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SOME PROPERTIES OF THE *UBAT*-SPACE AND A RELATED STRUCTURE

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Abstract. An *Ubat*-space is a nonempty set U together with a binary operation $*$ satisfying: $(U1)$ $x*(y*z) = (x*y)*z$ for all $x, y, z \in U$; $(U2)$ There exists $y \in U$ such that $x*y = y*x = y$ for all $x \in U$; And, $(U3)$ There exists $z \in U$ such that $x*z = z*x = x$ for all $x \in U$.

A *g-group* is a nonempty set G together with a binary operation $*$ satisfying: $(g1)$ $f*(g*h) = (f*g)*h$ for all $f, g, h \in G$; $(g2)$ for each $g \in G$, there is $e \in G$ such that $g*e = e*g = g$ (we call e an identity); and $(g3)$ for each $g \in G$, there exists $h \in G$ such that $g*h = h*g = e$ for some identity e described in $(g2)$.

In this paper, we present some important properties of the two algebraic structures (algebra).

1. Introduction

Let G be a non-empty set. A binary operation in G is a function $* : G \times G \to G$. We denote the image of (a, b) by $a*b$ or for brevity ab. An algebra $(G, *)$ (where $*$ is a binary operation in G) is a *group* if the following properties hold: $(G1)$ $x*(y*z) = (x*y)*z$ for all $x, y, z \in G$; $(G2)$ There exists an element $e \in G$ (called an *identity element*) such that $e*x = x*e = x$ for all $x \in G$; And, $(G3)$ For each a in G, there is an element a' in G such that $a*a' = a'*a = e$ (where e is an identity element mentioned in $G2$).

Let G be a non-empty set. An algebra $(G; *; A)$ (where $*$ is a binary operation in G and A is a non-empty subset of G) is an *e-group* if the following properties hold: $(E1)$ $x*(y*z) = (x*y)*z$ for all $x, y, z \in G$; $(E2)$ For every $x \in G$ there exists an element $a \in A$ such that $x*a = a*x = x$; And, $(E3)$ For every $x \in G$ there exists an element $y \in G$ such that $x*y, y*x \in A$ [1].

Let G be a non-empty set. An algebra $(G; *)$ (where $*$ is a binary operation in G) is a *g-group* if the following properties hold: $(g1)$ $f*(g*h) = (f*g)*h$ for all $f, g, h \in G$; $(g2)$ For each $g \in G$, there exists an element $e \in G$ (called an *identity element*) such that $g*e = e*g = g$; And, $(g3)$ For each $g \in G$, there exists an element $h \in G$ (called an *inverse* of g) such that $g*h = h*g = e$ for some identity element e.

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For example, the singleton sets $\{0\}$ and $\{1\}$ with respect to multiplication $\times$ are g-groups as shown in the Tables 1 and 2.

$\times$	0
0	0

Table 1. The g-group $\{0\}$

$\times$	1
1	1

Table 2. The g-group $\{1\}$

Similarly, the set $\{0,1\}$ is also a g-group under multiplication as shown in Table 3.

$\times$	0	1
0	0	0
1	0	1

Table 3. The g-group $\{0,1\}$

Let U be a non-empty set. An algebra $\langle U, *\rangle$ (where $*$ is a binary operation in U) is an *Ubat-space* if the following properties hold: $(U1)$ $x*(y*z) = (x*y)*z$ for all $x, y, z \in U$; $(U2)$ There exists $y \in U$ such that $x*y = y*x = y$ for all $x \in U$ (we call the element y a *zero* of U); And, $(U3)$ there exists $z \in U$ such that $x*z = z*x = x$ for all $x \in U$ (we call the element z an *identity* of U). An *Ubat*-space is *simple* if it is finite and if for each $x \in U$, $y*x$ is unique for all $y \in U$.

For example, the singleton set $\{0\}$ with respect to multiplication $\times$ in the Table 1, the singleton set $\{1\}$ with respect to multiplication $\times$ in the Table 2 and the set $\{0,1\}$ under multiplication in Table 3 are *Ubat*-spaces.

Let G be a non-empty set. An algebra $(G, *)$ (where $*$ is a binary operation in G) is a *generalized group* if the following properties hold: $(M1)$ $f*(g*h) = (f*g)*h$ for all $f, g, h \in G$; $(M2)$ for each $g \in G$, there exists a unique element $e(g)$ such that $g*e(g) = g = e(g)*g$; And, $(M3)$ for each $g \in G$, there exists an element $h \in G$ such that $g*h = h*g = e(g)$.

Hereafter, please refer to [2] for the other concepts.

It was not until the early decades of the twentieth century that algebra had evolved into the study of axiomatic systems referred to as *abstract algebra* [3]. About three millennia earlier, algebra only focused on solving polynomial equations. Although early mathematicians started contemplating on group theory in the late part of the 18th century, major developments in this area occurred in the 19th century [3].

The term *group* was introduced by Galois to refer to a collection of permutations that is closed under composition of functions [3]. Also, the concept implicitly led to the development of related theories across different branches of mathematics, e.g. Number Theory, Geometry and Analysis [4].

It was in 1854 when Cayley gave the first definition of a finite group. In such definition, the closure property, associativity and the notion of cyclic was given emphasis. Moreover, Weber provided another definition of groups in 1882 where he asserted three axioms; closure, associativity and cancellation. By this time, such definition applies only to finite groups. It was W. von Dyck later that year who consciously combined all major historical roots of group theory [2]. It was in von Dyck's definition where the existence of inverses was explicitly required [3].

From then on, special groups were discovered and gained popularity among mathematicians. In 1874, Lie introduced his general theory on continuous transformation groups known today as Lie groups [3]. In 1893, Holder introduced the concept of an automorphism of a group abstractly. He also introduced the concept of simple groups. In 1897, Dedekind and G.A. Miller characterized Hamiltonian groups, and non-Abelian groups [3].

Group theory has lots of applications in other areas of mathematics. For instance, in 1961, Grothendieck applied the concepts of group in additive categories and introduced the Grothendieck group [5], [6]. This was followed by the introduction of the Picard group later that year also applied in algebraic geometry particularly in smooth variety [2], [7], [8]. These were the predecessors of the p-Adic group introduced in 2003 [9].

There are also some mathematicians who tried to apply other mathematical principles in group theory. In 1971, the concept of fuzzy groups was introduced by A. Rosenfeld [10] where principles of fuzzy sets were applied to the elementary theory of groupoid and groups.

