

NONLOCAL ITERATIVE DIFFERENTIAL EQUATIONS UNDER GENERALIZED FRACTIONAL DERIVATIVES

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ABSTRACT. The objective of this paper is to study the existence and uniqueness of solutions of iterative differential equations with nonlocal conditions under ψ -type fractional derivative (or generalized fractional derivative).

1. INTRODUCTION

The nonlocal condition can be applied in physics with better effect than the classical initial value problem. Nonlocal conditions were initiated by Byszewski [4] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [5] and Deng [7], the nonlocal condition can be more useful than standard initial condition to describe some physical phenomena.

Many recent papers have dealt with the existence, uniqueness and other properties of solutions of special forms of the iterative fractional differential equations, for example [3, 9, 10]. Recently, Kharat and Jagtap [11] have investigated the existence and uniqueness of solutions of special form of iterative fractional differential equations.

Motivated by [11], in this paper, we consider the class of ψ -type fractional iterative differential equation of the type

$$(1) \quad \mathcal{D}^{\alpha;\psi} x(t) = f(t, x(x(t))), \quad 0 < \alpha < 1,$$

$$(2) \quad x(0) + g(x) = x_0,$$

where $\mathcal{D}^{\alpha;\psi}$ is the ψ -type Riemann-Liouville fractional derivative of order α , $f : J \times J \rightarrow J$, $g : C(J, J) \rightarrow J$ are continuous functions and $t, x_0 \in J = [0, T]$.

The aim of the present paper is to prove the existence and uniqueness of solutions of iterative differential equations under ψ -type fractional derivative. The main tools used in our analysis are based on the theory of fractional calculus and fixed point theorems.

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2. PREREQUISITES

First we state some definitions, results and hypotheses which are useful for our analysis.

Definition 2.1. [1] The ψ -type fractional order integral of the function f of order $\gamma > 0$ is defined by

$$(3) \quad \mathcal{I}_a^{\gamma;\psi} f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\gamma-1} f(s) ds,$$

where Γ is gamma function.

Definition 2.2. [1] The ψ -type fractional order derivative of the function f of order $0 \leq \gamma < 1$ is defined by

$$(4) \quad \mathcal{D}_a^{\gamma;\psi} = \frac{1}{\Gamma(1-\gamma)} \left(\frac{d}{dt} \right) \int_a^t \psi'(s) (\psi(t) - \psi(s))^{-\gamma} ds = \frac{d}{dt} \mathcal{I}_a^{1-\gamma;\psi} f(t).$$

Let (X, d) be a metric space. A mapping $P : X \rightarrow X$ is said to be ν -contraction if there exists $\nu \in [0, 1)$ such that

$$d(Px, Py) \leq \nu d(x, y), \quad \forall x, y \in X.$$

If $\nu = 1$, then the mapping P is said to be non-expansive.

Theorem 2.3. [2] Let K be a nonempty closed convex and bounded subset of a uniformly Banach space E . Then any non-expansive mapping $P : K \rightarrow K$ has at least a fixed point.

Definition 2.4. [11] Let K be a convex subset of a normal linear space E and let $P : K \rightarrow K$ be a self-mapping. Given an $x_0 \in K$ and a real number $\lambda \in [0, 1]$, the sequence x_n defined by the formula

$$x_{n+1} = (1 - \lambda)x_n + \lambda Px_n, \quad n = 0, 1, 2, \dots$$

is usually called Krasnoselskij iteration or Krasnoselskij-Mann iteration.

Definition 2.5. [11] Let K be a convex subset of a normed linear space E and let $P : E \rightarrow E$ be a self-mapping. Given an $x_0 \in K$ and a real number $\lambda_n \in [0, 1]$, the sequence x_n defined by the formula

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n Px_n, \quad n = 0, 1, 2, \dots$$

is usually called Mann iteration.

Edelstein [8] proved that strict convexity of E suffices for the Krasnoselskij iteration converges to a fixed point of P .

