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FIXED POINT THEOREMS IN A $(\alpha_{\mathbb{A}}, \eta_{\mathbb{A}})$-$C^*$-ALGEBRA VALUED $b-$QUASI-METRIC SPACES

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Abstract. In the present work, for a unital C^*-algebra $\mathbb{A}$, we introduce the notion of $(\alpha_{\mathbb{A}}, \eta_{\mathbb{A}})$-$C^*$-algebra valued b-quasi-metric spaces. Also, we discuss the existence and uniqueness of fixed points for a self-mapping defined on a such space. Our results extend and supplement several recent results in the literature. Some non-trivial examples are given to illustrate our results.

1. Introduction

It is well known that Banach contraction principle [2] played a central role in fixed point theory because of its application in many branches of mathematics and it has many applications. Various generalizations of it appeared in the literature [4–7, 10].

In 1930, Wilson [12] introduced the concept of quasi-metric spaces. Using this idea many researcher presented generalization of the renowned Banach fixed point theorem in the quasi-metric spaces.

The concept of b-metric spaces was initiated by Bakhtin [1] and Czerwik [3] where the triangle inequality of a metric spaces was replaced by another inequality, the so-called b-triangle inequality.

In [11], Shah and Hussain established the concept of b-quasi-metric space which generalizes the concept of quasi-metric space.

In 2014, Ma et al. [8] introduced the notion of C^*-algebra valued metric spaces by replacing the range set $\mathbb{R}$ with an unital C^*-algebra, which is more general class than the class of metric spaces.

This paper is aimed to generalization of some results on fixed point in a quasi-metric spaces and C^*-algebra valued b-quasi-metric spaces.

Throughout this paper, we use the concept of (α, η)-triangular-admissible of mappings defined on C^*-algebra valued b-quasi-metric space and we defined the generalized contractive on such spaces. Finally, some examples are provided to illustrate the results.

The following lemma will used to proof our main results.

Lemma 1.1. *[9] Suppose that A is a unital C^*-algebra with a unit I.*

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(1) *For any* $x \in A_+$ *we have* $x \preceq I, k \Leftrightarrow \|x\| \leq 1$;
(2) *If* $a \in A_+$ *with* $\|x\| < \frac{1}{2}$, *then* a *is invertible and* $\|a(I-1)^{-1}\| < 1$.
(3) *Suppose that* $a, b \in A$ *with* $a, b \succeq 0_{\mathbb{A}}$ *and* $ab = ba$, *then* $ab \succeq 0_{\mathbb{A}}$;
(4) *Let* $a \in A'$, *if* $b, c \in A$ *with* $b \succeq b \succeq 0_{\mathbb{A}}$, *and* $(I-a) \in A'_+$ *is an invertible operator, then* $(I-1)^{-1}b \succeq (I-1)^{-1}c$.

2. Main result

We now introduce the definition of a C^*-algebra-valued b-quasi-metric spaces.

Definition 2.1. Let X be a non empty set and $s \succeq I_{\mathbb{A}}$. Suppose the mapping $d : X \times X \to \mathbb{A}_+$ satisfies:

(i) $d(x, y) = 0_{\mathbb{A}}$ if and only if $x = y$; and $0_{\mathbb{A}} \preceq d(x, y)$ for all $x, y \in X$;
(ii) $d(x, y) \preceq s\,[d(x, z) + d(z, y)]$ for all $x, y, z \in X$, where $0_{\mathbb{A}}$ is zero-element in A and $I_{\mathbb{A}}$ is the unit element in $\mathbb{A}$.

Then $(X, \mathbb{A}_+, d)$ is called a C^*-algebra valued b-quasi-metric space.

Remark 2.2. The C^*-algebra-valued b-quasi-metric space generalise the C^*-algebra-valued b-metric space, C^*-algebra-valued quasi-metric space.

The following example illustrates that, in general, a C^*-algebra-valued b-quasi-metric space is not necessarily a C^*-algebra-valued metric space and is not necessarily a C^*-algebra-valued $b-$metric space.

Example 2.3. Let X be a Banach lattice, $d : X \times X \to \mathbb{A}_+$ given by

$$\begin{cases} d\,(x, y) = \|x - y\|^p.a \; if \; x \geq y \\ d\,(x, y) = \|y - x\|^p.a \; if \; y > x. \end{cases}$$

for all $x, y \in X$, $a \in \mathbb{A}_+, a \succeq 0$ *and* $p > 1$. Its easy to verify that is a C^*-algebra valued b-quasi-metric space.

Using the inequality $(a + b)^p \leq 2^p(a^p + b^p)$ for all $a, b \succeq 0, p > 1$, we have

$$\begin{cases} \|x - y\|^p \leq 2^p(\|x - z\|^p + \|z - y\|^p) \; if \; x \geq y \\ \|y - x\|^p \leq 2^p(\|y - z\|^p + \|z - x\|^p) \; if \; y > x. \end{cases}$$

for $x, y, z \in X$, which implies that

$$d(x, y) \leq 2^p(d(x - z) + d(z - y)).$$

Example 2.4. Let $X = \mathbb{R}$ and $\mathbb{A} = M_2(\mathbb{R})$ of all 2×2 matrices with the usual addition, scalar multiplication and multiplication. Define partial ordering on $\mathbb{A}$ as $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \succeq \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ $\Leftrightarrow a_i \geq b_i$ for $i = 1, 2, 3, 4$

For any $A \in \mathbb{A}$ we define its norm as ,$\|\mathbb{A}\| = max_{1 \leq i \leq 4}|a_i|$

Define $d : X \times X \to \mathbb{A}$ by

$$\begin{cases} d\,(x, y) = \begin{pmatrix} (x - y)^p & 0 \\ 0 & 0 \end{pmatrix} \; if \; x \geq y \\ d\,(x, y) = \begin{pmatrix} 0 & 0 \\ 0 & (x - y)^p \end{pmatrix} \; if \; y > x. \end{cases}$$

for all $x, y \in X$ and $p \geq 1$ is odd number.