In [11] Molaei introduced generalized groups. And in [12] Molaei et al. studied connected topological generalized groups. They showed that topological generalized groups with e-generalized subgroups are connected topological generalized groups.

In [13], Zand et al. introduced and studied the notion of a *pseudonorm* on a generalized group. And in [14], Aktas and Cagman introduced soft group theory to extend the notion of a group to include the algebraic structures of soft sets. They also showed that fuzzy groups may be considered a special case of the soft groups. Moreover, in [1] Saeid et al. introduced the concept of extended groups (e-groups) by considering a nonempty subset A instead of the element e.

In this study, we gave some properties of *Ubat*-spaces and g-groups. These structures may have important applications in microprocessor design. Specifically, it can be used to minimize digital circuits. For example, consider the digital circuit with three inputs, A, B, and C, given by $(A \vee B) \vee (A \vee C)$. By inspection, the expression $(A \vee B) \vee (A \vee C)$ suggest that a digital circuit needs three AND gates to give the desired output. However, using some properties of the *Ubat*-space or the g-group $(\mathbb{Z}_2, \cdot)$, the circuit can be minimized as follows. Identifying $\cdot$ with $\vee$, we have $(A \vee B) \vee (A \vee C) = (A \cdot B) \cdot (A \cdot C) = [(A \cdot B) \cdot A] \cdot C = [A \cdot (B \cdot A)] \cdot C = [A \cdot (A \cdot B)] \cdot C = [(A \cdot A) \cdot B] \cdot C = (A \cdot B) \cdot C = (A \vee B) \vee C$. Note that the expression $(A \vee B) \vee C$ uses only two AND gates, and still performs the same function as $(A \vee B) \vee (A \vee C)$. This simplifies the design of the circuit.

2. *Ubat*-Spaces

2.1. **Preliminary Results.** In this section, we present the rudimentary properties of *Ubat*-spaces.

In the foregoing examples, we see that $\langle \{0\}, * \rangle$ and $\langle \{1\}, * \rangle$ are *Ubat*-spaces. We shall call the two the trivial *Ubat*-spaces, otherwise an *Ubat*-space is *non-trivial*. A moments thought one may observe that the identity element and the zero element are distinct in a non-trivial *Ubat*-space.

Remarks 2.1 and 2.2 says that a group and an *Ubat*-space are two different structures.

Remark 2.1. *An non-trivial Ubat-space is not a group.*

To see this, let $\langle U, *\rangle$ be an *Ubat*-space. Then there exist $y, z \in U$ such that $xy = yx = y$ and $xz = zx = x$ for all $x \in U$. Note that if U is non-trivial, then $z \neq y$. Thus, $xy = yx = y \neq z$, that is y has no inverse. Thus, U can not be a group.

For example, consider the *Ubat*-space $\langle \{a, b, c, d, e, f\}, *\rangle$ given in Table 4. Observe that the elements a has no inverse.

$*$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	b	c	d	e	f
c	a	c	e	a	c	e
d	a	d	a	d	a	d
e	a	e	c	a	e	c
f	a	f	e	d	c	b

TABLE 4. The *Ubat*-space $\langle \{a, b, c, d, e, f\}, *\rangle$

Remark 2.2. *An non-trivial group is not an Ubat-space.*

To see this, let $(G, *)$ be a group. Suppose that there exists $0 \in G$ such that $x0 = 0x = 0$ for all $x \in G$. If x and y are distinct elements of G, then $0x = 0 = 0y$. By the Cancellation law, we have $x = y$. This is a contradiction.

Clearly, an *Ubat*-space is precisely a group if it is trivial.

An e-group can be constructed from a group. To see this, let $(G, *)$ be a group. Then, it is easy to see that $(G; *; G)$ is an e-group.

Remark 2.3. *An e-group can be constructed from an Ubat-space.*

To see this, let $\langle U, *\rangle$ be an *Ubat*-space. Then it is easy to see that $(U; *; \{0, 1\})$ is an e-group.

However, an e-group may not be an *Ubat*-space. To see this, let $(G, *)$ be a nontrivial group. Then, as presented earlier, $(G; *; G)$ is an e-group. But by theorem 2.2, a nontrivial group is not an *Ubat*-space. Thus, $\langle G, *\rangle$ is not an *Ubat*-space.

Remarks 2.4 and 2.5 says that a g-group and an *Ubat*-space are two different structures.

Remark 2.4. *An Ubat-space may not be a g-group.*

To see this, consider the *Ubat*-space $\langle \{a, b, c, d\}, *\rangle$ presented in Table 5. Notice that the element c has no inverse. Thus, $\langle \{a, b, c, d\}, *\rangle$ is not a g-group.

Remark 2.5. *A g-group may not be an Ubat-space.*

To see this, consider the g-group $G = \{a, b, c, d\}$ presented in Table 6. Note that there is no element $y \in G$ such that $xy = yx = y$ for all $x \in G$. Thus, $\langle \{a, b, c, d\}, *\rangle$ is not an *Ubat*-space.

Remarks 2.6 and 2.7 says that a generalized group and an *Ubat*-space are two different structures.

$*$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	a	a
d	a	d	c	d

Table 5. The *Ubat* space $\langle \{a, b, c, d\}, *\rangle$

$*$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

Table 6. The g-group $\{a, b, c, d\}$

Remark 2.6. *A generalized group may not be an Ubat-space.*

To see this, consider the generalized group in Table 5. Note that there is no element y such that $y \times x = y$ for all x. This implies that there is no zero element.

Remark 2.7. *An Ubat-space may not be a generalized group.*

To see this, consider the *Ubat*-space in Table 5. Note that b is the identity, while $a \times x = a$ for all x. This implies that a has no inverse. Infact, only b has an inverse.

Clearly, every group is a generalized group. However, a generalized groups may not be group. Similarly, it is clear that a generalized group is a g-group, but the converse is false.

Remark 2.8. *A g-group may not be a generalized group.*

To see this, consider the g-group in Table 3. Note that 1 is the identity, while $0 \times 0 = 0$ and $0 \times 1 = 0$. This implies that 0 has no inverse.

Remark 2.9. *A generalized group can be made an e-group.*

To see this, let $(G, *)$ be a generalized group. It is easy to see that $(G; *; G)$ is an e-group.

Remark 2.10. *An e-group may not be a generalized group.*

To see this, let $(G, *)$ be a group that is not a generalized group. Then $(G; *; G)$ is an e-group in which $(G, *)$ is not a generalized group.

