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ON f-KENMOTSU 3-MANIFOLDS ADMITTING W_2-CURVATURE TENSOR

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Abstract. In this study, we make the contribution of some results on 3-dimensional f-Kenmotsu manifolds under some certain conditions admitting on W_2-curvature tensor. We have also established an example of 3-dimensional f-Kenmotsu manifold.

1. Introduction

Let M^n be an almost contact manifold with an almost contact metric structure (ϕ, ξ, η, g) [1]. We denote by K, the fundamental 2-form of M^n, i.e., $K(X,Y)=g(X,\phi Y)$ for any vector fields $X.Y \in \chi(M^n)$, where $\chi(M^n)$ being the Lie algebra of differentiable vector fields on M^n. Furthermore, we recollect the following definitions [1,6,17].

The manifold M^n and its structure (ϕ, ξ, η, g) is said to be

i) normal if the almost complex structure defined on the product manifold $M^n \times \Re$ is integrable (equivalently,$[\phi, \phi]+2d\eta \otimes \xi = 0$),

ii) almost cosymplectic if $d\eta$=0 and dϕ=0,

iii) cosymplectic if it is normal and almost cosymplectic (equivalently, $\nabla\phi$=0, where ∇ is covariant differentiation with respect to the Levi-Civita connection). The manifold M^n is called locally conformal almost cosymplectic (respectively, locally conformal cosymplectic) if M^n has an open covering (V_t) endowed with differentiable functions $\delta_t : V_i \to R$ such that over each (V_t) the almost contact metric structure $(\phi_t, \xi_t, \eta_t, g_t)$ defined by

$$\phi_t = \phi,\ \xi_t = e^{\delta_t}\xi,\ \eta_t = e^{-\delta_t}\eta,\ g_t = e^{-2\delta_t}g$$

is almost cosymplectic (respectively, locally conformal cosymplectic). Normal locally conformal almost cosymplectic manifold were studied by Olszak and Rosca [12]. An almost contact metric manifold is said to be f-Kenmotsu if it is normal and locally conformal almost cosymplectic. Such type of manifold was also studied by several authors [2, 7, 8, 13, 19–23]. Olszak and Rosca [12] also gave a geometric interpretation of f-Kenmotsu manifolds and studied some curvature restrictions. Among others, they proved that a Ricci symmetric f-Kenmotsu manifold is an Einstein manifold.

Pokhariyal and Mishra [15] have introduced new tensor fields known as W_2 and E-tensor fields, in Riemannian manifold and studied its properties. After that Pokhariyal [14] has studied some certain properties of this tensor field in a Sasakian manifold. Moreover Matsumoto et.

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al. [?] have studied P-Sasakian manifolds with W_2 and E-tensor fields. After that De and Sarkar [3] have studied P-Sasakian manifolds equipped with W_2-tensor field. The curvature tensor W_2 is defined by

$$W_2(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Z)S(Y,U) - g(Y,Z)S(X,U)], \tag{1.1}$$

where S is a Ricci tensor of type $(0,2)$.
The outline of this paper is to study, some certain curvature conditions on 3-dimensional f-Kenmotsu manifolds. First we examine its geometric and relativistic properties in 3-dimensional f-Kenmotsu manifolds satisfying $W_2 = 0$ and W_2-semi-symmetric. Also we characterize such manifolds which satisfies some certain conditions, that is, $P \cdot W_2$=0, $T \cdot W_2$=0, $C \cdot W_2$=0 and $H \cdot W_2$=0 where P is the projective curvature tensor, T is the concircular curvature tensor, C is the conformal curvature tensor and H is the quasi- conformal curvature tensor.

2. f-Kenmotsu manifolds $(M^{2n+1}, \phi, \xi, \eta, g)$

An odd dimensional smooth manifold M^{2n+1} is said to be an almost contact metric manifold if there exist a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g on M^{2n+1} such that

$$\phi^2 = -I + \eta \otimes \xi, \ \ \eta(\xi) = 1, \ \ \eta(X) = g(X,\xi), \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

for any vector fields $X, Y \in \chi(M^{2n+1})$. Such a manifold of dimension $(2n+1)$ is denoted by $M^{2n+1}(\phi, \xi, \eta, g)$ and it is known as f-Kenmotsu manifold if the covariant differentiation of ϕ satisfies [9, 12].