It's clear that $0_{\mathbb{A}} \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0_{\mathbb{A}} \Leftrightarrow x = y$.

We will verify b-triangular inequality. Let x, y et $z \in \mathbb{R}$ then we have six cases.

Case 1: $x \geq y$

$$d\left(x,y\right) = \begin{pmatrix} (x-y)^p & 0 \\ 0 & 0 \end{pmatrix}$$

$(a)\ if\ y \geq z$

$$\begin{aligned} 2^p\left[d(x,z)+d(z,y)\right] &= \begin{pmatrix} 2^p\left[(x-z)^p\right] & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 2^p\left[(y-z)^p\right] \end{pmatrix} \\ &= \begin{pmatrix} 2^p\left[(x-z)^p\right] & 0 \\ 0 & 2^p\left[(y-z)^p\right] \end{pmatrix} \\ &\geq \begin{pmatrix} (x-y)^p & 0 \\ 0 & 0 \end{pmatrix} \\ &= d(x,y). \end{aligned}$$

$(b)\ if\ x \geq z \geq y$

$$\begin{aligned} 2^p\left[d(x,z)+d(z,y)\right] &= \begin{pmatrix} 2^p\left[(x-z)^p\right] & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2^p\left[(z-y)^p\right] & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2^p\left[(x-z)^p+(z-y)^p\right] & 0 \\ 0 & 0 \end{pmatrix} \\ &\succeq \begin{pmatrix} (x-y)^p & 0 \\ 0 & 0 \end{pmatrix} \\ &= d(x,y). \end{aligned}$$

$(c)\ if\ z \geq x$

$$\begin{aligned} 2^p\left[d(x,z)+d(z,y)\right] &= \begin{pmatrix} 0 & 0 \\ 0 & 2^p\left[(z-x)^p\right] \end{pmatrix} + \begin{pmatrix} 2^p\left[(z-y)^p\right] & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2^p\left[(z-y)^p\right] & 0 \\ 0 & 2^p\left[(z-y)^p\right] \end{pmatrix} \\ &\succeq \begin{pmatrix} (x-y)^p & 0 \\ 0 & 0 \end{pmatrix} \\ &= d(x,y). \end{aligned}$$

Case 2: $x \leq y$

$$d\left(x,y\right) = \begin{pmatrix} 0 & 0 \\ 0 & (y-x)^p \end{pmatrix}$$

(a) $if\ x \leq y \leq z$

$$
\begin{aligned}
2^p\left[d(x,z)+d(z,y)\right] &= \begin{pmatrix} 0 & 0 \\ 0 & 2^p\left[(z-x)^p\right] \end{pmatrix} + \begin{pmatrix} 2^p\left[(z-y)^p\right] & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 2^p\left[(z-y)^p\right] & 0 \\ 0 & 2^p\left[(z-x)^p\right] \end{pmatrix} \\
&\succeq \begin{pmatrix} 0 & 0 \\ 0 & (y-x)^p \end{pmatrix} \\
&= d(x,y).
\end{aligned}
$$

(b) $if\ x \leq z \leq y$

$$
\begin{aligned}
2^p\left[d(x,z)+d(z,y)\right] &= \begin{pmatrix} 0 & 0 \\ 0 & 2^p\left[(z-x)^p\right] \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 2^p\left[(y-z)^p\right] \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 2^p\left[(z-x)^p+(y-z)^p\right] \end{pmatrix} \\
&\succeq \begin{pmatrix} 0 & 0 \\ 0 & (y-x)^p \end{pmatrix} \\
&= d(x,y).
\end{aligned}
$$

(c) $if\ z \leq x \leq y$

$$
\begin{aligned}
2^p\left[d(x,z)+d(z,y)\right] &= \begin{pmatrix} 2^p\left[(x-z)^p\right] & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 2^p\left[(y-z)^p\right] \end{pmatrix} \\
&= \begin{pmatrix} 2^p\left[(x-z)^p\right] & 0 \\ 0 & 2^p\left[(y-z)^p\right] \end{pmatrix} \\
&\succeq \begin{pmatrix} 0 & 0 \\ 0 & (y-x)^p \end{pmatrix} \\
&= d(x,y).
\end{aligned}
$$

Then $(X, \mathbb{A}, d)$ is a C^*-algebra valued b-quasi-metric space. However we have the following:

1) $(X, \mathbb{A}, d)$ is not a C^*-algebra valued metric space, as $d(1,0) \neq d(0,1)$.
2) $(X, \mathbb{A}, d)$ is not a C^*-algebra valued quasi-metric space, as
$$d(2,0) = \begin{pmatrix} 2^p & 0 \\ 0 & 0 \end{pmatrix} \succ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = d(2,1) + d(1,0).$$

Definition 2.5. Let $(X, \mathbb{A}, d)$ is a C^*-algebra valued b -quasi-metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then