Figure 1, briefly summarizes the relationship of the different algebraic structure presented in the foregoing discussions. Solid arcs represent the fact that the family in the tail is a subset of the one in the head. On the other hand, dashed arcs represent the idea 'can be made'. For example, a dashed line is drawn from the family of Ubat-spaces to the family of e-groups since, although *Ubat*-spaces $(G, *)$ and e-groups are non comparable structures, a suitable subset A from G can be chosen, so that $(G; *; A)$ is an e-group.

At this point we present some statements about the zero element. The next theorem says that an *Ubat*-space has only one zero element.

Theorem 2.1. *Let U be an Ubat-space. Then the zero element of U is unique.*

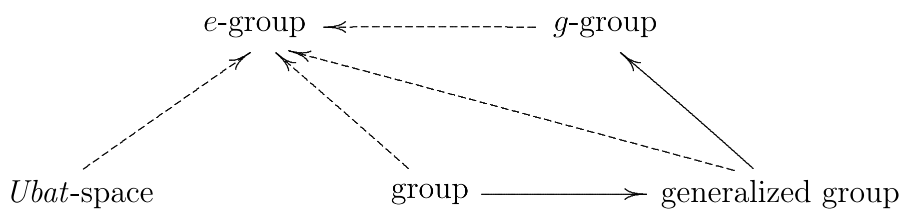

FIGURE 1. Relationship in terms of set theoretic inclusion of the classes of groups, g-groups, e-groups, generalized groups and $Ubat$-spaces

Proof. Suppose that 0 and 0' are the zero of U. Then $0x = x0 = 0$ and $0'x = x0' = 0'$ for all $x \in U$. Thus, $0 = 00' = 0'$. □

An $Ubat$-space has only one identity element. The next theorem presents this idea.

Theorem 2.2. *Let U be an Ubat-space. Then the identity element of U is unique.*

Proof. Suppose that 1 and 1' are identity elements of U. Then $1x = x1 = x$ and $1'x = x1' = x$ for all $x \in U$. Thus, $1 = 11' = 1'$. □

Since the identity element and the zero element are unique, we shall denote them by 1_U and 0_U (or simply 1 and 0, resp.), respectively.

If $\langle U, * \rangle$ is an $Ubat$-space and $x \in U$, and if there exists $y \in U$ such that $xy = yx = 1$, then we say that x is a *unit*, and we call y the *inverse* of x.

We observed that the identity element 1_U is a unit since $1_U = 1_U 1_U = 1_U$.

A unit has only one inverse. The next theorem presents this idea.

Theorem 2.3. *Let U be an Ubat-space. A unit in U has a unique inverse.*

Proof. Let a be a unit. Suppose that b and c are inverses of a. Then $ab = ba = 1_U$ and $ac = ca = 1_U$. Thus, $b = b1_U = b(ac) = (ba)c = 1_U c = c$. □

Since the inverse of a is unique, we shall denote it by a^{-1}.

Theorem 2.4 says that the inverse of a unit is a unit.

Theorem 2.4. *Let U be an Ubat-space and x be an element of U. If x is a unit, then so is its inverse. In particular, $(x^{-1})^{-1} = x$.*

Proof. Let $x \in U$. Note that $xx^{-1} = x^{-1}x = 1_U$. Hence, x is the inverse of x^{-1}, that is $(x^{-1})^{-1} = x$. Thus, x^{-1} is a unit. □

In an $Ubat$-space, the product of two units is a unit. The next theorem presents this idea.

Theorem 2.5. *Let U be an Ubat-space and $x, y \in U$. If a and b are units, then so is ab. In particular, $(ab)^{-1} = b^{-1}a^{-1}$.*

Proof. Let a and b be units. If a and b are units, then there exists $x^{-1}, y^{-1} \in U$ such that $aa^{-1} = a^{-1}a = 1_U$ and $bb^{-1} = b^{-1}b = 1_U$. Now, observe that, $(ab)(b^{-1}a^{-1}) = [a(bb^{-1})]\, a = (a1_U)a^{-1} = aa^{-1} = 1_U$. Hence, $(ab)^{-1} = b^{-1}a^{-1}$. Thus, ab is a unit. □

Theorem 2.6. *Let $\langle U, * \rangle$ be an Ubat-space. If $G = \{x \in U : x \text{ is a unit}\}$, then $(G, *)$ is a group.*

Proof. Let $G = \{x \in U : x \text{ is a unit}\}$. Then by Theorem 2.5, G is closed. Moreover, since the elements of G are elements of U, it follows that $G1$ holds. Since 1_U is a unit, that is $1_U \in G$, it follows that $G2$ holds. Finally, since every element of G is a unit, it follows that $G3$ holds. Accordingly, $(G, *)$ is a group. □

Corollary 2.1 and Corollary 2.2 follows from Theorem 2.6.

Corollary 2.1. *Let $\langle U, *\rangle$ be an Ubat-space, and $G = \{x \in U : x \text{ is a unit}\}$. If a and b are units, then the equations $ax = b$ and $xa = b$ has a unique solution in G.*

Corollary 2.2. *Let $\langle U, *\rangle$ be an Ubat-space, and, a, b, and c are units. If $ac = bc$ or $ca = cb$, then $a = b$.*

2.2. **Subspaces.** In this section we present the notion of subspaces of *Ubat*-spaces. Recall that $\langle U_\beta, *\rangle$ is a *subspace* of $\langle U_\alpha, *\rangle$ if: $(S1)$ $U_\beta \subseteq U_\alpha$; And, $(S2)$ $\langle U_\beta, *\rangle$ is an *Ubat*-space.

Remark 2.11 says that an *Ubat*-space and its subspace may have different zero elements. On the other hand, Remark 2.12 says that an *Ubat*-space and its subspace may have different identity elements.

Remark 2.11. *Let $\langle U, *\rangle$ be an Ubat-space, and V be a subspace of U. Then 0_U may not be in V.*

To see this, consider the *Ubat*-space of Table 5. Note that $\{b, d\}$ is a subspace as seen in Table 7 below. However, its zero is not a (the zero of the larger space) but rather it is d.

$*$	b	d
b	b	d
d	d	d

TABLE 7. The subspace $\{b, d\}$ of $\langle \{a, b, c, d\}, *\rangle$

Remark 2.12. *Let $\langle U, *\rangle$ be an Ubat-space, and V be a subspace of U. Then 1_U may not be in V.*

To see this, consider the *Ubat*-space of Table 4. Note that $\{a, c, e\}$ is a subspace as we can see in Table 8 below. However, its identity is not b (the identity of the mother space) but rather it is e.