$$(\nabla_X \phi)Y = f\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{2.3}$$

where $f \in C^\infty(M)$ such that $df \wedge \eta$=0. If f=$\alpha(\neq 0)$ is constant then the manifold is a α-Kenmotsu manifold [9]. Kenmotsu manifold is an example of f-Kenmotsu manifold with f=1 [10, 16]. If f=0, then the manifold is cosymplectic [9]. An f-Kenmotsu manifold is said to be regular if $f^2 + f^{'} \neq 0$, where $f^{'}$=ξf. For an f-Kenmotsu manifold it follows from (2.3) that

$$\nabla_X \xi = f\{X - \eta(X)\,\xi\}. \tag{2.4}$$

The condition $df \wedge \eta$=0 holds if dim. $M \geq 5$, in general and does not hold if dim. M=3 [16]. In a 3-dimensional Riemannian manifold, we have

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &- S(X,Z)Y - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{2.5}$$

In a 3-dimensional f-Kenmotsu manifold we have [12]:

$$\begin{aligned} R(X,Y)Z &= \left(\frac{r}{2} + 2f^2 + 2f^{'}\right)(X \wedge Y)Z \\ &- \left(\frac{r}{2} + 3f^2 + 3f^{'}\right)\{\eta(X)(\xi \wedge Y)Z + \eta(Y)(X \wedge \xi)Z\} \end{aligned} \tag{2.6}$$

$$S(X,Y) = \left(\frac{r}{2} + f^2 + f^{'}\right)g(X,Y) - \left(\frac{r}{2} + 3f^2 + 3f^{'}\right)\eta(X)\eta(Y), \tag{2.7}$$

(2.8) $$QX = \left(\frac{r}{2} + f^2 + f^{'}\right) X - \left(\frac{r}{2} + 3f^2 + 3f^{'}\right) \eta(X)\xi,$$

where r is the scalar curvature. From (2.6) and (2.7) we obtain

(2.9) $$R(X,Y)\xi = -\left(f^2 + f^{'}\right)\left[\eta(Y)X - \eta(X)Y\right],$$

(2.10) $$R(\xi,Y)Z = -\left(f^2 + f^{'}\right)\left[g(Y,Z)\xi - \eta(X)Y\right],$$

(2.11) $$S(X,\xi) = -2\left(f^2 + f^{'}\right)\eta(X),$$

(2.12) $$S(\xi,\xi) = -2\left(f^2 + f^{'}\right),$$

(2.13) $$Q\xi = -2\left(f^2 + f^{'}\right)\xi.$$

As a consequence of (2.4) we also have

(2.14) $$(\nabla_X\eta)(Y) = fg(\phi X, \phi Y).$$

Also from (2.10) it follows that

(2.15) $$S(\phi X,\phi Y) = S(X,Y) + 2\left(f^2 + f^{'}\right)\eta(X)\eta(Y)$$

for all vector fields X, Y.

The notion of a quasi-conformal curvature tensor H was given by Yano and Sawaki [18] and is defined by

(2.16) $$\begin{aligned} &H(X,Y)Z = \alpha R(X,Y)Z + \beta[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &-g(X,Z)QY] - \tfrac{r}{n}\left(\tfrac{\alpha}{n-1} + 2\beta\right)\{g(Y,Z)X - g(X,Z)Y\},\end{aligned}$$

where α and β are constant and R, S, Q and r are the Riemannian curvature tensor of type $(1,3)$, the Ricci tensor of type $(0,2)$, the Ricci operator defined by $g(QX,Y)$=$S(X,Y)$.

If α=1 and β=-$\frac{1}{n-2}$ then it reduces to conformal curvature tensor [5] which is defined by

(2.17) $$\begin{aligned} &H(X,Y)Z = R(X,Y)Z \\ &-\tfrac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &+\tfrac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] = C(X,Y)Z\end{aligned}$$

We define endomorphism $R(X,Y)$ and $X\wedge_A Y$ of $\aleph(M)$ by

(2.18) $$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

(2.19) $$(X\wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y,$$

respectively, where $X, Y, Z \in \aleph(M)$ and A is the symmetric $(0,2)$-tensor. Beside this, the projective curvature tensor P and the concircular curvature tensor T in a Riemannian manifold (M^n, g) are defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}(X \wedge_S Y)Z, \tag{2.20}$$

$$T(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}(X \wedge_g Y)Z, \tag{2.21}$$

respectively.

A regular f- manifold (M, ϕ, ξ, η, g) is said to be an Einstein manifold if its Ricci tensor S satisfies

$$S(X,Y) = c_1 g(X,Y), \tag{2.22}$$

for any vector fields X, Y and c_1 is a certain scalar.