(i) We say that $\{x_n\}_{n\in\mathbb{N}}$ forward converges to x with respect to $\mathbb{A}$ if and only if for given $\varepsilon \succ 0_{\mathbb{A}}$, there is N such that for all $n \geq N$, $d(x, x_n) \preceq \varepsilon$. We denote it by
$$\lim_{n\to+\infty} d(x, x_n) = 0_{\mathbb{A}}.$$

(ii) We say that $\{x_n\}_{n\in\mathbb{N}}$ backward converges to x with respect to $\mathbb{A}$ if and only if for given $\varepsilon \succ 0_{\mathbb{A}}$, there is N such that for all $n \geq N$, $d(x_n, x) \preceq \varepsilon$. We denote it by

$$\lim_{n\to+\infty} d(x_n, x) = 0_{\mathbb{A}}.$$

(iii) We say that $\{x_n\}_{n\in\mathbb{N}}$ converges to x with respect to $\mathbb{A}$ if and only if $\{x_n\}_{n\in\mathbb{N}}$ forward converges and backward converges to x.

Definition 2.6. Let $(X, \mathbb{A}, d)$ is a C^*-algebra valued b -quasi-metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then

(i) We say that $\{x_n\}_{n\in\mathbb{N}}$ forward Cauchy if

$$\lim_{n,m\to+\infty} d(x_n, x_m) = 0_{\mathbb{A}}.$$

(ii) We say that $\{x_n\}_{n\in\mathbb{N}}$ backward Cauchy if

$$\lim_{n,m\to+\infty} d(x_m, x_n) = 0_{\mathbb{A}}.$$

Example 2.7. Let $X = \mathbb{R}_+$ and $\mathbb{A} = M_2(\mathbb{R})$ of all 2×2 matrices with the usual addition, scalar multiplication and multiplication. Define partial ordering on $\mathbb{A}$ as $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \succeq \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ $\Leftrightarrow a_i \geq b_i$ for $i = 1, 2, 3, 4$
For any $A \in \mathbb{A}$ we define its norm as ,$\|\mathbb{A}\| = max_{1\leq i\leq 4}|a_i|$

Define $d : X \times X \to \mathbb{A}$ by

$$\begin{cases} d(x, y) = \begin{pmatrix} (x-y)^2 & 0 \\ 0 & 0 \end{pmatrix} \; if \;\; x \geq y \\ d(x-y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; if \;\; y > x. \end{cases}$$

Then $x, \mathbb{A}, d$ is an $(X, \mathbb{A}, d)$ a C^*-algebra valued b-quasi-metric space.

Let $x_n = x + \frac{1}{n+1}$. Then

$$\begin{cases} d(x_n, x) = \begin{pmatrix} (x_n - x)^2 & 0 \\ 0 & (x_n - x)^2 \end{pmatrix} \\ \qquad\qquad = \begin{pmatrix} \left(\frac{1}{n+1}\right)^2 & 0 \\ 0 & \left(\frac{1}{n+1}\right)^2 \end{pmatrix} \\ d(x, x_n) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{cases}$$

Then $\lim_{n\to+\infty} d(x_n, x) = 0_{\mathbb{A}}$ and $\lim_{n\to+\infty} d(x, x_n) = 1_{\mathbb{A}}$. Therefore the existence forward converges does not imply the existence backward converges.

Lemma 2.8. *Let $(X, \mathbb{A}, d)$ a C^*-algebra valued b-quasi-metric space and $\{x_n\}_n$ in X. If $\{x_n\}_n$ forward converges to $x \in X$ and backward converges to $y \in X$, then $x = y$.*

Proof. Let $\varepsilon \succ 0_{\mathbb{A}}$. Since $\{x_n\}_n$forward converges to x so there exists $n_1 \in \mathbb{N}$ such that $d(x_n, x) \preceq \frac{\varepsilon}{2s}$ for all $n \geq n_0$. Also $\{x_n\}_n$ forward converges to y so there exists $n_1 \in \mathbb{N}$ such that $d(y,x_n) \preceq \frac{\varepsilon}{2s}$ for all $n \geq n_1$. Then for all $N \geq \max\{n_0, n_1, \}$,

$$d(x,y) \preceq s\left[d(x,x_n) + d(x_n,y)\right] \preceq s\left[\frac{\varepsilon}{2s} + \frac{\varepsilon}{2s}\right] = \varepsilon.$$

As $\varepsilon \succ 0_{\mathbb{A}}$ was arbitrary, we deduce that $d(x,y) = 0_{\mathbb{A}}$, which implies $x = y$. □

Definition 2.9. Let $(X, \mathbb{A}, d)$ a C^*-algebra valued b-quasi-metric space. X is said to be forward (backward) complete if every forward (backward) Cauchy sequence $\{x_n\}_n$ in X forward (backward) converges to $x \in X$.

Definition 2.10. Let $(X, \mathbb{A}, d)$ a C^*-algebra valued b-quasi-metric space. X is said to be complete if X is forward and backward complete.