$*$	a	c	e
a	a	a	a
c	a	e	c
e	a	c	e

TABLE 8. The subspace $\{a, c, e\}$

Looking at how an *Ubat*-space is defined, one may be tempted right away to conclude that if the zero element and the identity element is in a subset V, then V is a subspace. However, this is not the case.

Remark 2.13. *Let $\langle U, * \rangle$ be an Ubat-space, and $V \subseteq U$. Even if $0_U, 1_U \in V$, V may still not be a subspace of U.*

To see this, consider the *Ubat*-space of Table 4. We note that its zero element is a and its identity element is b. Both a and b are in $\{a, b, c\}$, however, refering to Table 9 below, $\{a, b, c\}$ is not a subspace since it is not closed.

$*$	a	b	c
a	a	a	a
b	a	b	c
c	a	c	e

TABLE 9. The Cayley's table of the subset $\{a, b, c\}$

Given the insight that a subspace may have a different zero and identity from the mother space, the next statement seemed false, however we haven't found any counter-example yet. So we express as a conjecture the statement that the intersection of any family of subspaces is itself a subspace.

Conjecture 2.1. *Let $\langle U, * \rangle$ be an Ubat-space, and $\{\langle U_i, * \rangle : i \in I\}$ be a non-empty family of subspaces of $\langle U, * \rangle$. Then $\left\langle \bigcap_{i \in I} U_i, * \right\rangle$ is a subspace.*

2.3. **Homomorphism.** In this subsection we present the notion of homomorphism of *Ubat*-spaces, and gave some of its important properties. We recall that if $\langle U_\alpha, *_\alpha \rangle$ and $\langle U_\beta, *_\beta \rangle$ are *Ubat*-spaces, then a function $f : U_\alpha \to U_\beta$ is a *homomorphism* if $f(a *_\alpha b) = f(a) *_\beta f(b)$.

Theorem 2.7. *Let $\langle U_1, *_1 \rangle$ and $\langle U_2, *_2 \rangle$ be Ubat-spaces, and $f : U_1 \to U_2$ be a homomorphism, then:*

a. $f(1_{U_1}) = 1_{U_2}$;
b. If x is a unit, then $f(x)^{-1} = f(x^{-1})$; And,
c. $f(0_{U_1}) = 0_{U_2}$.

Proof. (a.) Let $x \in f(U_1)$. Then there exists $y \in U_1$ such that $f(y) = x$. Note that $xf(1_{U_1}) = f(y)f(1_{U_1}) = f(y1_{U_1}) = f(y) = x$, and $f(1_{U_1})x = f(1_{U_1})f(y) = f(1_{U_1}y) = f(y) = x$. Hence, $1_{U_2} = f(1_{U_1})$.

(b.) Let x be a unit of U_1. Then there exists $x^{-1} \in U_1$ such that $xx^{-1} = x^{-1}x = 1_{U_1}$. Note that $f(x)f(x^{-1}) = f(xx^{-1}) = f(1_{U_1}) = 1_{U_2}$, and $f(x^{-1})f(x) = f(x^{-1}x) = f(1_{U_1}) = 1_{U_2}$. Hence, $f(x)^{-1} = f(x^{-1})$.

(c.) Let $x \in f(U_1)$. Then there exists $y \in U_1$ such that $f(y) = x$. Note that $xf(0_{U_1}) = f(y)f(0_{U_1}) = f(y0_{U_1}) = f(0_{U_1})$, and $f(0_{U_1})x = f(0_{U_1})f(y) = f(0_{U_1}y) = f(0_{U_1})$. Hence, $0_{U_2} = f(0_{U_1})$. $\square$

The next theorem says that the homomorphic image of a zero is precisely a zero.

Theorem 2.8. *Let $\langle U_1, *_1 \rangle$ and $\langle U_2, *_2 \rangle$ be nontrivial Ubat-spaces. If $f : U_1 \to U_2$ is a homomorphism, then $f(x) = 0_{U_2}$ if and only if $x = 0_{U_1}$.*

Proof. Let $a \in U_1$ be a unit. Assume that $f(a) = 0_{U_2}$ and $a \neq 0_{U_1}$. Then by Theorem 2.7(a) $1_{U_2} = f(1_{U_1}) = f(aa^{-1}) = f(a)f(a^{-1}) = 0_{U_2}f(a^{-1}) = 0_{U_2}$. This is a contradiction. Therefore, $a = 0_{U_1}$.

Conversely, if $x = 0_{U_1}$, then by Theorem 2.9(3), $f(x) = 0_{U_2}$. □

The next theorem says that the homomorphic image of a unit is precisely a unit.

Theorem 2.9. *Let $\langle U_1, *_1\rangle$ and $\langle U_2, *_2\rangle$ be Ubat-spaces. If $f : U_1 \to U_2$ is a monomorphism, then x is a unit in U_1 if and only if $f(x)$ is a unit in U_2.*

Proof. Let $a \in U_1$ be a unit. Then, $a \in U_1$ is a unit if and only if there exists $a^{-1} \in U_1$ such that $aa^{-1} = 1_{U_1}$, if and only if $f(a)f(a^{-1}) = f(1_{U_1})$, if and only if $f(a)f(a^{-1}) = 1_{U_2}$, if and only if $f(a)f(a)^{-1} = 1_{U_2}$, if and only if $f(a)$ is a unit. □

The *kernel* of a homomorphism $f : U_1 \to U_2$, denoted by Ker f, is the set of all elements of U_1 mapped to 1_{U_2}. Given Theorem 2.9, the next corollary follows.

Corollary 2.3. *Let $\langle U_1, *_1\rangle$ and $\langle U_2, *_2\rangle$ be Ubat-spaces. If $f : U_1 \to U_2$ is a monomorphism, then $x \in$ Ker f implies that x is a unit.*

2.4. **Cyclic Spaces.** In this section, we show that if U is a simple *Ubat*-space, then $W = \{0\} \cup \{x^n : n \in \mathbb{Z}^+\}$ is a subspace.

Remark 2.14. *Let U be an Ubat-space and $x \in U$. Then $x^m x^n = x^{m+n}$ for all $m, n \in \mathbb{N}$.*

Remark 2.15. *Let U be an Ubat-space and $x \in U$. Then $(x^m)^n = x^{mn}$ for all $m, n \in \mathbb{N}$.*

Lemma 2.1. *Let U be a finite Ubat-space and $x \in U\backslash\{0\}$. If $W^* = \{x^n : n \in \mathbb{Z}^+\}$, then there exist positive integers i and j with $1 < i < j$ such that $x^i = x^j$.*

Proof. Suppose that $x^i \neq x^j$ for $i \neq j$. Then $W^* = \{x^n : n \in \mathbb{Z}^+\}$ must be infinite. Thus, it follows that U is infinite also. This is a contradiction. □

In the sense of Lemma 2.1, we let $S = \{k \in \mathbb{N} : k = j - i\}$. By Lemma 2.1 $S \neq \varnothing$. Hence, by the *Well-ordering Principle* S has a least element, say m. We will call m the *order* of W^*, denoted by $|x|$ or simply m. Hereafter, the x in $W^* = \{x^n : n \in \mathbb{Z}^+\}$ is a non-zero element of U.