A Riemannian or a semi-Riemannian manifold is said to semisymmetric if $R(X,Y)\cdot R{=}0$, where $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for the tangent vectors X, Y. Using (2.8), (2.9) and (2.12), the equation (2.17),(2.21) and (2.22), it follows that

$$P(\xi, X, Y) = -(f^2 + f^{'})g(X,Y)\xi - \frac{1}{2}S(X,Y)\xi, \tag{2.23}$$

$$T(\xi, X, Y) = \left\{(f^2 + f^{'}) + \frac{r}{6}\right\}(g(X,Y)\xi - \eta(Y)X), \tag{2.24}$$

$$\begin{aligned} C(\xi, X, Y) = &\left\{-(f^2 + f^{'}) + \tfrac{r}{2}\right\}(g(X,Y)\xi - \eta(Y)X) \\ &-S(X,Y)\xi + 2(f^2 + f^{'})(g(X,Y)\xi + 2\eta(Y)X), \end{aligned} \tag{2.25}$$

$$H(\xi, X, Y) = \lambda_1\{g(X,Y)\xi - \eta(Y)X\} + bS(X,Y)\xi + \lambda_2\{g(X,Y)\xi - 2\eta(Y)X\}, \tag{2.26}$$

respectively, where $\lambda_1 = -\left\{a(f^2 + f^{'}) + \frac{r}{3}\left(2b + \frac{a}{2}\right)\right\}$, $\lambda_2 = -2b(f^2 + f^{'})$.

Proposition 2.1. *In a 3-dimensional f-Kenmotsu manifold $(M^3, \phi, \xi, \eta, g,)$ the W_2-curvature tensor satisfies the condition*

$$W_2(X,Y,Z,\xi) = 0. \tag{2.27}$$

3. W_2-Flat f-Kenmotsu manifold$(M^{2n+1}, \phi, \xi, \eta, g)$

In this section, we study the geometric and relativistic properties of f-Kenmotsu manifold admitting vanishing W_2-curvature. Let $(M^3, \phi, \xi, \eta, g)$ be a f-Kenmostu manifold satisfying $W_2{=}0$. Then from (1.1), it follows

$$R(X,Y,Z,U) = \frac{1}{2}[g(Y,Z)S(X,U) - g(X,Z)S(Y,U)]. \tag{3.1}$$

Putting $X{=}Z{=}\xi$ in (3.1), using (2.1),(2.9)and (2.11), we get

$$S(Y,U) = 2(f^2 + f^{'})g(Y,U). \tag{3.2}$$

Thus M^3 is an Einstein manifold. Thus we have the following result.

Theorem 3.1. *Let $(M^3, \phi, \xi, \eta, g)$ be a 3-dimensional f-Kenmotsu manifold satisfying W_2=0. Then M^3 is an Einstein manifold.*

Also from (3.1) and (3.2), we have

$$R(X, Y, Z, U) = (f^2 + f^{'}) \{g(X, Z)g(Y, U) - g(Y, Z)g(X, U)\}. \tag{3.3}$$

If f=α=*constant* $\neq 0$. It follows that M^3 is of constant curvature $(-\alpha^2)$.Thus we state the following result.

Corollary 3.2. *A W_2-flat 3-dimensional α-Kenmotsu manifold $(M^3, \phi, \xi, \eta, g)$ is locally isometric to the hyperbolic space $H^3(-\alpha^2)$.*

Also if f=1. This implies that the manifold is Kenmotsu manifold.

Corollary 3.3. *A W_2-flat 3-dimensional Kenmotsu manifold $(M^3, \phi, \xi, \eta, g)$ is locally isometric to the hyperbolic space $H^3(-1)$.*

Corollary 3.4. *A W_2-flat 3-dimensional cosymplectic manifold $(M^3, \phi, \xi, \eta, g)$ is an Euclidean space.*

4. W_2-semisymmetric f-Kenmotsu manifold

This section is affectionate to the study of f-Kenmotsu manifold with W_2-semisymmetric. On that account, we can proof some certain the result.
A 3-dimensional f-Kenmotsu manifold is said to be W_2-semisymmetric if it satisfies the condition

$$R(X, Y) \cdot W_2 = 0, \tag{4.1}$$

where $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for the tangent vectors X, Y.
From (4.1), it follows that

$$\begin{aligned} &R(X, Y)W_2(Z, U)V - W_2(R(X, Y)Z, U)V \\ &\quad -W_2(Z, R(X, Y)U)V - W_2(Z, U)R(X, Y)V = 0. \end{aligned} \tag{4.2}$$