Definition 2.11. Let $(X, \mathbb{A}, d)$ a C^*-algebra valued b-quasi-metric space and $\alpha_{\mathbb{A}}, \eta_{\mathbb{A}} : X \times X \to \mathbb{A}_+'$ be two functions and $T : X \to X$. We say that T is a triangular $(\alpha_{\mathbb{A}}, \eta_{\mathbb{A}})$-admissible mapping if

(T_1) $\alpha_{\mathbb{A}}(x,y) \succeq I_{\mathbb{A}} \Rightarrow \alpha_{\mathbb{A}}(Tx,Ty) \succeq I_{\mathbb{A}}$, $x,y \in X$;
(T_2) $\eta_{\mathbb{A}}(x,y) \preceq I_{\mathbb{A}} \Rightarrow \eta_{\mathbb{A}}(Tx,Ty) \preceq I_{\mathbb{A}}$, $x,y \in X$;

$$(T_3) \begin{cases} \alpha_{\mathbb{A}}(x,y) \succeq I_{\mathbb{A}} \\ \alpha(y,z) \succeq I_{\mathbb{A}} \end{cases} \Rightarrow \alpha(x,z) \succeq I_{\mathbb{A}} \text{ for all } x,y,z \in X;$$

$$(T_4) \begin{cases} \eta_{\mathbb{A}}(x,y) \leq 1 \\ \eta_{\mathbb{A}}(y,z) \preceq I_{\mathbb{A}} \end{cases} \Rightarrow \eta_{\mathbb{A}}(x,z) \preceq I_{\mathbb{A}} \text{ for all } x,y,z \in X.$$

Definition 2.12. Let $(X, \mathbb{A}, d)$ a C^*-algebra valued b-quasi-metric space and $\alpha_{\mathbb{A}}, \eta_{\mathbb{A}} : X \times X \to \mathbb{A}_+'$ be two functions. $T : X \to X$.

(a) T is $\alpha_{\mathbb{A}}$-continuous mapping on $(X, \mathbb{A}, d)$, if for given point $x \in X$ and sequence $\{x_n\}$ in X, $x_n \to x$ and $\alpha_{\mathbb{A}}(x_n, x_{n+1}) \succeq I_{\mathbb{A}}$ for all $n \in \mathbb{N}$, imply that $Tx_n \to Tx$.
(b) T is $\eta_{\mathbb{A}}$ sub$-$continuous mapping on (X,d), if for given point $x \in X$ and sequence $\{x_n\}$ in X, $x_n \to x$ and $\eta_{\mathbb{A}}(x_n, x_{n+1}) \preceq I_{\mathbb{A}}$ for all $n \in \mathbb{N}$, imply that T$x_n \to Tx$.
(c) T is $(\alpha_{\mathbb{A}}, \eta_{\mathbb{A}})$ $-$continuous mapping on (X,d), if for given point $x \in X$ and sequence $\{x_n\}$ in X, $x_n \to x$ and $\alpha_{\mathbb{A}}(x_n, x_{n+1}) \succeq I_{\mathbb{A}}$ or $\eta_{\mathbb{A}}(x_n, x_{n+1}) \preceq I_{\mathbb{A}}$ for all $n \in \mathbb{N}$, imply that $Tx_n \to Tx$.

Definition 2.13. Let $(X, \mathbb{A}, d)$ a C^*-algebra valued b-quasi-metric space and $\alpha_{\mathbb{A}}, \eta_{\mathbb{A}} : X \times X \to \mathbb{A}_+'$ be two functions. The space X is said to be:

(a) $\alpha_{\mathbb{A}}-$complete, if every Cauchy sequence $\{x_n\}$ in X with $\alpha_{\mathbb{A}}(x_n, x_{n+1}) \succeq I_{\mathbb{A}}$ for all $n \in \mathbb{N}$, converges in X.
(b) $\eta_{\mathbb{A}}-$complete, if every Cauchy sequence $\{x_n\}$ in X with $\eta_{\mathbb{A}}(x_n, x_{n+1}) \preceq I_{\mathbb{A}}$ for all $n \in \mathbb{N}$, converges in X.
(c) $(\alpha_{\mathbb{A}}, \eta_{\mathbb{A}})$ $-$complete, if every Cauchy sequence $\{x_n\}$ in X with $\alpha_{\mathbb{A}}(x_n, x_{n+1}) \succeq I_{\mathbb{A}}$ or $\eta_{\mathbb{A}}(x_n, x_{n+1}) \preceq I_{\mathbb{A}}$ for all $n \in \mathbb{N}$, converges in X.

Definition 2.14. Let $(X, \mathbb{A}, d)$ a C^*-algebra valued b-quasi-metric space and $\alpha_{\mathbb{A}}, \eta_{\mathbb{A}} : X \times X \to \mathbb{A}_+'$ be two functions. The space X is said to be:

(a) $\alpha_{\mathbb{A}}$-regular, if $x_n \to x$, where $\alpha_{\mathbb{A}}(x_n, x_{n+1}) \succeq I_{\mathbb{A}}$ for all $n \in \mathbb{N}$, implies $\alpha_{\mathbb{A}}(x_n, x) \succeq I_{\mathbb{A}}$ for all $n \in \mathbb{N}$.

(b) (X,d) is $\eta_{\mathbb{A}}-$sub-regular, if $x_n \to x$, where $\eta_{\mathbb{A}}(x_n,x_{n+1}) \preceq I_{\mathbb{A}}$ for all $n \in \mathbb{N}$, implies $\eta_{\mathbb{A}}(x_n,x) \preceq I_{\mathbb{A}}$ for all $n \in \mathbb{N}$.