Lemma 2.2. *Let U be a finite Ubat-space and $W^* = \{x^n : n \in \mathbb{Z}^+\}$ be a subset of order m. Let i and j (with $1 \leq i < j$) be positive integers such that $x^i = x^j$ and $j - i = m$. Then the elements x^i, x^{i+1}, $x^{i+2}, \ldots, x^{i+(m-1)}$ are distinct.*

Proof. Suppose that there exist positive integers s and t with $i \leq s < t \leq i + m - 1$ such that $x^s = x^t$. Since $1 \leq t - s \leq m$, this contrary to our choice of m. □

Lemma 2.3. *Let U be a finite Ubat-space and $W^* = \{x^n : n \in \mathbb{Z}^+\}$ be a subset of order m. Let i and j (with $1 \leq i < j$) be positive integers such that $x^i = x^j$ and $j - i = m$. Then $x^{i+l} = x^{j+l}$ for $l = 1, 2, \ldots, m$.*

Proof. If $x^i = x^j$, then $x^i x^k = x^j x^k$ for all $k \in \mathbb{N}$, that is $x^{i+k} = x^{j+k}$ for all $k \in \mathbb{N}$. In particular, $x^{i+l} = x^{j+l}$ for $l = 1, 2, \ldots, m$. □

Corollary 2.4. *Let U be a finite Ubat-space and $W^* = \{x^n : n \in \mathbb{Z}^+\}$ be a subset of order m. Let i and j (with $1 \leq i < j$) be positive integers such that $x^i = x^j$ and $j - i = m$. Then $x^{i+l} = x^{i+nm+l}$ for $l = 1, 2, \ldots, m$, and for all $n \in \mathbb{N}$.*

Proof. We use induction. For $n = 1$, we have by Lemma 2.3 $x^{i+l} = x^{j+l}$ for $l = 1, 2, \ldots, m$. Hence, the assertion holds $n = 1$. Setting $l = m$, we have $x^{i+m} = x^{j+m}$, this is $x^i = x^j = x^{i+2m}$. By Lemma 2.3 $x^{i+l} = x^{i+2m+l}$ for $l = 1, 2, \ldots, m$, that is, the assertion holds for $n = 2$. Let $q \geq 2$ and assume that $x^{i+l} = x^{i+qm+l}$ for $l = 1, 2, \ldots, m$. By the inductive assumption $x^{i+(q-1)m+l} = x^{i+l} = x^{i+qm+l}$, that is $x^{i+(q-1)m+l} = x^{i+qm+l}$ for $l = 1, 2, \ldots, m$. Setting $l = m$, we have $x^{i+qm} = x^{i+(q+1)m}$. Hence, by Lemma 2.3 again $x^{i+qm+l} = x^{i+(q+1)m+l}$ for $l = 1, 2, \ldots, m$. Thus, $x^{i+l} = x^{j+(q+1)m+l}$ for $l = 1, 2, \ldots, m$. And so, the assertion holds for $q + 1$. Therefore, by the *Principle of Mathematical Induction* the lemma follows. □

Corollary 2.5. *Let U be a finite Ubat-space and $W^* = \{x^n : n \in \mathbb{Z}^+\}$ be a subset of order m. Then $x^i = x^{i+mn}$ for all $n \in \mathbb{N}$.*

Proof. In the sense of Corollary 2.4, let $l = m$. Then by Corollary 2.4, $x^i = x^j = x^{i+m} = x^{i+nm+m} = x^{i+(n+1)m} = x^{i+pm}$ for all $p \in \mathbb{N}$. □ □

Lemma 2.4. *Let U be a simple Ubat-space and $W^* = \{x^n : n \in \mathbb{Z}^+\}$ be a subset of order m. If i and j (with $1 \leq i < j$) are positive integers such that $x^i = x^j$ and $j - i = m$, then $x^{i-l} = x^{j-l}$ for $l = 1, 2, \ldots, m$.*

Proof. Suppose that $l \in \{1, 2, \ldots, m\}$ and $x^{i-l} \neq x^{j-l}$. Since U is simple, $x^{i-l}x^l \neq x^{j-l}x^l$, that is $x^i \neq x^j$. This is a contradiction. □ □

Corollary 2.6. *Let U be a simple Ubat-space and $W^* = \{x^n : n \in \mathbb{Z}^+\}$ be a subset of order m. If i and j (with $1 \leq i < j$) are positive integers such that $x^i = x^j$ and $j - i = m$, then $x^{j-l} = x^{j-nm-l}$ for $l = 1, 2, \ldots, m$, and for all n with $j - (n+1)m \geq 1$.*

Proof. Proved similarly as Corollary 2.4. □ □

Theorem 2.10. *Let U be a simple Ubat-space and $W^* = \{x^n : n \in \mathbb{Z}^+\}$ be a subset of order m. If $s \equiv t \pmod{m}$, then $x^s = x^t$.*

Proof. Let i and j (with $1 < i < j$) be positive integers such that $x^i = x^j$ and $j - i = m$. Without loss of generality, assume that $s = i$. If $s \equiv t (\text{mod } m)$, $m|(s - t)$. Hence, there exist $k \in \mathbb{Z}$ such that $mk = s - t$, that is $s = t + mk$. Then by Corollary 2.5 $x^s = x^{t+mk} = x^t$. □

Theorem 2.11. *Let U be a simple Ubat-space and $W^* = \{x^n : n \in \mathbb{Z}^+\}$ be a subset of order m. Then x^m is an identity of W^*.*

Proof. Let $v \in \langle x \rangle$. Then $v = x^a$ for some $a \in \mathbb{N}$. By the Division Algorithm, $a = mq + r$ with $0 \leq r < m$. Since $m(q+1) \equiv mq (\text{mod } m)$, we have by Theorem 2.10 $vx^m = x^a x^m = x^{mq+r}x^m = x^{mq+r+m} = x^{m(q+1)+r} = x^{m(q+1)}x^r = x^{mq}x^r = x^{mq+r} = x^a = v$. Similarly, $x^m v = x^m x^a = x^m x^{mq+r} = x^{m+mq+r} = x^{m(q+1)+r} = x^{m(q+1)}x^r = x^{mq}x^r = x^{mq+r} = x^a = v$. □