Which implies

$$\begin{aligned} &g(R(X, Y)W_2(Z, U)V, \xi) - g(W_2(R(X, Y)Z, U)V, \xi) \\ &\quad -g(W_2(Z, R(X, Y)U)V, \xi) - g(W_2(Z, U)R(X, Y)V, \xi) = 0. \end{aligned} \tag{4.3}$$

Taking X=ξ in (4.2), it yield

$$\begin{aligned} &g(R(\xi, Y)W_2(Z, U)V, \xi) - g(W_2(R(\xi, Y)Z, U)V, \xi) \\ &\quad -g(W_2(Z, R(\xi, Y)U)V, \xi) - g(W_2(Z, U)R(\xi, Y)V, \xi) = 0. \end{aligned} \tag{4.4}$$

With the help of (2.1) and (2.9), equation (4.4) take the form

$$\begin{aligned} &-(f^2 + f^{'}) \{g(Y, W_2(Z, U)V)\xi - \eta(W_2(Z, U)V)\eta(Y)\} \\ &+(f^2 + f^{'}) \{g(Y, Z)\eta(W_2(\xi, U)V) - \eta(Z)\eta(W_2(Y, U)V)\} \\ &+(f^2 + f^{'}) \{g(Y, U)\eta(W_2(Z, \xi)V) - \eta(U)\eta(W_2(Z, Y)V)\} \\ &+(f^2 + f^{'}) \{g(Y, V)\eta(W_2(Z, \xi)U) - \eta(V)\eta(W_2(Z, U)V)\} = 0. \end{aligned} \tag{4.5}$$

Taking the inner product of (4.5) with ξ and using (2.27), we get

$$-(f^2 + f^{'})W_2(Z, U, V, Y) = 0. \tag{4.6}$$

This implies

$$W_2(Z,U,V,Y)=0. \tag{4.7}$$

Therefore M^{2n+1} is W_2-flat. So according to Theorem 3.1, we can state the following result.

Theorem 4.1. *If a 3-dimensional regular f-Kenmotsu manifold (M^3,ϕ,ξ,η) is W_2-semisymmetric. Then it is an Einstein manifold.*

Corollary 4.2. *A W_2-semisymmetric 3-dimensional regular f-Kenmotsu manifold (M^3,ϕ,ξ,η,g) is W_2-flat.*

Corollary 4.3. *A W_2-semisymmetric 3-dimensional α-Kenmotsu manifold (M^3,ϕ,ξ,η,g) is locally isometric to the hyperbolic space $H^3(-\alpha^2)$.*

Corollary 4.4. *A W_2-semisymmetric 3-dimensional Kenmotsu manifold (M^3,ϕ,ξ,η,g) is locally isometric to the hyperbolic space $H^3(-1)$.*

Corollary 4.5. *A W_2-semisymmetric 3-dimensional cosymplectic manifold (M^3,ϕ,ξ,η,g) is an Euclidean space.*

5. f-Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)$ satisfying $P(X,Y)\cdot W_2=0$

This section concern with the study of f-Kenmotsu manifold bearing the condition

$$P(X,Y)\cdot W_2=0. \tag{5.1}$$

This implies

$$\begin{aligned}&P(X,Y)W_2(Z,U)V-W_2(P(X,Y)Z,U)V\\&-W_2(Z,P(X,Y)U)V-W_2(Z,U)P(X,Y)V=0.\end{aligned} \tag{5.2}$$

Substituting $X=\xi$ in (5.2), we have

$$\begin{aligned}&P(\xi,Y)W_2(Z,U)V-W_2(P(\xi,Y)Z,U)V\\&-W_2(Z,P(\xi,Y)U)V-W_2(Z,U)P(\xi,Y)V=0.\end{aligned} \tag{5.3}$$

In view of (2.23) and (5.3), it takes the form

$$\begin{aligned}&-2(f^2+f')\{g(Y,W_2(Z,U)V)\xi-g(Y,Z)\eta(W_2(\xi,U)V)\\&-g(Y,U)\eta(W_2(Z,\xi)V)-g(Y,U)\eta(W_2(Z,U)\xi)\\&-S(Y,W_2(Z,U)V)\xi+S(Y,Z)\eta(W_2(\xi,U)V)\\&+S(Y,U)\eta(W_2(Z,\xi)V)+S(Y,V)\eta(W_2(Z,U)\xi)=0.\end{aligned} \tag{5.4}$$

Taking the inner product of (5.4) with ξ and using (2.27), we get

$$-2(f^2+f')g(Y,W_2(Z,U)V)+S(Y,W_2(Z,U)V)=0. \tag{5.5}$$

With the help of (1.1), equation (5.5)) reduces to

$$\begin{aligned}&2(f^2+f')\left\{R(Z,U,V,Y)=\tfrac{1}{2}\left(g(Z,V)S(U,Y)-g(U,V)S(Z,Y)\right)\right\}\\&+\tfrac{1}{2}R(Z,U,V,QY)+\tfrac{1}{4}\left\{g(Z,V)S(QY,U)-g(U,V)S(Z,QY)\right\}=0,\end{aligned} \tag{5.6}$$

where $S(QY,Z)=S^2(Y,Z)$.