Lemma 2.6. [6] Let K be a convex and compact subset of Banach space E and let $P : K \rightarrow K$ be a non-expansive mapping. If the Mann iteration process x_n satisfies the assumptions

- (a) $x_n \in K$ for all positive integers n ,
- (b) $0 \leq \lambda_n \leq b < 1$ for all positive integer n ,
- (c) $\sum_{n=0}^{\infty} \lambda_n = \infty$.

Then x_n converges strongly to a fixed point of P .

Lemma 2.7. [6] *Let K be a closed bounded convex subset of a real normed space E and $P : K \rightarrow K$ be a non-expansive mapping. If $I - P$ maps closed bounded subset of E into closed subset of E and x_n is the Mann iteration with λ_n satisfying assumptions (a)-(c) in Lemma 2.6, then x_n converges strongly to a fixed point of P in K .*

Let $C(J, J)$ be the Banach space of all continuous functions from J into J endowed with the norm $\|x\| = \sup \{ |x(t)| : t \in J \}$, $M_t = \max \{t, T - t\}$ and

$$C_{L,\alpha;\psi} = \left\{ x : |x(t_1) - x(t_2)| \leq \frac{L}{\Gamma(\alpha + 1)} |(\psi(t_1)) - (\psi(t_2))|^\alpha, \quad \forall t_1, t_2 \in J, L > 0 \right\}.$$

It is clear that $C_{t,\alpha;\psi}$ is a nonempty convex and compact subset of the Banach space $(C[J], \|\cdot\|)$. For the convenience, we list the following hypotheses used in our further discussion.

(H1) For all $t, x_1, x_2 \in J$ there exists a constant $l > 0$ such that

$$|f(t, x_1) - f(t, x_2)| \leq l |x_1 - x_2|,$$

and a constant $l_g > 0$ such that

$$\|g(x_1) - g(x_2)\| \leq l_g \|x_1 - x_2\|.$$

Also for arbitrary $x \in C(J, J)$, there exists a constant $N > 0$ such that

$$\|g(x)\| \leq N.$$

(H2) If there exist a constant L such that $|x(t_1) - x(t_2)| \leq \frac{L}{\Gamma(\alpha+1)} |(\psi(t_1)) - (\psi(t_2))|^\alpha$ then

$$M = \max \{ |f(t, x)| : (t, x) \in J \times J \} \leq \frac{L}{2}.$$

(H3) One of the following conditions holds:

- (a) $M \frac{(\psi(T))^\alpha}{\Gamma(\alpha+1)} \leq M_{x_0}$, where $M_{x_0} = \max \{x_0 + N, T - (x_0 + N)\}$;
- (b) $x_0 - N = 0$, $M \frac{(\psi(T))^\alpha}{\Gamma(\alpha+1)} \leq T - (x_0 + N)$, $f(t, x \geq 0)$ for all $t, x \in J$;
- (c) $x_0 - N = T$, $M \frac{(\psi(T))^\alpha}{\Gamma(\alpha+1)} \leq x_0 - N$, $f(t, x) \geq 0$ for all $t, x \in J$,

(H4) $x_0 + N \leq \frac{\delta(\psi(t_0))^\alpha}{2\Gamma(\alpha+1)}$, $0 \neq t_0 \in J$, $\delta \in (0, 1)$.

(H5) If there exist a constant L such that $|x(t_1) - x(t_2)| \leq \frac{L}{\Gamma(\alpha+1)} |(\psi(t_1)) - (\psi(t_2))|^\alpha$ then $M = \min \left\{ \frac{\delta}{2}, \frac{L}{2} \right\}$.

3. MAIN RESULTS

In this section, we state and prove results related to existence and uniqueness of solutions of iterative differential equations under ψ -type fractional derivative.