Theorem 2.12. *Let U be a simple Ubat-space and $x \in U \backslash \{0\}$. Then $W = \{0\} \cup \{x^n : n \in \mathbb{Z}^+\}$ is a subspace of U.*

Proof. Let $x, y \in W$. If $x = 0$, then $xy = 0 \in W$. If $x \neq 0$ and $y \neq 0$, then $x = x^s$ and $y = x^t$ for some $s, t \in \mathbb{N}$. Thus, $xy = x^s x^t = x^{s+t} \in W$. Therefore, W is closed. Next, observe that $x0 = 0x = 0$ for all $x \in W$. Hence, the zero element is 0. If m is the order of $W \backslash \{0\}$, then by Theorem 2.11, x^m is the identity element of $W \backslash \{0\}$, and must be the order of W. Finally, since $W \subseteq U$, the operation must be associative in W. Accordingly, W is a subspace of U. □

The subspace $W = \{0\} \cup \{x^n : n \in \mathbb{Z}^+\}$ is called the *cyclic* subspace of U generated by x, denoted by $\langle x \rangle$. An element x of an *Ubat*-space U *generates* U if $\langle x \rangle = U$. In this case, the element x is called the *generator* of U.

2.5. **Normal Spaces.** In this section, we present a *First Isomorphism Theorem* for *Ubat*-spaces.

Remark 2.16. *Let U be an Ubat-space and T be a normal subspace of U. If V is a subspace of U with $T \subseteq V$, then T is a normal subspace of V.*

To see this, let T be a normal subspace of U. Tthen $xT = Tx$ for all $x \in U$. Since $V \subseteq U$, $xT = Tx$ for all $x \in U$. Since T is a subspace of V, T is also a normal subspace of V.

Theorem 2.13. *Let U be an Ubat-space and T be a normal subspace of U. If U/T is the set of all left cosets of T in U, then U/T is an Ubat-space with $xT * yT = xyT$.*

Proof. Let $x, y \in U$. Since U is an *Ubat*-space, $xy \in U$. Hence, $xTyT = xyT \in U/T$. This shows that U/T is closed. Next, let $x, y, z \in U$. Since U is an *Ubat*-space, $x(yz) = (xy)z$. Hence, $xT(yTzT) = xT(yzT) = x(yz)T = (xy)zT = (xyT)zT = (xTyT)zT$. Thus, the operation is associative. Next, let 0 be the zero element of U. Then $0x = x0 = 0$ for all $x \in U$. Hence, $0TxT = 0xT = 0T = \{0\} = 0T = x0T = xT0T$ for all $x \in U$. Thus, the zero of U/T is $\{0\}$. Finally, let 1 be the identity of U. Then $1x = x1 = x$ for all $x \in U$. Observe that $1TxT = 1xT = xT = x1T = xT1T$ for all $x \in U$. Thus, the identity of U/T is $1T = T$. Accordingly, U/T is an *Ubat*-space. □

Theorem 2.14. *Let U and V be Ubat-spaces. If $f : U \to V$ is a monomorphism, then the kernel of f is a normal subspace of U.*

Proof. Let $x \in U$. We show that $x\text{Ker}f = \text{Ker}f\, x$. If $z \in x\text{Ker}f$, then $z = xy$ for some $y \in \text{Ker}f$. Hence, $f(z) = f(xy) = f(x)f(y) = f(x)1 = 1f(x) = f(y)f(x) = f(yx)$. Since f is a monomorphism, $z = yx$. Thus, $z = yx \in \text{Ker}f x$. On the other hand, if $w \in \text{Ker}f x$, then $w = yx$ for some $y \in \text{Ker}f$. Hence, $f(w) = f(yx) = f(y)f(x) = 1f(x) = f(x)1 = f(x)f(y) = f(xy)$. Since f is a monomorphism, $w = xy$. Thus, $w = xy \in x\text{Ker}f$. This shows that $\text{Ker}f$ is normal. □

Theorem 2.15. *Let U be an Ubat-space. If T is a normal subspace of U, then the map $\mu : U \to U/T$ given by $x \mapsto xT$ is an epimorphism with kernel T.*

Proof. Let $x, y \in U$. Then $\mu(xy) = xyT = xTyT = \mu(x)\mu(y)$. This shows that μ is a homomorphism. Next, we show that μ is surjective. Clearly, $\mu(U) \subseteq U/T$. Let $xT \in U/T$, and consider x (which is in U). Observe that $\mu(x) = xT$, that is $xT \in \mu(U)$. Hence, $\mu(U) \supseteq U/T$. Thus, $\mu(U) = U/T$, that is μ is an epimorphism. Finally, $x \in T$ if and only if $\mu(x) = xT = T$ if and only if $x \in \text{Ker}\mu$. Thus, $\text{Ker}\mu = T$. □

Theorem 2.16. *(First Isomorphism Theorem) Let U and V be Ubat-spaces. If $f : U \to V$ be a monomorphism, then f induces an isomorphism $U/Kerf \cong Imf$.*

Proof. Define $\mu(U) : U/\mathrm{Ker}f \to \mathrm{Im}f$ by $x\mathrm{Ker}f = f(x)$. Let $u, v \in U/\mathrm{Ker}f$. Then there exists $x, y \in U$ such that $u = x\mathrm{Ker}f$ and $v = y\mathrm{Ker}f$. Thus, $\mu(uv) = \mu(x\mathrm{Ker}f\, y\mathrm{Ker}f) = \mu(xy\mathrm{Ker}f) = f(xy) = f(x)f(y) = \mu(x\mathrm{Ker}f)\mu(y\mathrm{Ker}f) = \mu(u)\mu(v)$. This shows that μ is a homomorphism. Next, we show that μ is injective. Let $u, v \in U/\mathrm{Ker}f$ such that $\mu(u) = \mu(v)$. If $u, v \in U/\mathrm{Ker}f$, then there exists $x, y \in U$ such that $u = x\mathrm{Ker}f$ and $v = y\mathrm{Ker}f$. If $\mu(u) = \mu(v)$, then $\mu(x\mathrm{Ker}f) = \mu(y\mathrm{Ker}f)$, that is $f(x) = f(y)$. This implies that $x = y$. Thus, $x\mathrm{Ker}f = y\mathrm{Ker}f$, that is $u = v$. This shows that μ is injective. Lastly, we show that μ is surjective. Let $w \in \mathrm{Im}f$. Then there exists $x \in U$ such that $w = f(x)$. Now, consider $x\mathrm{Ker}f$. Observe that $\mu(x\mathrm{Ker}f) = f(x) = w$. This shows that μ is surjective. Accordingly, μ is an isomorphism, that is $U/\mathrm{Ker}f \cong \mathrm{Im}f$. □

3. g-Groups

This section presents some important properties of g-groups. We note that a g-group can have two or more identity elements. To see this, consider the g-group $\{0, 1\}$. Note that both elements are identity.