Putting $Z=V=\xi$ in (5.6), using (2.1), (2.10)and (2.11), we obtain

$$S^2(Y,U)=-4(f^2+f')S(Y,U)-4(f^2+f')^2g(Y,U). \tag{5.7}$$

So, we can state the following result.

Theorem 5.1. *If in a 3-dimensional regular f-Kenmotsu manifold satisfies the condition $P(X,Y)\cdot W_2$=0. Then the equation(5.7) is holds on M^3.*

Again, we consider the Lemma that was proved by Deszcz et al. as follows

Lemma 5.2. *[4] Let A be a symmetric $(0,2)$-tensor at point x of a semi-Riemannian manifold (M^n,g), $n>1$, and let T=$g\wedge A$ be the Kulkarni-Nomizu product of g and A. Then the relation*

$$T\cdot T=\alpha Q(g,T),\ \ \alpha\in\Re \tag{5.8}$$

is satisfied at x if and only if the condition

$$A^2=A\alpha+\lambda g,\ \ \lambda\in\Re \tag{5.9}$$

holds at x.

With the help of Theorem 5.1 and Lemma 5.2 we have the following result.

Corollary 5.3. *Let M be an 3-dimensional α-Kenmostu manifold satisfying the condition $P(X,Y)\cdot W_2=0$. Then $T\cdot T$=$\alpha Q(g,T)$, where T=$g\wedge S$ and α=-4α^2.*

Corollary 5.4. *Let M be an 3-dimensional Kenmostu manifold satisfying the condition $P(X,Y)\cdot W_2=0$. Then $T\cdot T$=$\alpha Q(g,T)$, where T=$g\wedge S$ and α=-4.*

6. f-Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)$ satisfying $T(X,Y)\cdot W_2$=0

This segment is affectionate to the study of f-Kenmotsu manifold satisfying the condition $T(X,Y)\cdot W_2$=0 and deduce some certain result.
Let (M^3,ϕ,ξ,η,g) be a f-Kenmotsu manifold satisfying the condition

$$T(X,Y)\cdot W_2=0. \tag{6.1}$$

This implies

$$\begin{aligned}&T(X,Y)W_2(Z,U)V-W_2(T(X,Y)Z,U)V\\&-W_2(Z,T(X,Y)U)V-W_2(Z,U)T(X,Y)V=0.\end{aligned} \tag{6.2}$$

Let $X=\xi$. Then (6.2) implies

$$\begin{aligned}&T(\xi,Y)W_2(Z,U)V-W_2(T(\xi,Y)Z,U)V\\&-W_2(Z,T(\xi,Y)U)V-W_2(Z,U)T(\xi,Y)V=0.\end{aligned} \tag{6.3}$$

Using (2.24) in (6.3), we have

$$\begin{aligned}&-\left\{(f^2+f')+\tfrac{r}{6}\right\}\{g(Y,W_2(Z,U)V)\xi-g(Y,Z)W_2(\xi,U)V\\&-g(Y,U)W_2(Z,\xi)V-g(Y,V)W_2(Z,U)\xi.\\&+\left\{(f^2+f')+\tfrac{r}{6}\right\}\{\eta(W_2(Z,U)V)Y-\eta(Z)W_2(Y,U)V\\&-\eta(U)W_2(Z,Y)V-\eta(V)W_2(Z,U)Y=0.\end{aligned} \tag{6.4}$$

Taking the inner product of (6.4)) with ξ together with (2.27), we obtain

$$\left\{(f^2+f')+\frac{r}{6}\right\}W_2(Z,U,V,Y)=0. \tag{6.5}$$

It is obvious from (6.5) that either $r=-6(f^2+f')$ or

$$W_2(Z,U,V,Y)=0. \tag{6.6}$$

It means the manifold is W_2-flat. Thus with the help of Theorem 3.1, Theorem 4.1 and Corollary 3.2 we state the following result.