(c) (X,d) is $(\alpha_{\mathbb{A}},\eta_{\mathbb{A}})$-regular, if $x_n \to x$, where $\alpha_{\mathbb{A}}(x_n,x_{n+1}) \succeq I_{\mathbb{A}}$ or $\eta(x_n,x_{n+1}) \preceq I_{\mathbb{A}}$ for all $n \in \mathbb{N}$, imply that $\alpha_{\mathbb{A}}(x_n,x) \succeq I_{\mathbb{A}}$ or $\eta_{\mathbb{A}}(x_n,x) \succeq I_{\mathbb{A}}$ for all $n \in \mathbb{N}$.

Definition 2.15. Let $(X,\mathbb{A},d)$ a C^*-algebra valued b-quasi-metric space. A mapping $T : X \to X$ is a C^*-valued contractive mapping on X, if there exists an $a \in \mathbb{A}$ with $\| \mathbb{A} \| < 1$ such that

$$d(Tx,Ty) \preceq a^* d(x,y) a \tag{2.1}$$

for all $x, y \in \mathbb{A}$.

Theorem 2.16. *Let $(X,\mathbb{A},d)$ a C^*-algebra valued b-quasi-metric space $(\alpha_{\mathbb{A}},\eta_{\mathbb{A}})$-complete, with $s \succeq I_{\mathbb{A}}, s \in \mathbb{A}'$,$\|s\|\|a\|^2 < 1$ suppose that $T : X \to X$, be a contractive mapping satisfies the following conditions:*

(i) *T is triangular $(\alpha_{\mathbb{A}},\eta_{\mathbb{A}})$-admissible.*
(ii) *There exists $x_0 \in X$ such that $\alpha_{\mathbb{A}}(x_0,Tx_0) \succeq I_{\mathbb{A}}$ or $\eta_{\mathbb{A}}(x_0,Tx_0) \preceq I_{\mathbb{A}}$.*
(iii) *T is $(\alpha_{\mathbb{A}},\eta_{\mathbb{A}})$-continuous.*

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha_{\mathbb{A}}(z,u) \succeq I_{\mathbb{A}}$ or $\eta_{\mathbb{A}}(z,u) \preceq I_{\mathbb{A}}$ for all $z, u \in Fix(T)$.

Proof. Let $x_0 \in X$ such that $\alpha_{\mathbb{A}}(x_0,Tx_0) \succeq I_{\mathbb{A}}$ or $\eta_{\mathbb{A}}(x_0,Tx_0) \preceq I_{\mathbb{A}}$. We define the sequence $\{x_n\}$ in X by $x_n = Tx_{n+1}$, for all $n \in \mathbb{N}$.

Since T is an triangular $(\alpha_{\mathbb{A}},\eta_{\mathbb{A}})$-admissible mapping, then

$$\alpha_{\mathbb{A}}(x_0,x_1) = \alpha_{\mathbb{A}}(x_0,Tx_0) \succeq I_{\mathbb{A}} \Rightarrow \alpha_{\mathbb{A}}(Tx_0,Tx_1) = \alpha_{\mathbb{A}}(x_1,x_2) \succeq I_{\mathbb{A}}$$

or

$$\eta_{\mathbb{A}}(x_0,x_1) = \eta_{\mathbb{A}}(x_0,Tx_0) \preceq I_{\mathbb{A}} \Rightarrow \eta_{\mathbb{A}}(Tx_0,Tx_1) = \eta_{\mathbb{A}}(x_1,x_2) \preceq I_{\mathbb{A}}$$

Continuing this process we have

$$\alpha_{\mathbb{A}}(x_{n-1},x_n) \succeq I_{\mathbb{A}}$$

or

$$\eta_{\mathbb{A}}(x_{n-1},x_n) \preceq I_{\mathbb{A}},$$

for all $n \in \mathbb{N}$. By (T_3) and (T_4), one has.

$$\alpha_{\mathbb{A}}(x_m,x_n) \succeq I_{\mathbb{A}} \text{ or } \eta_{\mathbb{A}}(x_m,x_n) \preceq I_{\mathbb{A}}, \qquad \forall m,n \in \mathbb{N},\ m \neq n. \tag{2.2}$$

Suppose that there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = Tx_{n_0}$. Then x_{n_0} is a fixed point of T and the prove is finished. Hence, we assume that $x_n \neq Tx_n$, i.e. $d(x_{n-1},x_n) > 0$ for all $n \in \mathbb{N}$.

***Step* 1**: Applying inequality 2.1 with $x = x_{n-1}$ and $y = x_n$, we obtain

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \\ &\preceq a^* d(x_{n-1},x_n) a \\ &\preceq (a^*)^2 d(x_{n-2},x_{n-1}) a^2 \\ &\preceq ... \\ &\preceq (a^*)^n d(x_0,x_1) a^n. \end{aligned}$$