The set of all real numbers $\mathbb{R}$ is a g-group with respect to multiplication. To see this, we must be able to show that $\mathbb{R}$ together with multiplication satisfies $g1$, $g2$, and $g3$. Clearly, $g1$ holds. To show $g2$, let $g \in \mathbb{R}$. If $g = 0$, then we take $e = 0$. Note that $ge = eg = 0(0) = 0 = g$. On the other hand, if $g \neq 0$, then we take $e = 1$. Note that $ge = g(1) = g$ and $eg = 1(g) = g$. Thus, in each case, there exist $e \in \mathbb{R}$ such that $ge = eg = g$. This shows $g2$. Finally, to show $g3$, we again let $g \in \mathbb{R}$. If $g = 0$, then consider $h = 0$. Note that $gh = hg = 0(0) = 0 = e$. On the other hand, if $g \neq 0$, then we consider $h = 1/g$. Note that $gh = g(1/g) = 1 = e$ and $hg = (1/g)g = 1 = e$. Thus, in each case, there exist $h \in \mathbb{R}$ such that $gh = hg = e$ for some identity e. This shows $g3$.

It is easy to see that a group is a g-group. However, the converse of is not true. The next remark presents this idea.

Remark 3.1. *A g-group may not be a group.*

To see this, we note that the set of all real numbers $\mathbb{R}$ is a g-group with respect to multiplication. However, it is not a group since the element 0 has no multiplicative inverse.

Remark 3.2. *A g-group can be made an e-group.*

To see this, let G be a g-group under a binary operation $*$, and let A be the set of all identity elements. It is easy to see that $(G; *; A)$ is an e-group.

Remark 3.3. *An e-group is not necessarily a g-group.*

To see this, consider the e-group $(\{a, b, c, d\}; *; \{a, b\})$ of Example 2.5 in [1]. Note that $(\{a, b, c, d\}, *)$ does not satisfy $g3$ since the element c has no inverse.

Remark 3.4. *A simple e-group is not necessarily a g-group.*

$*$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	a	a
d	a	d	c	d

TABLE 10. The e-group $(\{a, b, c, d\}; *; \{a, b\})$

$*$	a	b	c	d
a	a	a	c	d
b	a	b	c	d
c	c	c	d	c
d	d	d	c	c

TABLE 11. The simple e-group $(\{a, b, c, d\}; *; \{a, b\})$

To see this, we may consider the simple e-group $(\{a, b, c, d\}; *; \{a, b\})$ is presented in Table 11. Note that the elements c and d have no inverse, hence it doe not satisfy $g3$.

The identity element and inverse element may not be unique. For example, consider the g-group $\mathbb{R}$ under multiplication. Note that each element in $\mathbb{R}$ is an identity and an inverse of 0.

If G be a g-group, then we will call an element of G having a unique identity element a *unit*. In this case, we denote by e_a the identity element of a. For example, in the g-group $\mathbb{R}$ under multiplication. All the elements of $\mathbb{R}$ except 0 is a unit. Note that 0 is not a unit since it has many identity elements, in fact all the real numbers is its identity.

Lemma 3.1. *Let G be a g-group and $x \in G$. If x is a unit, then so is any inverse y of x. In particular, $e_x = e_y$.*

Proof. Let e_x be the identity of x, and e_y be an identity of y. Suppose $e_x \neq e_y$. Since x has a unique identity, $xe_y \neq x$. Since $x = xe_x$, $xe_y \neq xe_x$. Thus, $(xe_x)e_y \neq x(xy)$. But $(xe_x)e_y = [x(xy)]\, e_y = (xx)(ye_y) = (xx)y = x(xy)$. Hence, $(xe_x)e_y \neq (xe_x)e_y$. This is a contradiction. Therefore, we must have $e_x = e_y$. □

The inverse of a unit is unique. This idea is presented in the next theorem.

Theorem 3.1. *A unit has a unique inverse.*

Proof. Let G be a g-group and x be a unit. Suppose that y and z are inverse of x. Then by Theorem 3.1, $y = ye_y = ye_x = y(xz) = (yx)z = e_xz = e_zz = z$. □

Since the inverse of a unit x is unique, we may now denote it by x^{-1}.

The converse of Theorem 3.1 is not true. To see this consider the g-group $S_6 = \{0, 1, 2, 3, 4, 5\}$ given in Table 12. Note that 4 has a unique inverse (which is itself) but is not a unit.

The next statement says that the inverse of the inverse of a unit is the unit itself.

Corollary 3.1. *Let G be a g-group and $x \in G$. If x is a unit, then $(x^{-1})^{-1} = x$.*

Proof. If x is a unit, then by Lemma 3.1, $xx^{-1} = x^{-1}x = e_x = e_{x^{-1}}$. Hence, x is an inverse of x^{-1}. By Lemma 3.1, x^{-1} is a unit. Thus, by Theorem 3.1, the inverse of x^{-1} is unique. Therefore, $(x^{-1})^{-1} = x$. □

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

TABLE 12. The g-group $S_6 = \{0, 1, 2, 3, 4, 5\}$

The next theorem says that two units have the same inverse precisely when the two are equal.

Theorem 3.2. *Let G be a g-group. If x and y are units, then $x^{-1} = y^{-1}$ if and only if $x = y$.*

Proof. If $x^{-1} = y^{-1}$, then $(x^{-1})^{-1} = (y^{-1})^{-1}$. By Corollary 3.1, $x = y$. The converse follows immediately from Lemma 3.1 and Theorem 3.1. □

The next theorem says that cancellation law also holds in g-groups under some conditions.

Theorem 3.3. *Let G be a g-group and $x, y, z \in G$ be units with $e_x = e_y = e_z$. If $xy = xz$ and $yx = zx$, then $y = z$.*

Proof. If $xy = xz$, then $y = e_y y = e_x y = (x^{-1}x)y = x^{-1}(xy) = x^{-1}(xz) = (x^{-1}x)z = e_x z = e_z z = z$. The second equation is proved similarly. □

The identity of an identity is itself. This is presented in the next theorem.