Theorem 6.1. *If a 3-dimensional regular f-Kenmotsu manifold (M^3, ϕ, ξ, η) satisfying the condition $T(X,Y)\cdot W_2$=0. Then either r=-6(f^2+f') or it is an Einstein manifold.*

Corollary 6.2. *If a 3-dimensional regular f-Kenmotsu manifold (M^3, ϕ, ξ, η) satisfying the condition $T(X,Y)\cdot W_2$=0 is W_2-flat provided $r \neq -6(f^2+f')$.*

Corollary 6.3. *A 3-dimensional α-Kenmotsu manifold $(M^3, \phi, \xi, \eta, g)$ bearing the condition $T(X,Y)\cdot W_2$=0. Then it is an Einstein manifold. Moreover it is locally isometric to the hyperbolic space $H^3(-\alpha^2)$.*

Corollary 6.4. *A 3-dimensional Kenmotsu manifold $(M^3, \phi, \xi, \eta, g)$ with the condition $T(X,Y)\cdot W_2$=0 is it is locally isometric to the hyperbolic space $H^3(-1)$*

Corollary 6.5. *A 3-dimensional cosymplectic manifold $(M^3, \phi, \xi, \eta, g)$ satisfying the condition $T(X,Y)\cdot W_2$=0 is an Euclidean space.*

7. f-Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ satisfying $C(X,Y)\cdot W_2$=0

In this constituent, we study f-Kenmotsu manifold with $C(X,Y)\cdot W_2$=0, and deduce some certain result.
Let $(M^3, \phi, \xi, \eta, g,)$ be a f-Kenmotsu manifold satisfying the condition

$$C(X,Y)\cdot W_2 = 0 \tag{7.1}$$

This equation implies

$$\begin{aligned} &C(X,Y)W_2(Z,U)V - W_2(C(X,Y)Z,U)V \\ &-W_2(Z,C(X,Y)U)V - W_2(Z,U)C(X,Y)V = 0. \end{aligned} \tag{7.2}$$

Putting X=ξ in (7.2), we obtain

$$\begin{aligned} &C(\xi,Y)W_2(Z,U)V - W_2(C(\xi,Y)Z,U)V \\ &-W_2(Z,C(\xi,Y)U)V - W_2(Z,U)C(\xi,Y)V = 0. \end{aligned} \tag{7.3}$$

In view of (2.25) and (7.3) we have

$$\begin{gathered} k_1[g(Y,W_2(Z,U)V)\xi - g(Y,Z)W_2(\xi,U)V - g(Y,U)W_2(Z,\xi,V) \\ -g(Y,V)W_2(Z,U)\xi] + k_2[g(Y,W_2(Z,U)V)\xi - 2\eta(Z)W_2(Z,U)V)Y \\ -g(Y,Z)W_2(\xi,U,V) + 2\eta(Z)W_2(Y,U)V \\ -g(Y,U)W_2(Z,\xi)V - 2\eta(U)W_2(Z,Y)V - S(Y,(W_2(Z,U)V)\xi \\ -S(Y,Z)W_2(\xi,U)V - S(Y,U)W_2(Z,\xi,V) - S(Y,V)W_2(Z,U,\xi)], \end{gathered}$$

where $k_1 = \left\{-(f^2+f') + \frac{r}{2}\right\}$, $k_2 = 2(f^2+f')$.
Taking the inner product of above equation with ξ and using (2.27), it yield

$$\left\{-(f^2+f') + \frac{r}{2}\right\} W_2(Z,U,V,Y) = 0. \tag{7.4}$$

It is obvious from (7.5) either r=2(f^2+f') or

$$W_2(Z,U,V,Y) = 0. \tag{7.5}$$

It means the manifold is W_2-flat. Thus with the help of Theorem 3.1, Theorem 4.1 and Corollary 3.2, we state the following result.

Theorem 7.1. *If a 3-dimensional regular f-Kenmotsu manifold (M^3, ϕ, ξ, η) satisfying the condition $C(X,Y)\cdot W_2{=}0$. Then either $r{=}2(f^2+f')$ or it is an Einstein manifold.*

Corollary 7.2. *If a 3-dimensional regular f-Kenmotsu manifold (M^3, ϕ, ξ, η) satisfying the condition $C(X,Y)\cdot W_2{=}0$ is W_2-flat provided $r \neq 2(f^2+f')$.*

Corollary 7.3. *A 3-dimensional α-Kenmotsu manifold $(M^3, \phi, \xi, \eta, g)$ bearing the condition $C(X,Y)\cdot W_2{=}0$. Then it is an Einstein manifold. Moreover it is locally isometric to the hyperbolic space $H^3(-\alpha^2)$.*