Notice that in C^*-algebra, if $a, b \in \mathbb{A}^+$ and $0_{\mathbb{A}} \preceq a \preceq b$, then for any $x \in A$ both x^*ax and x^*bx are positive elements and

$$0_{\mathbb{A}} \preceq x^*ax \preceq x^*bx.$$

By property (ii) of the C^*-algebra valued b-quasi-metric space, we have

$$\begin{aligned}
d(x_n, x_{n+m}) &\preceq s\left[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+m})\right] \\
&\preceq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) \\
&+ \dots \\
&+ s^{m-1} d(x_{n+m-2}, x_{n+m-1}) + s^{m-1} d(x_{n+m-1}, x_{n+m}) \\
&\preceq s(a^*)^n d(x_0, x_1)\, a^n + s^2 (a^*)^{n+1} d(x_0, x_1)\, a^{n+1} \\
&+ \dots + s^{m-1}(a^*)^{n+m-2} d(x_0, x_1)\, a^{n+m-2} + s^{m-1}(a^*)^{n+m-1} d(x_0, x_1)\, a^{n+m-1} \\
&= \sum_{i=1}^{i=m-1} s^k (a^*)^{n+k-1} d(x_0, x_1)\, a^{n+k-1} + s^{m-1}(a^*)^{n+m-1} d(x_0, x_1)\, a^{n+m-1} \\
&= \sum_{i=n}^{i=n+k-1} (s^{\frac{k}{2}} a^{n+k-1} d(x_0, x_1)^{\frac{1}{2}})^* s^{\frac{k}{2}} d(x_0, x_1)^{\frac{1}{2}}\, a^{n+k-1} \\
&+ s^{\frac{m-1}{2}} (a^*)^{n+m-1} d(x_0, x_1)^{\frac{1}{2}}\, s^{\frac{m-1}{2}} a^{n+m-1} d(x_0, x_1)^{\frac{1}{2}} \\
&= \sum_{i=n}^{i=n+k-1} (s^{\frac{k}{2}} a^{n+k-1} d(x_0, x_1)^{\frac{1}{2}})^* s^{\frac{k}{2}} d(x_0, x_1)^{\frac{1}{2}}\, a^{n+k-1} \\
&+ (s^{\frac{m-1}{2}} (a)^{n+m-1} d(x_0, x_1)^{\frac{1}{2}})^* s^{\frac{m-1}{2}} a^{n+m-1} d(x_0, x_1)^{\frac{1}{2}} \\
&= \sum_{i=n}^{i=n+k-1} |s^{\frac{k}{2}} d(x_0, x_1)^{\frac{1}{2}}\, a^{n+k-1}|^2 + |s^{\frac{m-1}{2}} a^{n+m-1} d(x_0, x_1)^{\frac{1}{2}}\,|^2 \\
&\preceq \sum_{i=n}^{i=n+k-1} \|s^{\frac{k}{2}} d(x_0, x_1)^{\frac{1}{2}}\, a^{n+k-1}\|^2 . I_{\mathbb{A}} + \|s^{\frac{m-1}{2}} a^{n+m-1} d(x_0, x_1)^{\frac{1}{2}}\,\|^2 . I_{\mathbb{A}} \\
&\preceq \|d(x_0, x_1)^{\frac{1}{2}}\,\|^2 \sum_{i=n}^{i=n+k-1} \|s^{\frac{k}{2}} a^{n+k-1}\|^2 . I_{\mathbb{A}} + \|s^{\frac{m-1}{2}} a^{n+m-1} d(x_0, x_1)^{\frac{1}{2}}\,\|^2 . I_{\mathbb{A}} \\
&= \|d(x_0, x_1)\| \sum_{i=n}^{i=n+k-1} \|s^k\| \|a^{2(n+k-1)}\| . I_{\mathbb{A}} + \|d(x_0, x_1)\| \|\|s\|^{m-1} \|a\|^{2(n+m-1)} . I_{\mathbb{A}} \\
&\preceq \|d(x_0, x_1)\| \left[\|s\| \|a\|^{2n}\right] \left(\frac{1 - (\|s\| \|a^2\|)^{m-1}}{1 - \|s\|a\|^2\|} \right) . I_{\mathbb{A}} \\
&+ \|d(x_0, x_1)\| \|\|s\|^{m-1} \|a\|^{2(n+m-1)} . I_{\mathbb{A}} \to 0_{\mathbb{A}} (n \to \infty)
\end{aligned}$$

with the condition $\|s\|a\|^2 < 1$ and at $n \to \infty$. Therefore x_n is a forward Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$ there exists an $z \in X$ such that

Step 2: Substituting $x = x_n$ and $y = x_{n-1}$, from (2.1), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d\left(x_{n+1}, x_n\right) &= d\left(Tx_n, Tx_{n-1}\right) \\ &\preceq a^* d\left(x_n, x_{n-1}\right) a \\ &\preceq (a^*)^2 d\left(x_{n-1}, x_{n-2}\right) a^2 \\ &\preceq ... \\ &\preceq (a^*)^n d\left(x_1, x_0\right) a^n. \end{aligned}$$

By property (ii) of the C^*-algebra valued b-quasi-metric space, we have

$$\begin{aligned} d\left(x_{n+m}, x_n\right) &\preceq s\left[d\left(x_{n+m}, x_{n+m-1}\right) + d\left(x_{n+m-2}, x_n\right)\right] \\ &\preceq sd\left(x_{n+m}, x_{n+m-1}\right) + s^2 d\left(x_{n+m-1}, x_{n+m-2}\right) \\ &+ ... \\ &+ s^{m-1} d\left(x_{n+2}, x_{n+1}\right) + s^{m-1} d\left(x_{n+1}, x_n\right) \\ &\preceq s(a^*)^n d\left(x_1, x_0\right) a^n + s^2 (a^*)^{n+1} d\left(x_1, x_0\right) a^{n+1} \\ &+ ... + s^{m-1}(a^*)^{n+m-2} d\left(x_1, x_0\right) a^{n+m-2} + s^{m-1}(a^*)^{n+m-1} d\left(x_1, x_0\right) a^{n+m-1} \\ &= \sum_{i=1}^{i=m-1} s^k (a^*)^{n+k-1} d\left(x_1, x_0\right) a^{n+k-1} + s^{m-1}(a^*)^{n+m-1} d\left(x_1, x_0\right) a^{n+m-1} \\ &\preceq \|d\left(x_1, x_0\right)\| \left[\|s\|\|a\|^{2n}\right] \left(\frac{1 - (\|s\|\|a^2\|)^{m-1}}{1 - \|s\|a\|^2\|}\right).I_{\mathbb{A}} \\ &+ \|d\left(x_1, x_0\right)\|\|s\|^{m-1}\|a\|^{2(n+m-1)}.I_{\mathbb{A}} \to 0_{\mathbb{A}} (n \to \infty) \end{aligned}$$

