(\Omega_1)} \sum_{y_j \in V(\Omega_2)} \frac{((n_2+1)\gamma_{\Omega_1}(x_l) + n_2 + (n_2+1)\gamma_{\Omega_1}(x_m) + n_2)(\gamma_{\Omega_2}(y_j)+2)}{((n_2+1)\gamma_{\Omega_1}(x_l) + n_2 + (n_2+1)\gamma_{\Omega_1}(x_m) + n_2) + \gamma_{\Omega_2}(y_j) + 2} \\
&= \sum_{e_i \in E(\Omega_1)} \sum_{y_j \in V(\Omega_2)} \frac{((n_2+1)(\gamma_{\Omega_1}(x_l) + \gamma_{\Omega_1}(x_m)) + 2n_2)(\gamma_{\Omega_2}(y_j)+2)}{(n_2+1)(\gamma_{\Omega_1}(x_l) + \gamma_{\Omega_1}(x_m)) + \gamma_{\Omega_2}(y_j) + (2n_2+2)} \\
&\leq \sum_{e_i \in E(\Omega_1)} \sum_{y_j \in V(\Omega_2)} \frac{1}{4}((n_2+1)(\gamma_{\Omega_1}(x_l) + \gamma_{\Omega_1}(x_m)) + 2n_2)(\gamma_{\Omega_2}(y_j)+2) \\
&\Big(\frac{1}{(n_2+1)(\gamma_{\Omega_1}(x_l) + \gamma_{\Omega_1}(x_m))} + \frac{1}{\gamma_{\Omega_2}(y_j) + 2n_2 + 2} \Big) \\
&= \frac{1}{4} \sum_{e_i \in E(\Omega_1)} \sum_{y_j \in V(\Omega_2)} \Big((\gamma_{\Omega_2}(y_j)+2) + \frac{2n_2(\gamma_{\Omega_2}(y_j)+2)}{(n_2+1)(\gamma_{\Omega_1}(x_l) + \gamma_{\Omega_1}(x_m))} \Big) \\
&+ \frac{(n_2+1)(\gamma_{\Omega_1}(x_l) + \gamma_{\Omega_1}(x_m)) + 2n_2(\gamma_{\Omega_2}(y_j)+2)}{\gamma_{\Omega_2}(y_j) + 2n_2 + 2} \\
&= I_3^{'} + I_3^{''}, \text{ where}
\end{aligned}
$$

$$
\begin{aligned}
I_3^{'} &= \frac{1}{4} \sum_{e_i \in E(\Omega_1)} \sum_{y_j \in V(\Omega_2)} \Big(\gamma_{\Omega_2}(y_j) + 2 + \frac{2n_2(\gamma_{\Omega_2}(y_j)+2)}{(n_2+1)(\gamma_{\Omega_1}(x_l) + \gamma_{\Omega_1}(x_m))} \Big) \\
&= \frac{1}{4} \sum_{e_i \in E(\Omega_1)} \Big(2m_2 + 2n_2 + \frac{2n_2(2m_2+2n_2)}{(n_2+1)(\gamma_{\Omega_1}(x_l) + \gamma_{\Omega_1}(x_m))} \Big) \\
&= \frac{1}{4} \Big((2m_2+2n_2)m_1 + \frac{n_2(2m_2+2n_2)}{(n_2+1)} H(\Omega_1) \Big).
\end{aligned}
$$

$$\begin{aligned}
I_3'' &= \frac{1}{4}\sum_{e_i\in E(\Omega_1)}\sum_{y_j\in V(\Omega_2)}\frac{((n_2+1)(\gamma_{\Omega_1}(x_l)+\gamma_{\Omega_1}(x_m))+2n_2)(\gamma_{\Omega_2}(y_j)+2)}{\gamma_{\Omega_2}(y_j)+2n_2+2}\\
&\leq \frac{1}{16}\sum_{e_i\in E(\Omega_1)}\sum_{y_j\in V(\Omega_2)}\Big(((n_2+1)(\gamma_{\Omega_1}(x_l)+\gamma_{\Omega_1}(x_m))+2n_2)(\gamma_{\Omega_2}(y_j)+2)\Big)\\
&\quad \Big(\frac{1}{\gamma_{\Omega_2}(y_j)}+\frac{1}{2n_2+2}\Big)\\
&= \frac{1}{16}\sum_{e_i\in E(\Omega_1)}\sum_{y_j\in V(\Omega_2)}\Big((n_2+1)(\gamma_{\Omega_1}(x_l)+\gamma_{\Omega_1}(x_m))+2n_2\\
&+ \frac{2((n_2+1)(\gamma_{\Omega_1}(x_l)+\gamma_{\Omega_1}(x_m))+2n_2)}{\gamma_{\Omega_2}(x_j)}\Big)\\
&+ \frac{((n_2+1)(\gamma_{\Omega_1}(x_l)+\gamma_{\Omega_1}(x_m))+2n_2)(\gamma_{\Omega_2}(y_j)+2)}{2n_2+2}\\
&= \frac{1}{16}\sum_{e_i\in E(\Omega_1)}\Big(n_2(n_2+1)(\gamma_{\Omega_1}(x_l)+\gamma_{\Omega_1}(x_m))+2n_2^2\Big)\\
&+ 2\Big((n_2+1)(\gamma_{\Omega_1}(x_l)+\gamma_{\Omega_1}(x_m)+2n_2)ID(\Omega_2)\\
&+ ((n_2+1)(\gamma_{\Omega_1}(y_l)+\gamma_{\Omega_1}(x_m))+2n_2)(2m_2+2n_2)\Big)\\
&= \frac{1}{16}\Big(n_2(n_2+1)M_1(\Omega_1)+2n_2^2m_1+(2(n_2+1)M_1(\Omega_1)+4n_2m_1)ID(\Omega_2)\\
&+ ((n_2+1)M_1(\Omega_1)+2n_2m_1)(2m_2+2n_2)\Big).
\end{aligned}$$