Theorem 3.1. *Suppose that hypotheses (H1)-(H3) are satisfied and*

$$(5) \quad \left\{ l_g + \frac{(\psi(T))^{\alpha l}}{\Gamma(\alpha + 1)} \left[\frac{L}{\Gamma(\alpha + 1)} + 1 \right] \right\} \leq 1,$$

then there exists at least one solution of the system (1)-(2) in $C_{L,\alpha;\psi}$ which can be approximated by the Krasnoselskij iteration

$$x_{n+1} = (1 - \lambda)x_n + \lambda x_0 - \lambda g(x) + \lambda \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) ((\psi(t)) - (\psi(s)))^{\alpha-1} f(\tau, x_n(x_n(t))) d\tau,$$

where $\lambda \in (0, 1)$ and $x_1 \in C_{L,\alpha;\psi}$ is arbitrary.

Proof. Consider the integral operator $P : C_{L,\alpha;\psi} \rightarrow C(J)$ defined by

$$Px(t) = x_0 + g(x) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) ((\psi(t)) - (\psi(s)))^{\alpha-1} f(\tau, x_n(x_n(t))) d\tau, \quad t \in J.$$

Any fixed point of the equation $x = Px$ is a solution of initial value problem (1)-(2).

First we show that $C_{L,\alpha;\psi}$ is invariant set with respect to P , i.e., $P(C_{L,\alpha;\psi}) \subset C_{L,\alpha;\psi}$. Making use of hypothesis (H3)(a), we have

$$\begin{aligned} |Px(t)| &\leq x_0 + |g(x)| + \left| \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) ((\psi(t)) - (\psi(s)))^{\alpha-1} f(\tau, x_n(x_n(t))) d\tau \right| \\ &\leq x_0 + |g(x)| + M \frac{(\psi(T))^\alpha}{\Gamma(\alpha + 1)} \\ &\leq x_0 + |g(x)| + M_{x_0} \\ &\leq (\psi(T)) \end{aligned}$$

and

$$\begin{aligned} |Px(t)| &\geq x_0 + |g(x)| - \left| \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) ((\psi(t)) - (\psi(s)))^{\alpha-1} f(\tau, x_n(x_n(t))) d\tau \right| \\ &\leq x_0 + |g(x)| - M \frac{(\psi(T))^\alpha}{\Gamma(\alpha + 1)} \\ &\leq x_0 + |g(x)| - M_{x_0} \geq 0. \end{aligned}$$

Thus $Px(t) \in J, t \in J$.

Similarly, we get the result using hypotheses (H3)(b) and (H3)(c).

Using hypotheses (H2), for every $t_1, t_2 \in J$, we obtain

$$\begin{aligned} &|(Px)(t_1) - (Px)(t_2)| \\ &= \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s) ((\psi(t_1)) - (\psi(\tau)))^{\alpha-1} f(\tau, x_n(x_n(t))) d\tau \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \psi'(s) ((\psi(t_2)) - (\psi(\tau)))^{\alpha-1} f(\tau, x_n(x_n(t))) d\tau \right| \\ &\leq 2M \frac{|(\psi(t_1)) - (\psi(t_2))|^\alpha}{\Gamma(\alpha + 1)} \\ &\leq L \frac{|(\psi(t_1)) - (\psi(t_2))|^\alpha}{\Gamma(\alpha + 1)}. \end{aligned}$$

Hence $Px \in C_{L,\alpha;\psi}$ whenever $x \in C_{L,\alpha;\psi}$. Therefore, $P : C_{L,\alpha;\psi} \rightarrow C_{L,\alpha;\psi}$, i.e., P is a self mapping.