Therefore x_n is a backward Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$ there exists an $u \in X$ such that

$$\lim_{n\to\infty} d(u, x_n) = 0_{\mathbb{A}}.$$

So, from Lemma 2.8, we get $z = u$.

On has Since T is $(\alpha_{\mathbb{A}}, \eta_{\mathbb{A}})$ $-$continuous, we have

$$\lim_{n\to\infty} d(Tx_n, Tz) = 0_{\mathbb{A}}$$

and

$$\lim_{n\to\infty} d(Tz, Tx_n) = 0_{\mathbb{A}}.$$

Then $d(Tz, z) = 0_{\mathbb{A}}$ *or* $d(z, Tz) = 0_{\mathbb{A}}$. Thus $z = Tz$ is a fixed point for T.

Uniqueness: Suppose that $u \neq z$ is another fixed point of T. Since

$$\begin{aligned} 0_{\mathbb{A}} \preceq d\left(z, u\right) = d\left(Tz, Tu\right) &\preceq a^* d(z, u) a \\ &\preceq \|a^* d(z, u) a\| \\ &\preceq \|a^*\|\|d(z, u)\|\|a\| \\ &= \|a\|^2 \|d(z, u)\| \\ &< \|d(z, u)\|, \ \textit{which is a contradiction.} \end{aligned}$$

Hence $d\left(z, u\right) = 0_{\mathbb{A}}$ and $z = u$, which implies that the fixed point is unique. □

Now, we replace the assumption of $(\alpha_{\mathbb{A}},\eta_{\mathbb{A}})$continuoity of T in the above theorem by another condition.

Theorem 2.17. *Let $(X,\mathbb{A},d)$ a C^*-algebra valued b-quasi-metric space $(\alpha_{\mathbb{A}},\eta_{\mathbb{A}}$ complete, with $b\succeq I_{\mathbb{A}}, b\in\mathbb{A}'$,$\|s\|\|a\|^2<1$ suppose that $T:X\to X$, be a contractive mapping satisfies the following conditions:*

(i) *T is triangular $(\alpha_{\mathbb{A}},\eta_{\mathbb{A}})$-admissible.*
(ii) *There exists $x_0\in X$ such that $\alpha_{\mathbb{A}}(x_0,Tx_0\succeq I_{\mathbb{A}}$ or $\eta_{\mathbb{A}}(x_0,Tx_0\preceq I_{\mathbb{A}}$.*
(iii) *(X,d) is a $(\alpha_{\mathbb{A}},\eta_{\mathbb{A}})$-$C^*$-algebra valued b-quasi-metric space.*

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha_{\mathbb{A}}(z,u)\succeq_{\mathbb{A}} I$ or $\eta_{\mathbb{A}}(z,u)\preceq I_{\mathbb{A}}$ for all $z,u\in Fix(T)$.

Proof. Similar to the proof of Theorem 2.16, we can conclude that

$$(\alpha_{\mathbb{A}}(x_n,x_{n+1})\succeq I_{\mathbb{A}} \text{ or } \eta_{\mathbb{A}}(x_n,x_{n+1})\preceq I_{\mathbb{A}}),\ \text{ and } x_n\to z \text{ as } n\to\infty,$$

Since $(X,,\mathbb{A},d)$ is $(\alpha_{\mathbb{A}},\eta_{\mathbb{A}})$-regular, then $\alpha_{\mathbb{A}}(x_n,x)\succeq I_{\mathbb{A}}$ or $\eta_{\mathbb{A}}(x_n,x)\succeq I_{\mathbb{A}}$ for all $n\in\mathbb{N}$. We can apply (2.1), to x_n and z for all $n>n_0$ to get

$$\begin{aligned}0_{\mathbb{A}}\preceq d(z,Tz)&\preceq s\left[d(z,Tx_n)+d(Tx_n,Tz)\right]\\ &\preceq s\left[(d(z,Tz)+a^*d(x_n,z)a\right]\to s\left(d(z,Tz)\right)(as\ n\to\infty).\end{aligned}$$

It is a contradiction. Hence $d(z,Tz)=0$.

New, Applying inequality (2.1) with $x=z$ and $y=x_n$, we obtain

$$\begin{aligned}0_{\mathbb{A}}\preceq d(Tz,z)&\preceq s\left[d(Tz,Tx_n)+d(Tx_n,z)\right]\\ &\preceq s\left[a^*d(z,x_n)a+d(Tx_n,z)\right]\to s\left(d(z,Tz)\right)(as\ n\to\infty).\end{aligned}$$

It is a contradiction. Hence $d(Tz,z)=0$. Thus $z=Tz$ is a fixed point for T.