Corollary 7.4. *A 3-dimensional Kenmotsu manifold $(M^3, \phi, \xi, \eta, g)$ with the condition $C(X,Y)\cdot W_2{=}0$ is it is locally isometric to the hyperbolic space $H^3(-1)$.*

Corollary 7.5. *A 3-dimensional cosymplectic manifold $(M^3, \phi, \xi, \eta, g)$ satisfying the condition $C(X,Y)\cdot W_2{=}0$ is an Euclidean space.*

8. f-Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ satisfying $H(X,Y)\cdot W_2{=}0$

In this Section, we study f-Kenmotsu manifold with $H(X,Y)\cdot W_2{=}0$, and deduce some result.
Let $(M^3, \phi, \xi, \eta, g)$ be a f-Kenmootsu manifold satisfying the condition

$$H(X,Y)\cdot W_2 = 0. \tag{8.1}$$

Above equation takes the form

$$\begin{aligned} &H(X,Y)W_2(Z,U)V - W_2(H(X,Y)Z,U)V \\ &-W_2(Z,H(X,Y)U)V - W_2(Z,U)H(X,Y)V = 0. \end{aligned} \tag{8.2}$$

Putting $X{=}\xi$ in (8.2), we obtain

$$\begin{aligned} &H(\xi,Y)W_2(Z,U)V - W_2(H(\xi,Y)Z,U)V \\ &-W_2(Z,H(\xi,Y)U)V - W_2(Z,U)H(\xi,Y)V = 0. \end{aligned} \tag{8.3}$$

Using (2.26) in (8.3), and taking associate with ξ together with (2.27), we get

$$\lambda_1 g(Y, W_2(Z,U)V) + bS(Y, W_2(Z,U)V) = 0. \tag{8.4}$$

Again putting $Z{=}V{=}\xi$ in (8.4), using (1.1) and (2.9), we obtain

$$bS(QU,Y) + \left\{\lambda_1 + 2b(f^2+f')\right\} S(Y,U) + \lambda_3\eta(Y)\eta(U) + 2\lambda_1(f^2+f')g(Y,U) = 0. \tag{8.5}$$

where $\lambda_3{=}\{4b(f^2+f')^2 - 4b(f^2+f')\}$.

If $b{=}0$, then (8.5), we get

$$\lambda_1\left\{S(Y,U) + 2(f^2+f')g(Y,U)\right\} = 0. \tag{8.6}$$

This implies either $\lambda_1{=}0$ or $S(Y,U)$=-$2(f^2+f')g(Y,U)$, respectively.

Again, if $b \neq 0$, then (8.5)) we have

$$\begin{aligned} &S(QU,Y) + \left\{\tfrac{\lambda_1}{b} + 2(f^2+f')\right\} S(Y,U) + \tfrac{\lambda_3}{b}\eta(Y)\eta(U) \\ &+2\tfrac{\lambda_1}{b}(f^2+f')g(Y,U) = 0. \end{aligned} \tag{8.7}$$

So it leads to the following result.

Theorem 8.1. *If a 3-dimensional regular f-Kenmotsu manifold satisfying the condition $H(X,Y)\cdot W_2{=}0$. Then*
(i) if $b{=}0$, then either $\lambda_1{=}0$ on M, or it is an Einstein manifold.
(ii) if $b \neq 0$, then equation (8.7) holds on M.

Corollary 8.2. *Let M be a 3-dimensional Kenmotsu manifold satisfying the condition $H(X,Y)\cdot W_2{=}0$. Then $T\cdot T{=}\alpha Q(g,T)$, where $T{=}g\wedge S$ and α=-$\{2+\frac{\lambda_1}{b}\}$.*

9. Example

9.1. **When f is a constant function.** Let $M^3{=}\{(u,v,w)\in\mathbb{R}^3 : u,v,z(\neq 0)\in\mathbb{R}\}$ be a Riemannian manifold, where(u,v,w) denotes the standard coordunates of a point in $\mathbb{R}^3$. Let us suppose that

$$e_1 = w\frac{\partial}{\partial u},\quad e_2 = w\frac{\partial}{\partial v},\quad e_3 = -w\frac{\partial}{\partial w}$$

be linearly independent vector fields at each point of M^3 and therefore it form a basis for the tangent space $T(M^3)$. We also define the Riemannian metric g of the manifold M^3 as $g(e_i,e_j)=\delta_{ij}$, where δ_{ij} denotes the Kronecker delta and $i,j=1,2,3$, and given by

$$g = \frac{1}{w^2}\left[du\otimes du + dv\otimes dv + dw\otimes dw\right].$$

Let η be the 1-form have the significance

$$\eta(U) = g(U,e_3)$$

for any $U\in\Gamma(TM)$ and ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = -e_2,\quad \phi e_2 = -e_1,\quad \phi e_3 = 0.$$