Using hypotheses (H1) for $x, y \in C_{L,\alpha;\psi}$ and $t \in J$, we have

$$\begin{aligned} & |(Px)(t) - (Py)(t)| \\ & \leq \|g(x) - g(y)\| \\ & + \left| \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) ((\psi(t)) - (\psi(\tau)))^{\alpha-1} f(\tau, x_n(x_n(t))) d\tau \right. \\ & \left. - \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) ((\psi(t)) - (\psi(\tau)))^{\alpha-1} f(\tau, y_n(y_n(t))) d\tau \right| \\ & \leq l_g \|x - y\| + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) ((\psi(t)) - (\psi(\tau)))^{\alpha-1} |f(\tau, x_n(x_n(t))) - f(\tau, y_n(y_n(t)))| d\tau \\ & \leq l_g \|x - y\| + \frac{l}{\Gamma(\alpha)} \int_0^t \psi'(s) ((\psi(t)) - (\psi(\tau)))^{\alpha-1} |x(x(\tau)) - y(y(\tau))| d\tau \\ & \leq l_g \|x - y\| \\ & + \frac{l}{\Gamma(\alpha)} \int_0^t \psi'(s) ((\psi(t)) - (\psi(\tau)))^{\alpha-1} [|x(x(\tau)) - x(y(\tau))| + |x(y(\tau)) - y(y(\tau))|] d\tau \\ & \leq \left\{ l_g + \frac{(\psi(T))^{\alpha l}}{\Gamma(\alpha + 1)} \left[\frac{L}{\Gamma(\alpha + 1)} + 1 \right] \right\} \|x - y\|. \end{aligned}$$

Now, by taking supremum we get,

$$\|(Px) - (Py)\| \leq \left\{ l_g + \frac{(\psi(T))^{\alpha l}}{\Gamma(\alpha + 1)} \left[\frac{L}{\Gamma(\alpha + 1)} + 1 \right] \right\} \|x - y\|.$$

In view of equation (5)

- (i) if $\left\{ l_g + \frac{(\psi(T))^{\alpha l}}{\Gamma(\alpha + 1)} \left[\frac{L}{\Gamma(\alpha + 1)} + 1 \right] \right\} < 1$, then P is a contraction mapping and hence by Banach fixed point theorem equation (1)-(2) has a unique solution.
 - (ii) if $\left\{ l_g + \frac{(\psi(T))^{\alpha l}}{\Gamma(\alpha + 1)} \left[\frac{L}{\Gamma(\alpha + 1)} + 1 \right] \right\} = 1$, then P is nonexpansive and hence it is continuous.
- Thus Theorem 2.3 implies that equation (1)-(2) has a solution in $C_{L,\alpha;\psi}$.

Finally, by applying Lemma 2.6 and Lemma 2.7, we obtain the second part of the theorem. \square

Next we establish the solution of equation (1)-(2) in a subset of $C_{L,\alpha;\psi}$ defined by

$$C_{L,\alpha;\psi,\delta} = \left\{ x \in C_{L,\alpha;\psi} : x(t) \leq \frac{\delta(\psi(t))^\alpha}{\Gamma(\alpha + 1)}, \text{ for all } t \in J \right\}, \quad \delta \in (0, 1).$$

It is clear that $C_{L,\alpha;\psi,\delta}$ is non-empty, convex and compact subset in $C(J)$.

Theorem 3.2. *Assume that hypotheses (H1), (H3)-(H5) are satisfied. If*

$$\begin{aligned} L_1 = & \left\{ l_g + \frac{(\psi(T))^{\alpha-1} l}{q\Gamma(\alpha)} \left(\frac{L}{\Gamma(\alpha + 1)} |(1 - \exp(-q(\psi(t) - \psi(t_0))))| \right. \right. \\ & \left. \left. + \frac{1}{\delta} |\exp(q(\delta - 1)(\psi(t))) - \exp(q(\delta\psi(t_0) - \psi(t)))| \right) \right\} \\ (6) \quad & \leq l, \end{aligned}$$

then there exists at least one solution of the system (1)-(2) in $C_{L,\alpha;\psi,\delta}$ which can be approximated by the Krasnoselskij iteration

$$x_{n+1} = (1 - \lambda)x_n + \lambda x_0 - \lambda g(x) + \lambda \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) ((\psi(t)) - (\psi(\tau)))^{\alpha-1} f(\tau, x_n(x_n(t))) d\tau,$$

where $\lambda \in (0, 1)$ and $x_1 \in C_{L,\alpha;\psi,\delta}$ is arbitrary.