The proof of the uniqueness is similarly to that of Theorem 2.16. □

Example 2.18. Let $X=[1,+\infty[$, $\mathbb{A}=M_2(\mathbb{R})$ of all 2×2 matrices with the usual addition ,scalar multiplication and multiplication.

Define partial ordering on $\mathbb{A}$ as $\begin{pmatrix}a_1 & a_2\\ a_3 & a_4\end{pmatrix}\succeq\begin{pmatrix}b_1 & b_2\\ b_3 & b_4\end{pmatrix}\Leftrightarrow a_i\geq b_i$ for $i=1,2,3,4$. For any $A\in\mathbb{A}$ we define its norm as, $\|\begin{pmatrix}a_1 & a_2\\ a_3 & a_4\end{pmatrix}\|=\left[\sum_{i=1}^{i=4}|a_i|^2\right]^{\frac{1}{2}}$.

We define $d:X\times X\to[0,+\infty[$ as

$$\begin{cases} d(x,y)=\begin{pmatrix}(x-y)^3 & 0\\ 0 & (x-y)^3\end{pmatrix} & if\ x\preceq y\\ d(x,y)=\begin{pmatrix}(y-x)^3 & 0\\ 0 & (y-x)^3\end{pmatrix} & if\ x\prec y\end{cases}$$

Then $(X,\mathbb{A}_+,d)$ is a C^*-algebra valued rectangular quasi-metric space. Define mapping $T:X\to X$ by

$$T(x)=\sqrt{x}$$

and

$$\alpha_{\mathbb{A}}(x,y)=\begin{pmatrix}\frac{x+y}{\max\{x,y\}+1} & 0\\ 0 & \frac{x+y}{\max\{x,y\}+1}\end{pmatrix}.$$

$$\eta_{\mathbb{A}}(x,y)=\begin{pmatrix}\frac{|x-y|}{\max\{x,y\}+1} & 0\\ 0 & \frac{|x-y|}{\max\{x,y\}+1}\end{pmatrix}.$$

Evidently, $T(x)\in X$ and Then, T is an $((\alpha_{\mathbb{A}},\eta_{\mathbb{A}})$ −continuous triangular (α,η) −admissible mapping.

Consider the following possibilities:

case 1 : $x\succeq y$. Then

$$T(x)=\sqrt{x},\ T(y)=\sqrt{x},\ d(Tx,Ty)=\begin{pmatrix}(\sqrt{x}-\sqrt{y})^3 & 0\\ 0 & (\sqrt{x}-\sqrt{y})^3\end{pmatrix}.$$

On the other hand

$$d(x,y)=\begin{pmatrix}(x-y)^3 & 0\\ 0 & (x-y)^3\end{pmatrix}.$$

it follows that

$$d(Tx,Ty)\preceq a^*d(x,y)a. \tag{2.3}$$

Indeed

$$\begin{aligned}d(Tx,Ty)&=\begin{pmatrix}(\sqrt{x}-\sqrt{y})^3 & 0\\ 0 & (\sqrt{x}-\sqrt{y})^3\end{pmatrix}\\ &\preceq\begin{pmatrix}\frac{1}{\sqrt{8}} & 0\\ 0 & \frac{1}{\sqrt{8}}\end{pmatrix}\begin{pmatrix}(x-y)^3 & 0\\ 0 & (x-y)^3\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{8}} & 0\\ 0 & \frac{1}{\sqrt{8}}\end{pmatrix}\\ &=a^*d(x,y)a.\end{aligned}$$

where

$$a=\begin{pmatrix}\frac{1}{\sqrt{8}} & 0\\ 0 & \frac{1}{\sqrt{8}}\end{pmatrix}$$

with verify

$$\|a\|=\frac{\sqrt{2}}{\sqrt{8}}=\frac{1}{4}<1.$$

case 2 : $x\prec y$. Then

$$T(x)=\sqrt{x},\ T(y)=\sqrt{x},\ d(Tx,Ty)=\begin{pmatrix}(\sqrt{y}-\sqrt{x})^3 & 0\\ 0 & (\sqrt{y}-\sqrt{x})^3\end{pmatrix}.$$

On the other hand

$$d(x,y)=\begin{pmatrix}(y-x)^3 & 0\\ 0 & (y-x)^3\end{pmatrix}.$$

it follows that

$$d(Tx,Ty)\preceq a^*d(x,y)a. \tag{2.4}$$

Indeed

$$\begin{aligned} d(Tx,Ty) &= \begin{pmatrix} (\sqrt{y}-\sqrt{x})^3 & 0 \\ 0 & (\sqrt{y}-\sqrt{x})^3 \end{pmatrix} \\ &\preceq \begin{pmatrix} \frac{1}{\sqrt{8}} & 0 \\ 0 & \frac{1}{\sqrt{8}} \end{pmatrix} \begin{pmatrix} (y-x)^3 & 0 \\ 0 & (y-x)^3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{8}} & 0 \\ 0 & \frac{1}{\sqrt{8}} \end{pmatrix} \\ &= a^* d(x,y) a. \end{aligned}$$

where

$$a = \begin{pmatrix} \frac{1}{\sqrt{8}} & 0 \\ 0 & \frac{1}{\sqrt{8}} \end{pmatrix}$$

with verify

$$\|a\| = \frac{\sqrt{2}}{\sqrt{8}} = \frac{1}{4} < 1.$$

Hence, the condition (2.1) is satisfied. Therefore, T has a unique fixed point $z = 1$.

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