By the linearity properties of ϕ and g we can easily verify the following relations

$$\eta(e_3) = 1, \phi^2(U) = -U + \eta(U)e_3$$
$$g(\phi U,\phi V) = g(U,V) - \eta(U)\eta(V),$$

for arbitrary vector fields $U,W\in T(M^3)$. This shows that $\xi{=}e_3$ and the structure (ϕ,ξ,η,g) defines an almost contact metric structure on M^3. If ∇ be the Levi-Civita connection with respect to the Riemannian metric g, then with the help of above, we can easily calculate that

$$[e_1,e_2]=0,\quad [e_1,e_3]=e_1,\quad [e_2,e_3]=e_2.$$

Now we recall the Koszul's formula as

$$\begin{aligned}2g(\nabla_U V,W) &= U(g(V,W)) + V(g(W,X)) - W(g(U,V))\\ &- g(U,[V,W]) - g(V,[U,W]) + g(W,[U,V])\end{aligned}$$

for arbitrary vector fields $U,V,W\in T(M^3)$. Making use Koszul's formula we get the following:

$$\nabla_{e_2}e_3 = e_2,\quad \nabla_{e_2}e_2 = -e_3,\quad \nabla_{e_2}e_1 = 0,$$
$$\nabla_{e_3}e_3 = 0,\quad \nabla_{e_3}e_2 = 0,\quad \nabla_{e_3}e_1 = 0,$$
$$\nabla_{e_1}e_3 = e_1,\quad \nabla_{e_1}e_2 = 0,\quad \nabla_{e_1}e_1 = -e_3.$$

From the above calculation it is clear that M^3 satisfies the condition $\nabla_U\xi{=}f\{U-\eta(U)\xi\}$ for $e_3{=}\xi$, where $f{=}1{=}\alpha$ = constant$(\neq 0)$. Thus we conclude that M^3 leads to f-Kenmotsu

(Kenmotsu) manifold. Also $f^2 + f^{'} \neq 0$. That implies M^3 is a 3-dimensional regular f-Kenmotsu manifold.

9.2. **When f is a smooth function.** We consider the 3-dimensional manifold M=$\{(u, v, w) \in \mathbb{R}^3, w \neq 0\}$, where (u, v, w) are the standard coordinate in $\mathbb{R}^3$. Let (e_1, e_3, e_3) be linearly independent vector fields at each point of M, given by

$$e_1 = \frac{1}{w}\frac{\partial}{\partial u}, \quad e_2 = \frac{1}{w}\frac{\partial}{\partial v}, \quad e_3 = -\frac{\partial}{\partial w}.$$

Let g be the Riemannian metric such that

$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

and given by

$$g = w^2 \left[du \otimes du + dv \otimes dv + \frac{1}{w^2} dw \otimes dw \right].$$

Let η be the 1-form have the significance

$$\eta(U) = g(U, e_3)$$

for any $U \in \Gamma(TM)$ and ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = -e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0.$$

Making use of the linearity of ϕ and g we have

$$\eta(e_3) = 1, \quad \phi^2(U) = -U + \eta(U)e_3$$
$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V),$$

for any $U, W \in \Gamma(TM)$. Now we can easily calculate

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -\frac{1}{w}e_2, \quad [e_2, e_3] = -\frac{1}{w}e_1.$$

Making use Koszul's formula we get the following:

$$\nabla_{e_2} e_3 = -\frac{1}{w}e_2, \quad \nabla_{e_2} e_2 = \frac{1}{w}e_3, \quad \nabla_{e_2} e_1 = 0,$$
$$\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_1 = 0,$$
$$\nabla_{e_1} e_3 = -\frac{1}{w}e_1, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_1 = \frac{1}{w}e_3.$$

Consequently it is clear that M satisfies the condition $\nabla_U \xi$=$f\{U - \eta(U)\xi\}$ for e_3=ξ, where f=-$\frac{1}{w}$. Thus we conclude that M leads to f-Kenmotsu manifold. Also $f^2 + f^{'}$=$\frac{2}{w^2} \neq 0$. That implies M is a 3-dimensional regular f-Kenmotsu manifold.

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