Theorem 3.4. *Let G be a g-group and $x \in G$. If x is a unit, then $(e_x)^n = e_x$ for all positive integer n.*

Proof. For $n = 1$, we have $(e_x)^1 = e_x$. Let $k \geq 1$, and assume that $(e_x)^k = e_x$. Then $(e_x)^{k+1} = (e_x)^k e_x = e_x e_x = (e_x)^2$. But, $(e_x)^2 x = (e_x e_x)x = e_x(e_x x) = e_x x = x$. Hence, $(e_x)^{k+1} = e_x$. By the Principle of Mathematical Induction the theorem follows. □

Corollary 3.2 and Corollary 3.3 follows from Theorem 3.4.

Corollary 3.2. *Let G be a g-group and $x \in G$. If x is a unit, then $e_{e_x} = e_x$ and $(e_x)^{-1} = e_x$.*

Corollary 3.3. *Let G be a g-group and $x \in G$. If x is a unit, then e_x is a unit.*

3.1. **Abelian g-groups.** An g-group G is *Abelian* if for all $x, y \in G$, we have $xy = yx$. For example, $\mathbb{R}$ is an Abelian g-group under multiplication.

It is clear that if G is an Abelian g-group and a is a unit, and if $ae = a$, then $e = e_a$. This idea is used in the next theorem.

Theorem 3.5. *Let G be an Abelian g-group and $a, b \in G$. If x and y are identity elements of a and b, respectively, then xy is an identity of ab. In particular, if a, b, and ab are units, then $e_{ab} = e_a e_b$.*

Proof. Let $a, b \in G$, and, x and y be identity elements of a and b, respectively. Since G is Abelian, $(ab)(xy) = (ax)(by) = ab = (xa)(yb) = (xy)(ab)$. Hence, xy is an identity of ab. □

Note that if G be an Abelian g-group and a is a unit, and if $ab = e_a$, then $a^{-1} = b$. This idea is used in the next theorem.

Theorem 3.6. *Let G be an Abelian g-group and $a, b \in G$. If a, b, and ab are units, then $(ab)^{-1} = a^{-1}b^{-1}$.*

Proof. By Theorem 3.5, we have $(ab)(a^{-1}b^{-1}) = (aa^{-1})(bb^{-1}) = e_a e_b = e_{ab}$. Hence, $(ab)^{-1} = a^{-1}b^{-1}$. □

Let G be a g-group, and $H = \{h \in G : h \text{ is a unit}\}$. We say that H has a *unique identity* if the following condition holds: $a, b \in H$ implies that $e_a = e_b$. Let $\mathbb{Z}_n$ be the g-group of integers modulo n under multiplication, and $H = \{x \in \mathbb{Z}_n : x \text{ is a unit}\}$. Then H has a unique identity. To see this, we observe that 1 is a common identity. Hence, if x is a unit, then 1 must be its only identity.

Let G be a g-group and $x \in G$. Then x is called a *zero* of G if $xy = yx = x$ for all $y \in G$. Clearly, 0 is not a unit. If $\mathbb{Z}_n$ is the g-group of integers modulo n under multiplication. Then the element 0 is is the zero of $\mathbb{Z}_n$.

Theorem 3.7 says that if $H = \{h \in G : h \text{ is a unit}\}$ has a unique identity, then a linear equation has a unique solution in H.

Theorem 3.7. *Let G be an Abelian g-group, and $H = \{h \in G : h \text{ is a unit}\}$. If H has a unique identity, then the linear equations $ax = b$ and $xa = b$ has a unique solution in H.*

Proof. We have $ax = b$ if and only if $a^{-1}(ax) = a^{-1}b$ if and only if $(a^{-1}a)x = a^{-1}b$ if and only if $ex = a^{-1}b$ if and only if $x = a^{-1}b$. This shows that $x = a^{-1}b$ is a unique solution to $ax = b$. The second is proved similarly. □

A cancellation law also holds in g-groups under certain conditions.

Theorem 3.8. *Let G be an Abelian g-group, and $H = \{h \in G : h \text{ is a unit}\}$ has a unique identity element. If $a \in H$ and, $ab = ac$ or $ba = ca$, then $b = c$.*

Proof. Let $a, b, c \in H$ with $ab = ac$. If $ab = ac$, then b is a solution to the equation $ax = ac$. Since $ac = ac$, c is also a solution to the equation $ax = ac$. Now by Theorem 3.7, we must have $b = c$. The second is proved similarly. □

Even if x is not a unit, only one of its identity gives an inverse. The next theorem crystallize this idea.

Theorem 3.9. *Let G be an Abelian g-group and $H = \{h \in G : h \text{ is a unit}\}$. If $x \in G \backslash H$, then x has only one identity element e such that there exists y with $xy = e$.*

Proof. Let $x \in G\backslash H$, and, e_1 and e_2 be distinct identities of x. Suppose that $xy = e_1$ and $xz = e_2$. Then $xy \neq xz$. Hence, $(xe_2)y \neq (xe_1)z$. Thus, $[x(xz)]\, y \neq [x(xy)]\, z$. This is a contradiction. □

Although only one identity produce an inverse, Theorem 3.9 does not imply that there is only one inverse. Consider the g-group $S_6 = \{0, 1, 2, 3, 4, 5\}$ under multiplication modulo 6 of Table 12. Observe that 2 have two identity elements, namely, 1 and 4. Now, only one of them has an inverse, only 4. However, there are two inverses which corresponds to 4, namely 2 and 5.

Remark 3.5 says that elements with identity e are contained in $H = \{h \in G : h \text{ is a unit, and } e_h = e\}$, while Theorem 3.10 says that the inverse of the elements of $H = \{h \in G : h \text{ is a unit}\}$ are in H.

Remark 3.5. *Let G be an Abelian g-group, e be an identity element, and $H = \{h \in G : h \text{ is a unit, and } e_h = e\}$. If $x \in G$ and $e_x = e$, then $x \in H$.*

Theorem 3.10. *Let G be an Abelian g-group and $H = \{h \in G : h \text{ is a unit}\}$. If H is a trunk (in which the identity of the elements is e) and $xy = e$, then $y \in H$.*

Proof. Let $x \in G$ and $xy = e$. Suppose that $xe' = x$. Then $e'e = e'(xy) = (e'x)y = xy = e$. Hence, e' is an identity of e. Since e is a unit and $e^2 = e$, we must have $e' = e$. Therefore, $x \in G$. □

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