$$\begin{aligned}
I_4 &= \sum_{y_i\in V(\Omega_1)}\sum_{w_{ij}w_{ik}\in E_{iv}(\Omega_2)}\frac{\gamma_{\Omega_1\bullet\Omega_2}(w_{ij})\gamma_{\Omega_1\bullet\Omega_2}(w_{ik})}{\gamma_{\Omega_1\bullet\Omega_2}(w_{ij})+\gamma_{\Omega_1\bullet\Omega_2}(w_{ik})}\\
&= \sum_{y_i\in V(\Omega_1)}\sum_{w_jw_k\in E(\Omega_2)}\frac{(\gamma_{\Omega_2}(w_j)+1)(\gamma_{\Omega_2}(w_k)+1)}{\gamma_{\Omega_2}(w_j)+\gamma_{\Omega_2}(w_k)+2}\\
&\leq \frac{1}{4}\sum_{y_i\in V(\Omega_1)}\sum_{w_jw_k\in E(\Omega_2)}\Big(\gamma_{\Omega_2}(w_j)\gamma_{\Omega_2}(w_k)+(\gamma_{\Omega_2}(w_j)+\gamma_{\Omega_2}(w_k))+1\Big)\\
&\quad \Big(\frac{1}{\gamma_{\Omega_2}(w_j)+\gamma_{\Omega_2}(w_k)}+\frac{1}{2}\Big),
\end{aligned}$$

by Jensen's inequality and equality holds if and only if $\gamma_{\Omega_2}(w_j)+\gamma_{\Omega_2}(w_k)=2$.

$$= \frac{1}{4}\Big(n_1ISI(\Omega_2)+\frac{n_1}{2}M_2(\Omega_2)+n_1m_2+\frac{n_1}{2}M_1(\Omega_2)+\frac{n_1}{2}H(\Omega_2)+\frac{n_1m_1}{2}\Big).$$

$$\begin{aligned}
I_5 &= \sum_{x_i\in V(\Omega_1)}\sum_{w_{ij}\in V_{iv}(\Omega_2)}\frac{\gamma_{\Omega_1\bullet\Omega_2}(x_i)\gamma_{\Omega_1\bullet\Omega_2}(w_{ij})}{\gamma_{\Omega_1\bullet\Omega_2}(x_i)+\gamma_{\Omega_1\bullet\Omega_2}(w_{ij})}\\
&= \sum_{x_i\in V(\Omega_1)}\sum_{w_j\in V(\Omega_2)}\frac{((n_2+1)\gamma_{\Omega_1}(x_i)+n_2)(\gamma_{\Omega_2}(w_j)+1)}{(n_2+1)\gamma_{\Omega_1}(x_i)+n_2+\gamma_{\Omega_2}(w_j)+1}\\
&\leq \frac{1}{4}\sum_{x_i\in V(\Omega_1)}\sum_{w_j\in V(\Omega_2)}\Big(((n_2+1)\gamma_{\Omega_1}(x_i)+n_2)(\gamma_{\Omega_2}(w_j)+1)\Big)\Big(\frac{1}{4(n_2+1)\gamma_{\Omega_1}(x_i)}+\frac{1}{4(n_2+1)}\\
&\quad +\frac{1}{\gamma_{\Omega_2}(w_j)}\Big),
\end{aligned}$$

by Jensen's inequality

$$= \frac{1}{4}\sum_{x_i\in V(\Omega_1)}\sum_{w_j\in V(\Omega_2)}\Big(\frac{\gamma_{\Omega_2}(w_j)}{4}+\frac{\gamma_{\Omega_1}(x_i)\gamma_{\Omega_2}(w_j)}{4}+(n_2+1)\gamma_{\Omega_1}(x_i)+\frac{\gamma_{\Omega_1}(v_i)}{4}+\frac{(n_2+1)\gamma_{\Omega_1}(x_i)}{\gamma_{\Omega_2}(w_j)}$$

$$+\frac{n_2 d_{\Omega_2}(w_j)}{4(n_2+1)\gamma_{\Omega_1}(x_i)}+\frac{n_2 d_{\Omega_2}(w_j)}{4(n_2+1)}+n_2+\frac{n_2}{4(n_2+1)\gamma_{\Omega_1}(x_i)}+\frac{n_2}{4(n_2+1)}+\frac{n_2}{\gamma_{\Omega_2}(w_j)}\Big)$$

$$= \frac{1}{16}\Big(2n_1m_2+2n_2m_1+4m_1m_2+8m_1m_2(n_2+1)+n_1n_2+\frac{2n_1n_2m_2}{n_2+1}+4n_1n_2^2+\frac{n_1n_2^2}{n_2+1}$$

$$+\frac{(2n_2m_2+n_2^2)ID(\Omega_1)}{n_2+1}+(4n_2n_1+8m_1(n_2+1))ID(\Omega_2)\Big).$$

Adding I_1 to I_5 we arrive the required result. □

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