Proof. Let $C(J)$ be the Banach space endowed with the Bieleckis norm given by the formula

$$\|x\|_B = \max_{t \in J} \{ \|x(t)\| \exp(-q(\psi(t) - \psi(t_0))), \quad q > 0, \quad t > t_0 \}$$

Let P be defined as in the proof of Theorem 3.1, by assumptions (H1), (H3)-(H5), it is sufficient to prove that, if $x \in C_{L,\alpha;\psi,\delta}$ then $Px \in C_{L,\alpha;\psi,\delta}$.

For $x \in C_{L,\alpha;\psi,\delta}$ and $t \in J$, we have

$$\begin{aligned} Px(t) &\leq x_0 + N + M \frac{(\psi(t))^\alpha}{\Gamma(\alpha + 1)} \\ &\leq x_0 + N + \frac{M((\psi(t))^\alpha - (\psi(t_0))^\alpha) + M(\psi(t_0))^\alpha}{\Gamma(\alpha + 1)} \\ &\leq \frac{\delta(\psi(t))^\alpha}{2\Gamma(\alpha + 1)}, \quad t > t_0. \end{aligned}$$

This shows that $Px \in C_{L,\alpha;\psi,\delta}$ and hence $C_{L,\alpha;\psi,\delta}$ is invariant under P .

Now, for $x, y \in C_{L,\alpha;\psi,\delta}$ and $t \in J$, we get

$$\begin{aligned} & |(Px)(t) - (Py)(t)| \\ & \leq \|g(x) - g(y)\| \\ & + \left| \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) ((\psi(t)) - (\psi(\tau)))^{\alpha-1} f(\tau, x_n(x_n(t))) d\tau \right. \\ & \left. - \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) ((\psi(t)) - (\psi(\tau)))^{\alpha-1} f(\tau, y_n(y_n(t))) d\tau \right| \\ & \leq l_g \|x - y\| \\ & + \frac{(\psi(T))^\alpha}{\Gamma(\alpha)} \left| \int_0^t \left(\frac{L}{\Gamma(\alpha)} |x(\tau) - y(\tau)| + |x(y(\tau)) - y(y(\tau))| \right) d\tau \right| \\ & \leq \left\{ l_g \exp(q((\delta\psi(t)) - (\psi(t_0)))) + \frac{(\psi(T))^{\alpha-1} l}{\Gamma(\alpha)} \left(\left| \frac{L}{q\Gamma(\alpha + 1)} (\exp(q(\psi(t) - \psi(t_0))) - 1) \right| \right. \right. \\ & \left. \left. + \frac{1}{q\delta} |\exp(q(\delta\psi(t) - \psi(t_0))) - \exp(q(\delta\psi(t_0) - \psi(t_0)))| \right) \right\} \|x - y\|_B \end{aligned}$$

Hence,

$$\begin{aligned} & |(Px)(t) - (Py)(t)| \exp(-q(\psi(t) - \psi(t_0))) \\ & \leq \left\{ l_g + \frac{(\psi(T))^{\alpha-1} l}{q\Gamma(\alpha)} \left(\frac{L}{\Gamma(\alpha + 1)} |(1 - \exp(-q(\psi(t) - \psi(t_0))))| \right. \right. \\ & \left. \left. + \frac{1}{\delta} |\exp(q(\delta - 1)(\psi(t))) - \exp(q(\delta\psi(t_0) - \psi(t_0)))| \right) \right\} \|x - y\|_B. \end{aligned}$$

Now, by taking maximum in the last inequality, we obtain

$$\|Px - Py\|_B \leq L_1 \|x - y\|_B.$$

In view of equation (6)

- (i) if $L_1 < 1$, then P is a contraction mapping and hence by Banach fixed point theorem equation (1)-(2) has a unique solution in $C_{L,\alpha;\psi,\delta}$.

- (ii) if $L_1 = 1$ then P is nonexpansive and hence it is continuous. Thus Theorem 2.3 implies that equation (1)-(2) has a solution in $C_{L,\alpha;\psi,\delta}$.

Finally, by applying Lemma 2.6 and Lemma 2.7, we obtain the second part of the theorem. This completes the proof. \square

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